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Mathématique

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Volume 362 (2024), p. 275-278

Online since: 2 May 2024

https://doi.org/10.5802/crmath.555

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Académie des Sciences
Institut de France

Comptes Rendus. Mathématique
2024, Vol. 362, p. 275-278
https://doi.org/10.5802/crmath.555

Research article / *Article de recherche*
Number theory / *Théorie des nombres*

On weakly prime-additive numbers with length $4k+3$

Jin-Hui Fang *,[a] **and Fang-Gang Xue** [b]

[a] School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, P.R. China
[b] Nanjing University of Information Science & Technology, Nanjing 210044, P.R. China
E-mails: fangjinhui1114@163.com, xfg950118@126.com

Abstract. If a positive integer n has at least two distinct prime divisors and can be written as $n = p_1^{\alpha_1} + \cdots + p_t^{\alpha_t}$, where $p_1 < \cdots < p_t$ are prime divisors of n and $\alpha_1, \ldots, \alpha_t$ are positive integers, then we define such n as weakly prime-additive. Obviously, $t \geqslant 3$. Following Erdős and Hegyvári's work, Fang and Chen [J. Number Theorey 182(2018), 258-270] obtained the following result: for any positive integer m, there exist infinitely many weakly prime-additive numbers n with $m \mid n$ and $n = p_1^{\alpha_1} + \cdots + p_5^{\alpha_5}$, where $p_1, \ldots, p_5$ are distinct prime divisors of n and $\alpha_1, \ldots, \alpha_5$ are positive integers. In this paper, we prove the existence of such n with general length t, where $t \equiv 3 \pmod 4$ and $t > 3$. The main result is summarized as follows: for any positive integers m, t with $t \equiv 3 \pmod 4$ and $t > 3$, there exist infinitely many weakly prime-additive numbers n with $m \mid n$ and $n = p_1^{\alpha_1} + \cdots + p_t^{\alpha_t}$, where $p_1, \ldots, p_t$ are distinct prime divisors of n and $\alpha_1, \ldots, \alpha_t$ are positive integers.

Keywords. weakly prime-additive numbers, Dirichlet's theorem, the Chinese remainder theorem.

2020 Mathematics Subject Classification. 11A07, 11A41.

Funding. This work is supported by the National Natural Science Foundation of China, Grant No. 12171246 and the Natural Science Foundation of Jiangsu Province, Grant No. BK20211282.

Manuscript received 28 July 2023, revised 14 August 2023, accepted 17 August 2023.

1. Introduction

If a positive integer n has at least two distinct prime divisors and can be written as $n = p_1^{\alpha_1} + \cdots + p_t^{\alpha_t}$, where $p_1 < \cdots < p_t$ are prime divisors of n and $\alpha_1, \ldots, \alpha_t$ are positive integers, then we define such n as weakly prime-additive. Obviously, $t \geqslant 3$. In 1992, Erdős and Hegyvári [2] firstly considered weakly prime-additive numbers and proved that, for any prime p, there exist infinitely many weakly prime-additive numbers n such that $p|n$.

In 2018, Fang and Chen [5] made further research on weakly prime-additive numbers and obtained the following result:

Theorem 1. *For any positive integer m, there exist infinitely many weakly prime-additive numbers with $t = 3$ which are divisible by m if and only if $8 \nmid m$.*

In [5], Fang and Chen also proved that:

* Corresponding author.

Theorem 2. *For any positive integer m, there exist infinitely many weakly prime-additive numbers n with $m \mid n$ and*

$$n = p_1^{\alpha_1} + \cdots + p_5^{\alpha_5},$$

where $p_1, \ldots, p_5$ are distinct prime divisors of n and $\alpha_1, \ldots, \alpha_5$ are positive integers.

Afterwards, Fang [3] proved that, for any positive integers m, t with $t \equiv 1 \pmod 4$ and $t > 1$, there exist infinitely many weakly prime-additive numbers n with $m \mid n$ and $n = p_1^{\alpha_1} + \cdots + p_t^{\alpha_t}$, where $p_1, \ldots, p_t$ are distinct prime divisors of n and $\alpha_1, \ldots, \alpha_t$ are positive integers. In this paper, by adding some tricky arguments, we solve the case $t \equiv 3 \pmod 4$, where $t > 3$, that is:

Theorem 3. *For any positive integers m, t with $t \equiv 3 \pmod 4$ and $t > 3$, there exist infinitely many weakly prime-additive numbers n with $m \mid n$ and*

$$n = p_1^{\alpha_1} + \cdots + p_t^{\alpha_t}, \tag{1}$$

where $p_1, \ldots, p_t$ are distinct prime divisors of n and $\alpha_1, \ldots, \alpha_t$ are positive integers.

Let k, l be integers with $k \geqslant 2$ and $l \geqslant 3$. Write

$$S_k = \left\{ n : n = \sum_{p \mid n} p^k, \omega(n) \geqslant 2 \right\},$$

where $\omega(n)$ is the number of distinct prime factors of n. In 2005, De Koninck and Luca [1] obtained many nice results about S_k. As a main result, they considered the nature of S_k and identified all integers n in S_3 with $\omega(n) = 3$. Let $\{P_m\}_{m \geqslant 0}$ denote the Pell sequence given by $P_0 = 0$, $P_1 = 1$ and $P_{m+1} = 2P_m + P_{m-1}$ for $m \geqslant 1$. In 2022, Fang [4] proved that, a positive integer n can be expressed as $n = 2^2 + p^2 + q^2$, where p, q are distinct odd prime factors of n, if and only if, $n = 2^2 + P_{2m+1}^2 + P_{2m+3}^2$ for some positive integer m, where P_{2m+1}, P_{2m+3} are both primes. One may refer to [1] and [4] for details.

2. Proof of Theorem 3

The idea is from [5, Theorem 1.3] and [3, Theorem 1].

Proof. Write $t = 4k+3$ and $m = 2^a m_1$, where $a \geqslant 0$ and $2 \nmid m_1$. Then $k \geqslant 1$. Let ϕ denote the Euler totient function. Similar to [3, Theorem 1], we consider the integers

$$n = 2^u + p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_{4k+1}^{\alpha_{4k+1}} + p_{4k+2}, \tag{2}$$

where p_i are primes fixed later,

$$u = (a+2)\phi(m_1) \prod_{i=1}^{4k+2} (p_i - 1) \text{ and } \alpha_i = \prod_{j=i+1}^{4k+2} \frac{p_j - 1}{2} \text{ for } i = 1, 2, \ldots, 4k+1.$$

(The method of this proof and the construction are the same as [3, Theorem 1], but we use a tricky idea during the choices of $p_1, p_2, \ldots, p_{4k+2}$, which is the key point during the proof.)

By the Chinese remainder theorem and Dirichlet's theorem there exists a prime $p_1 > \max\{2^{a+2}, m_1, 5\}$ such that

$$p_1 \equiv 1 \pmod{2^{a+2}}, \quad p_1 \equiv -1 \pmod{m_1}.$$

By the Chinese remainder theorem and Dirichlet's theorem there exists a prime $p_2 > p_1$ such that

$$p_2 \equiv -1 \pmod{2^{a+2}}, \quad p_2 \equiv 1 \pmod{m_1}, \quad p_2 \equiv 1 \pmod{p_1}.$$

By the Chinese remainder theorem and Dirichlet's theorem there exists a prime $p_3 > p_2$ such that

$$p_3 \equiv -1 \pmod{2^{a+2}}, \quad p_3 \equiv 1 \pmod{m_1}, \quad p_3 \equiv -1 \pmod{p_1}, \quad p_3 \equiv 1 \pmod{p_2}.$$

By the Chinese remainder theorem and Dirichlet's theorem there exists a prime $p_4 > p_3$ such that

$$p_4 \equiv -1 \pmod{2^{a+2}},\ p_4 \equiv -1 \pmod{m_1},\ p_4 \equiv 1 \pmod{p_1},\ p_4 \equiv -1 \pmod{p_2},\ p_4 \equiv 1 \pmod{p_3}.$$

For $i \geqslant 5$, we will choose p_i successively according to the parity of i. For $j \geqslant 3$, we firstly choose p_{2j-1}, where $3 \leqslant j \leqslant 2k$. By the Chinese remainder theorem and Dirichlet's theorem there exists a prime $p_{2j-1} > p_{2j-2}$ such that

$$\begin{aligned}
&p_{2j-1} \equiv -1 \pmod{2^{a+2}}, \quad p_{2j-1} \equiv -1 \pmod{m_1},\\
&p_{2j-1} \equiv -1 \pmod{p_1}, \quad p_{2j-1} \equiv 1 \pmod{p_2}, \quad p_{2j-1} \equiv 1 \pmod{p_3}, \quad p_{2j-1} \equiv 1 \pmod{p_4},\\
&p_{2j-1} \equiv 1 \pmod{p_{2s-1}}, \quad p_{2j-1} \equiv -1 \pmod{p_{2s}} \quad \text{for } s = 3,4,\dots,j-1.
\end{aligned}$$

(If $j = 3$, then we omit the last procedure).

Now we choose p_{2j}. For $3 \leqslant j \leqslant 2k$, by the Chinese remainder theorem and Dirichlet's theorem there exists a prime $p_{2j} > p_{2j-1}$ such that

$$\begin{aligned}
&p_{2j} \equiv -1 \pmod{2^{a+2}}, \quad p_{2j} \equiv 1 \pmod{m_1},\\
&p_{2j} \equiv 1 \pmod{p_1}, \quad p_{2j} \equiv -1 \pmod{p_2}, \quad p_{2j} \equiv -1 \pmod{p_3}, \quad p_{2j} \equiv -1 \pmod{p_4},\\
&p_{2j} \equiv -1 \pmod{p_{2s-1}}, \quad p_{2j} \equiv 1 \pmod{p_{2s}} \quad \text{for } s = 3,4,\dots,j-1, \quad p_{2j} \equiv -1 \pmod{p_{2j-1}}.
\end{aligned}$$

Furthermore, we add the restriction

$$p_2 \equiv p_3 \equiv 1 \pmod 3,\ p_4 \equiv 2 \pmod 3,\ p_{2j} \equiv 1 \pmod 5 \text{ for } j = 3,4,\dots,2k. \tag{3}$$

(If $3 \mid m_1$, then the congruences $p_2 \equiv p_3 \equiv 1 \pmod 3$ and $p_4 \equiv 2 \pmod 3$ follow from the condition $p_2 \equiv p_3 \equiv 1 \pmod{m_1}$ and $p_4 \equiv -1 \pmod{m_1}$; if $5 \mid m_1$, then the congruences $p_{2j} \equiv 1 \pmod 5$ follow from the condition $p_{2j} \equiv 1 \pmod{m_1}$ for $j = 3,4,\dots,2k$; if $(3, m_1) = 1$ or $(5, m_1) = 1$, then (3) follows from Dirichlet's theorem.)

By the Chinese remainder theorem and Dirichlet's theorem there exists a prime $p_{4k+1} > p_{4k}$ such that

$$\begin{aligned}
&p_{4k+1} \equiv -1 \pmod{2^{a+2}}, \quad p_{4k+1} \equiv 1 \pmod{m_1},\\
&p_{4k+1} \equiv -1 \pmod{p_1}, \quad p_{4k+1} \equiv 1 \pmod{p_2}, \quad p_{4k+1} \equiv 1 \pmod{p_3}, \quad p_{4k+1} \equiv 1 \pmod{p_4},\\
&p_{4k+1} \equiv 1 \pmod{p_{2s-1}}, \quad p_{4k+1} \equiv -1 \pmod{p_{2s}} \quad \text{for } s = 3,4,\dots,2k.
\end{aligned}$$

Finally, we will choose p_{4k+2}. By the Chinese remainder theorem and Dirichlet's theorem there exists a prime $p_{4k+2} > p_{4k+1}$ such that

$$\begin{aligned}
&p_{4k+2} \equiv 4k-1 \pmod{2^{a+2}}, \quad p_{4k+2} \equiv -2 \pmod{m_1},\\
&p_{4k+2} \equiv -1 \pmod{p_1}, \quad p_{4k+2} \equiv -3 \pmod{p_2}, \quad p_{4k+2} \equiv -3 \pmod{p_3},\\
&p_{4k+2} \equiv -3 \pmod{p_4}, \quad p_{4k+2} \equiv 1 \pmod{p_5},\\
&p_{4k+2} \equiv -5 \pmod{p_{2s}}, \quad p_{4k+2} \equiv 1 \pmod{p_{2s+1}} \quad \text{for } s = 3,4,\dots,2k.
\end{aligned}$$

(If $k = 1$, then we delete the middle stage and choose p_5 and p_6 by the last process).

Obviously,

$$p_{4k+2} > p_{4k+1} > \cdots > p_2 > p_1 > \max\{2^{a+2}, m_1, 5\}.$$

Noting that $p_1 \equiv 1 \pmod 4$, $p_{4k+2} \equiv 4k-1 \equiv 3 \pmod 4$ and $p_i \equiv 3 \pmod 4$ $(i = 2,\dots,4k+1)$, we could deduce from the law of quadratic reciprocity that

$$\left(\frac{p_1}{p_j}\right) = \left(\frac{p_j}{p_1}\right) \quad \text{for } j = 2,\dots,4k+2, \qquad \left(\frac{p_i}{p_j}\right) = -\left(\frac{p_j}{p_i}\right) \quad \text{for } 2 \leqslant i < j \leqslant 4k+2. \tag{4}$$

It follows from the definition of α_i and

$$p_i^{\frac{1}{2}(p_j-1)} \equiv \left(\frac{p_i}{p_j}\right) \pmod{p_j}, \quad 1 \leqslant i < j \leqslant 4k+2$$

that

$$p_i^{\alpha_i} \equiv \left(\frac{p_i}{p_j}\right) \pmod{p_j}, \quad 1 \leqslant i < j \leqslant 4k+2.$$

Thus, for each positive integer n with the form (2), we could deduce from (4) and the fact α_i is odd that

$$\begin{aligned}
&n \equiv 0 + 1 + \underbrace{(-1) + \cdots + (-1)}_{4k \text{ times}} + (4k-1) \equiv 0 \pmod{2^{a+2}},\\
&n \equiv 1 + (-1) + 1 + 1 + (-1) + (-1) + 1 + \cdots + (-1) + 1 + 1 + (-2) \equiv 0 \pmod{m_1},\\
&n \equiv 1 + 0 + 1 + (-1) + 1 + (-1) + \cdots + 1 + (-1) + (-1) \equiv 0 \pmod{p_1},\\
&n \equiv 1 + \left(\frac{p_1}{p_2}\right) + 0 + 1 + (-1) + \cdots + 1 + (-1) + 1 + (-3) \equiv 0 \pmod{p_2},\\
&n \equiv 1 + \left(\frac{p_1}{p_3}\right) + \left(\frac{p_2}{p_3}\right) + 0 + 1 + 1 + (-1) + \cdots + 1 + (-1) + 1 + (-3) \equiv 0 \pmod{p_3},\\
&n \equiv 1 + \left(\frac{p_1}{p_4}\right) + \left(\frac{p_2}{p_4}\right) + \left(\frac{p_3}{p_4}\right) + 0 + 1 + (-1) + \cdots + 1 + (-1) + 1 + (-3) \equiv 0 \pmod{p_4}.
\end{aligned}$$

For $j \geqslant 3$, we firstly consider p_{2j-1}, where $3 \leqslant j \leqslant 2k+1$, we have

$$n \equiv 1 + \left(\frac{p_1}{p_{2j-1}}\right) + \cdots + \left(\frac{p_{2j-2}}{p_{2j-1}}\right) + 0 + (-1) + 1 + \cdots + (-1) + 1 + 1 \equiv 0 \pmod{p_{2j-1}}.$$

Now we consider p_{2j}, where $3 \leqslant j \leqslant 2k$, we have

$$n \equiv 1 + \left(\frac{p_1}{p_{2j}}\right) + \cdots + \left(\frac{p_{2j-1}}{p_{2j}}\right) + 0 + (-1) + 1 + \cdots + 1 + (-1) + (-5) \equiv 0 \pmod{p_{2j}}.$$

Finally, we consider p_{4k+2}, we could deduce from (3) and (4) that

$$\begin{aligned}
n &\equiv 1 + \left(\frac{p_1}{p_{4k+2}}\right) + \left(\frac{p_2}{p_{4k+2}}\right) + \cdots + \left(\frac{p_{4k}}{p_{4k+2}}\right) + \left(\frac{p_{4k+1}}{p_{4k+2}}\right)\\
&\equiv 1 + 1 - \left(\frac{p_2}{3}\right) - \left(\frac{p_3}{3}\right) - \left(\frac{p_4}{3}\right) + (-1) + \left(\frac{p_6}{5}\right) + \cdots + (-1) + \left(\frac{p_{4k}}{5}\right) + (-1)\\
&\equiv 0 \pmod{p_{4k+2}}.
\end{aligned}$$

To sum up,

$$n = 2^u + p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_{4k+1}^{\alpha_{4k+1}} + p_{4k+2}$$
$$\text{and} \quad 4mp_1p_2\dots p_{4k+1}p_{4k+2} = 2^{a+2}m_1p_1p_2\dots p_{4k+1}p_{4k+2} \mid n.$$

This completes the proof of Theorem 3. □

Acknowledgments

The authors sincerely thank the referees for their invaluable suggestions which improve the quality of this manuscript to a large extent.

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