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Volume 362 (2024), p. 275-278

Online since: 2 May 2024

https://doi.org/10.5802/crmath.555

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The Comptes Rendus. Mathématique are a member of the
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www.centre-mersenne.org — e-ISSN : 1778-3569
Research article / Article de recherche
Number theory / Théorie des nombres

On weakly prime-additive numbers with length $4k + 3$

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Abstract. If a positive integer $n$ has at least two distinct prime divisors and can be written as $n = p_1^{\alpha_1} + \cdots + p_t^{\alpha_t}$, where $p_1 < \cdots < p_t$ are prime divisors of $n$ and $\alpha_1, \ldots, \alpha_t$ are positive integers, then we define such $n$ as weakly prime-additive. Obviously, $t \geq 3$. In 1992, Erdős and Hegyvári [2] firstly considered weakly prime-additive numbers and proved that, for any prime $p$, there exist infinitely many weakly prime-additive numbers with length $t = 3$ which are divisible by $p$. In 2018, Fang and Chen [5] made further research on weakly prime-additive numbers and obtained the following result:

Theorem 1. For any positive integer $m$, there exist infinitely many weakly prime-additive numbers with $t = 3$ which are divisible by $m$ if and only if $8 \nmid m$.

In [5], Fang and Chen also proved that:

1. Introduction

If a positive integer $n$ has at least two distinct prime divisors and can be written as $n = p_1^{\alpha_1} + \cdots + p_t^{\alpha_t}$, where $p_1 < \cdots < p_t$ are prime divisors of $n$ and $\alpha_1, \ldots, \alpha_t$ are positive integers, then we define such $n$ as weakly prime-additive. Obviously, $t \geq 3$. In 1992, Erdős and Hegyvári [2] firstly considered weakly prime-additive numbers and proved that, for any prime $p$, there exist infinitely many weakly prime-additive numbers $n$ such that $p|n$.

In 2018, Fang and Chen [5] made further research on weakly prime-additive numbers and obtained the following result:

Theorem 1. For any positive integer $m$, there exist infinitely many weakly prime-additive numbers with $t = 3$ which are divisible by $m$ if and only if $8 \nmid m$.
Theorem 2. For any positive integer $m$, there exist infinitely many weakly prime-additive numbers $n$ with $m \mid n$ and
\[ n = p_1^{a_1} + \cdots + p_5^{a_5}, \]
where $p_1, \ldots, p_5$ are distinct prime divisors of $n$ and $a_1, \ldots, a_5$ are positive integers.

Afterwards, Fang [3] proved that, for any positive integers $m, t$ with $t \equiv 1 \pmod{4}$ and $t > 1$, there exist infinitely many weakly prime-additive numbers $n$ with $m \mid n$ and $n = p_1^{a_1} + \cdots + p_t^{a_t}$, where $p_1, \ldots, p_t$ are distinct prime divisors of $n$ and $a_1, \ldots, a_t$ are positive integers. In this paper, by adding some tricky arguments, we solve the case $t \equiv 3 \pmod{4}$, where $t > 3$, that is:

Theorem 3. For any positive integers $m, t$ with $t \equiv 3 \pmod{4}$ and $t > 3$, there exist infinitely many weakly prime-additive numbers $n$ with $m \mid n$ and
\[ n = p_1^{a_1} + \cdots + p_t^{a_t}, \]
where $p_1, \ldots, p_t$ are distinct prime divisors of $n$ and $a_1, \ldots, a_t$ are positive integers.

Let $k, l$ be integers with $k \geq 2$ and $l \geq 3$. Write
\[ S_k = \left\{ n : n = \sum_{p \mid n} p^k, \omega(n) \geq 2 \right\}, \]
where $\omega(n)$ is the number of distinct prime factors of $n$. In 2005, De Koninck and Luca [1] obtained many nice results about $S_k$. As a main result, they considered the nature of $S_k$ and identified all integers $n$ in $S_3$ with $\omega(n) = 3$. Let $(P_m)_{m \geq 0}$ denote the Pell sequence given by $P_0 = 0, P_1 = 1$ and $P_{m+1} = 2P_m + P_{m-1}$ for $m \geq 1$. In 2022, Fang [4] proved that, a positive integer $n$ can be expressed as $n = 2^2 + p^2 + q^2$, where $p, q$ are distinct odd prime factors of $n$, if and only if, $n = 2^2 + P_{2m+1}^2 + P_{2m+3}^2$ for some positive integer $m$, where $P_{2m+1}, P_{2m+3}$ are both primes. One may refer to [1] and [4] for details.

2. Proof of Theorem 3

The idea is from [5, Theorem 1.3] and [3, Theorem 1].

Proof. Write $t = 4k + 3$ and $m = 2^a m_1$, where $a \geq 0$ and $2 \nmid m_1$. Then $k \geq 1$. Let $\phi$ denote the Euler totient function. Similar to [3, Theorem 1], we consider the integers
\[ n = 2^a + p_1^{a_1} + p_2^{a_2} + \cdots + p_{4k+1}^{a_{4k+1}} + p_{4k+2}, \]
where $p_i$ are primes fixed later,
\[ u = (a + 2)\phi(m_1) \prod_{i=1}^{4k+2} (p_i - 1) \quad \text{and} \quad a_i = \prod_{j=i+1}^{4k+2} \frac{p_j - 1}{2} \quad \text{for} \quad i = 1, 2, \ldots, 4k + 1. \]
(The method of this proof and the construction are the same as [3, Theorem 1], but we use a tricky idea during the choices of $p_1, p_2, \ldots, p_{4k+2}$, which is the key point during the proof.)

By the Chinese remainder theorem and Dirichlet’s theorem there exists a prime $p_1 > \max\{2^{a+2}, m_1, 5\}$ such that
\[ p_1 \equiv 1 \pmod{2^{a+2}}, \quad p_1 \equiv -1 \pmod{m_1}. \]
By the Chinese remainder theorem and Dirichlet’s theorem there exists a prime $p_2 > p_1$ such that
\[ p_2 \equiv -1 \pmod{2^{a+2}}, \quad p_2 \equiv 1 \pmod{m_1}, \quad p_2 \equiv 1 \pmod{p_1}. \]
By the Chinese remainder theorem and Dirichlet’s theorem there exists a prime $p_3 > p_2$ such that
\[ p_3 \equiv -1 \pmod{2^{a+2}}, \quad p_3 \equiv 1 \pmod{m_1}, \quad p_3 \equiv -1 \pmod{p_1}, \quad p_3 \equiv 1 \pmod{p_2}. \]
By the Chinese remainder theorem and Dirichlet’s theorem there exists a prime $p_4 > p_3$ such that
\[ p_4 \equiv -1 \pmod{2^{a+2}}, p_4 \equiv 1 \pmod{m_1}, p_4 \equiv -1 \pmod{p_1}, p_4 \equiv -1 \pmod{p_2}, p_4 \equiv 1 \pmod{p_3}. \]

For $i \geq 5$, we will choose $p_i$ successively according to the parity of $i$. For $j \geq 3$, we firstly choose $p_{2j-1}$, where $3 \leq j \leq 2k$. By the Chinese remainder theorem and Dirichlet’s theorem there exists a prime $p_{2j-1} > p_{2j-2}$ such that
\[ p_{2j-1} \equiv -1 \pmod{2^{a+2}}, \quad p_{2j-1} \equiv -1 \pmod{m_1}, \]
\[ p_{2j-1} \equiv 1 \pmod{p_1}, \quad p_{2j-1} \equiv 1 \pmod{p_2}, \quad p_{2j-1} \equiv 1 \pmod{p_3}, \quad p_{2j-1} \equiv 1 \pmod{p_4}. \]

For $j \geq 3$, then we omit the last procedure.

Now we choose $p_{2j}$. For $3 \leq j \leq 2k$, by the Chinese remainder theorem and Dirichlet’s theorem there exists a prime $p_{2j} > p_{2j-1}$ such that
\[ p_{2j} \equiv -1 \pmod{2^{a+2}}, \quad p_{2j} \equiv 1 \pmod{m_1}, \]
\[ p_{2j} \equiv 1 \pmod{p_1}, \quad p_{2j} \equiv 1 \pmod{p_2}, \quad p_{2j} \equiv -1 \pmod{p_3}, \quad p_{2j} \equiv -1 \pmod{p_4}. \]

Furthermore, we add the restriction
\[ p_2 \equiv p_3 \equiv 1 \pmod{3}, \quad p_4 \equiv 2 \pmod{3}, \quad p_{2j} \equiv 1 \pmod{5} \quad \text{for} \quad j = 3, 4, \ldots, 2k. \quad (3) \]

(If $3 \mid m_1$, then the congruences $p_2 \equiv p_3 \equiv 1 \pmod{3}$ and $p_4 \equiv 2 \pmod{3}$ follow from the condition $p_2 \equiv p_3 \equiv 1 \pmod{m_1}$ and $p_4 \equiv -1 \pmod{m_1}$; if $5 \mid m_1$, then the congruences $p_{2j} \equiv 1 \pmod{5}$ follow from the condition $p_{2j} \equiv 1 \pmod{m_1}$ for $j = 3, 4, \ldots, 2k$; if $(3, m_1) = 1$ or $(5, m_1) = 1$, then (3) follows from Dirichlet’s theorem.)

By the Chinese remainder theorem and Dirichlet’s theorem there exists a prime $p_{4k+1} > p_{4k}$ such that
\[ p_{4k+1} \equiv -1 \pmod{2^{a+2}}, \quad p_{4k+1} \equiv 1 \pmod{m_1}, \]
\[ p_{4k+1} \equiv 1 \pmod{p_1}, \quad p_{4k+1} \equiv 1 \pmod{p_2}, \quad p_{4k+1} \equiv 1 \pmod{p_3}, \quad p_{4k+1} \equiv 1 \pmod{p_4}, \]
\[ p_{4k+1} \equiv 1 \pmod{p_{2s-1}}, \quad p_{4k+1} \equiv -1 \pmod{p_{2s}} \quad \text{for} \quad s = 3, 4, \ldots, 2k. \]

Finally, we will choose $p_{4k+2}$. By the Chinese remainder theorem and Dirichlet’s theorem there exists a prime $p_{4k+2} > p_{4k+1}$ such that
\[ p_{4k+2} \equiv 4k-1 \pmod{2^{a+2}}, \quad p_{4k+2} \equiv -2 \pmod{m_1}, \]
\[ p_{4k+2} \equiv -1 \pmod{p_1}, \quad p_{4k+2} \equiv -3 \pmod{p_2}, \quad p_{4k+2} \equiv -3 \pmod{p_3}, \]
\[ p_{4k+2} \equiv -3 \pmod{p_4}, \quad p_{4k+2} \equiv 1 \pmod{p_5}, \]
\[ p_{4k+2} \equiv -5 \pmod{p_{2s}}, \quad p_{4k+2} \equiv 1 \pmod{p_{2s+1}} \quad \text{for} \quad s = 3, 4, \ldots, 2k. \]

(If $k = 1$, then we delete the middle stage and choose $p_5$ and $p_6$ by the last process).

Obviously,
\[ p_{4k+2} > p_{4k+1} > \cdots > p_2 > p_1 > \max[2^{a+2}, m_1, 5]. \]

Noting that $p_1 \equiv 1 \pmod{4}$, $p_{4k+2} \equiv 4k-1 \equiv 3 \pmod{4}$ and $p_i \equiv 3 \pmod{4}$ ($i = 2, \ldots, 4k+1$), we could deduce from the law of quadratic reciprocity that
\[ \left( \frac{p_1}{p_j} \right)_p = \left( \frac{p_{j-1}}{p_1} \right)_p \quad \text{for} \quad j = 2, \ldots, 4k+2, \quad \left( \frac{p_i}{p_j} \right)_p = -\left( \frac{p_j}{p_i} \right)_p \quad \text{for} \quad 2 \leq i < j \leq 4k+2. \quad (4) \]

It follows from the definition of $\alpha_i$ and
\[ p_i^{\frac{1}{2}}(p_j-1) \equiv \left( \frac{p_1}{p_j} \right)_p \pmod{p_j}, \quad 1 \leq i < j \leq 4k+2 \]

that
\[ p_i^{a_i} \equiv \left( \frac{p_i}{p_j} \right) \pmod{p_j}, \quad 1 \leq i < j \leq 4k + 2. \]

Thus, for each positive integer \( n \) with the form (2), we could deduce from (4) and the fact \( \alpha_i \) is odd that
\[
\begin{align*}
    n &\equiv 0 + 1 + (-1) + \cdots + (-1) + (4k - 1) \equiv 0 \pmod{2^{a+2}}, \\
    n &\equiv 1 + (-1) + 1 + (-1) + \cdots + (-1) + 1 + (-2) \equiv 0 \pmod{m_1}, \\
    n &\equiv 1 + 0 + 1 + (-1) + \cdots + 1 + (-1) + (1) \equiv 0 \pmod{p_1}, \\
    n &\equiv 1 + \left( \frac{p_1}{p_2} \right) + 0 + 1 + (-1) + \cdots + 1 + (-1) + (1) \equiv 0 \pmod{p_2}, \\
    n &\equiv 1 + \left( \frac{p_1}{p_3} \right) + \left( \frac{p_2}{p_3} \right) + 0 + 1 + (-1) + \cdots + 1 + (-1) + (1) \equiv 0 \pmod{p_3}, \\
    n &\equiv 1 + \left( \frac{p_1}{p_4} \right) + \left( \frac{p_2}{p_4} \right) + \left( \frac{p_3}{p_4} \right) + 0 + 1 + (-1) + \cdots + 1 + (1) \equiv 0 \pmod{p_4}.
\end{align*}
\]

For \( j \geq 3 \), we firstly consider \( p_{2j-1} \), where \( 3 \leq j \leq 2k + 1 \), we have
\[
    n \equiv 1 + \left( \frac{p_1}{p_{2j-1}} \right) + \cdots + \left( \frac{p_{2j-2}}{p_{2j-1}} \right) + 0 + (-1) + 1 + \cdots + (1) + 1 \equiv 0 \pmod{p_{2j-1}}.
\]

Now we consider \( p_{2j} \), where \( 3 \leq j \leq 2k \), we have
\[
    n \equiv 1 + \left( \frac{p_1}{p_{2j}} \right) + \cdots + \left( \frac{p_{2j-1}}{p_{2j}} \right) + 0 + (-1) + 1 + \cdots + (1) + (1) \equiv 0 \pmod{p_{2j}}.
\]

Finally, we consider \( p_{4k+2} \), we could deduce from (3) and (4) that
\[
\begin{align*}
    n &\equiv 1 + \left( \frac{p_1}{p_{4k+2}} \right) + \left( \frac{p_2}{p_{4k+2}} \right) + \cdots + \left( \frac{p_{4k-1}}{p_{4k+2}} \right) + \left( \frac{p_{4k}}{p_{4k+2}} \right) \\
    &\equiv 1 + 1 - \left( \frac{p_2}{3} \right) - \left( \frac{p_3}{3} \right) - \left( \frac{p_4}{3} \right) + (1) + \left( \frac{p_5}{5} \right) + \cdots + (1) + \left( \frac{p_{4k}}{5} \right) + (1) \\
    &\equiv 0 \pmod{p_{4k+2}}.
\end{align*}
\]

To sum up,
\[
    n = 2^u + p_1^{a_1} + p_2^{a_2} + \cdots + p_{4k+1}^{a_{4k+1}} + p_{4k+2}
\]

and \( 4m_1 p_1 \cdots p_{4k+1} p_{4k+2} = 2^{a+2} m_1 p_1 \cdots p_{4k+1} p_{4k+2} \mid n \).

This completes the proof of Theorem 3. \( \square \)

**Acknowledgments**

The authors sincerely thank the referees for their invaluable suggestions which improve the quality of this manuscript to a large extent.

**References**


