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A simplified counterexample to the integral representation of the relaxation of double integrals

Un contre-exemple simplifié à la représentation intégrale de la relaxation des intégrales doubles

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Abstract. We show that the lower-semicontinuous envelope of a non-convex double integral may not admit a representation as a double integral. By taking an integrand with value $+\infty$ except at three points (say -1, 0 and 1) we give a simple proof and an explicit formula for the relaxation that hopefully may shed some light on this type of problems. This is a simplified version of examples by Mora-Corral and Tellini, and Kreisbeck and Zappale, who characterize the lower-semicontinuous envelope via Young measures.

Résumé. Nous montrons que l'enveloppe semi-continue inférieure d'une intégrale double non convexe peut ne pas admettre de représentation sous forme d'intégrale double. En prenant un intégrande avec une valeur infinie sauf en trois points (disons -1, 0 et 1), nous donnons une preuve simple et une formule explicite pour la relaxation qui, espérons-le, pourra éclairer ce type de problèmes. Ceci est une version simplifiée des exemples de Mora-Corral et Tellini, et de Kreisbeck et Zappale, qui caractérisent l'enveloppe semi-continue inférieure via les mesures de Young.

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Double-integral functionals defined in L^p spaces of the form

$$F(u) := \int_{\Omega\times\Omega} f(u(x)-u(y))\,dx\,dy \tag{1}$$

can be treated using the direct methods of the Calculus of Variations. To that end, necessary and sufficient conditions for the lower semicontinuity of F with respect to weak L^p topologies turn out to be the convexity and lower semicontinuity of f, exactly as in the case of single-integral functionals (see e.g. [1, 5]). In the case of non-convex f the parallel is lost. Indeed, in [4] it is shown that the lower-semicontinuous envelope of F cannot be represented as a double integral of the same form when the function f is a simple double-well potential. The proof in [4] relies on the representation of the relaxed functional in terms of Young measures and on the study of the optimality conditions satisfied by such measure-valued minimizers. We now give a simple explanation of the non-representability of the relaxed functional when f is a double-well potential (or rather a “triple-well” potential with wells in -1, 0 and 1) with “infinite depth”; namely,

$$f(z)=\begin{cases}0 & \text{if } z\in\{-1,1\}\\ 1 & \text{if } z=0\\ +\infty & \text{otherwise.}\end{cases} \tag{2}$$

For simplicity we chose $\Omega=(0,1)$. We remark that we can extend this example to everywhere finite integrands f by approximation.

We note that other examples are shown in [3] when the integrand is of the form $f(u(x),u(y))$. In that case the functionals are not invariant by translations, so the parallel with local functionals would be with integrands depending on $(u(x),\nabla u(x))$, for which lower-semicontinuity conditions are more complex [2].

We now turn to the analysis of the counterexample.

1. Characterization of the lower-semicontinuous envelope

Note preliminarily that a lower bound for the lower-semicontinuous envelope $\overline{F}$ of F with respect to the weak L^1-convergence is

$$F_0(u):=\int_{\Omega\times\Omega} f^{**}(u(x)-u(y))\,dx\,dy, \tag{3}$$

where the lower-semicontinuous convex envelope of f is

$$f^{**}(z)=\begin{cases}0 & \text{if } z\in[-1,1]\\ +\infty & \text{otherwise;}\end{cases} \tag{4}$$

that is,

$$F_0(u)=\begin{cases}0 & \text{if ess-sup}\,u-\text{ess-inf}\,u\le 1\\ +\infty & \text{otherwise.}\end{cases} \tag{5}$$

This lower bound implies that $\overline{F}$ is finite at most on functions $u\in L^\infty(0,1)$ such that

$$\text{ess-sup}\,u-\text{ess-inf}\,u\le 1. \tag{6}$$

Let $u\in L^\infty(0,1)$ satisfy (6), and let u_j be a sequence weakly converging to u and such that $F(u_j)<+\infty$ for all j. Note that for fixed j the function u_j can take at most two values almost everywhere and these values are at distance 1. Indeed by Fubini's theorem for almost all $y\in(0,1)$ we have $u_j(x)\in\{u_j(y),u_j(y)-1,u_j(y)+1\}$ for almost every $x\in(0,1)$. Hence, there exists z^j such that $u_j(x)\in\{z^j,z^j-1,z^j+1\}$ for almost every $x\in(0,1)$. If both values z^j-1 and z^j+1 were taken on sets of positive measure, then we would have $F(u_j)=+\infty$, and a contradiction. Hence, we can suppose that there exist z^j such that $u_j(x)\in\{z^j,z^j+1\}$ almost everywhere. We can assume, up to subsequences, that $z_j\to z$, so that

$$z\le\text{ess-inf}\,u\quad\text{and}\quad\text{ess-sup}\,u\le z+1, \tag{7}$$

and that, if we let $A^j:=\{x:u_j(x)=z+1\}$, there exists $t\in[0,1]$ such that $\lim_{j\to+\infty}|A^j|=t$. Hence, we obtain

$$\lim_{j\to+\infty}F(u_j)=\lim_{j\to+\infty}\Big(|A^j|^2+\Big(1-|A^j|\Big)^2\Big)=t^2+(t-1)^2=2t^2-2t+1. \tag{8}$$

Note that the minimum of $t^2+(1-t)^2$ is $\frac12$ so that (8) implies that $\overline{F}(u)\ge\frac12$ for all u.

Since by the convergence of $\int_{(0,1)}u_j\,dx$ to $\int_{(0,1)}u\,dx$ we have

$$t=\int_{(0,1)}u\,dx-z, \tag{9}$$

the limit of $F(u_j)$ can be described in terms of $\int_{(0,1)} u\,dx$ and z only, and is independent of the particular sequence u_j.

Note conversely that if u and z are such that (7) holds, then there exist u_j with $u_j \in \{z, z+1\}$ and weakly converging to u, so that the value $2t^2+2t+1$ is achieved on this sequence with t given by (9). By optimizing in z we then have a description of $\overline{F}(u)$ as

$$\begin{aligned}\overline{F}(u) &= \min\Big\{\Big(\int_{(0,1)} u\,dx - z\Big)^2 + \Big(\int_{(0,1)} u\,dx - z - 1\Big)^2 \\ &\qquad\qquad : z \le \text{ess-inf}\,u,\ \text{ess-sup}\,u \le z+1\Big\} \\ &= \min\Big\{2\Big(\int_{(0,1)} u\,dx\Big)^2 - 2(2z+1)\Big(\int_{(0,1)} u\,dx\Big) + 2z^2 + 2z + 1 \\ &\qquad\qquad : z \le \text{ess-inf}\,u,\ \text{ess-sup}\,u \le z+1\Big\}.\end{aligned} \tag{10}$$

We can make this formula more symmetric by the change of variables $w = z + \frac{1}{2}$, so that

$$\begin{aligned}\overline{F}(u) = \min\Big\{2\Big(\int_{(0,1)} u\,dx\Big)^2 - 4w\Big(\int_{(0,1)} u\,dx\Big) + 2w^2 + \frac{1}{2} \\ : \text{ess-sup}\,u - \frac{1}{2} \le w \le \text{ess-inf}\,u + \frac{1}{2}\Big\}.\end{aligned} \tag{11}$$

Furthermore, noting that the functionals are invariant if we add a constant to u, replacing u by $u - \int_{(0,1)} u\,dx = 0$ we also have

$$\overline{F}(u) = \min\Big\{2w^2 + \frac{1}{2} : \text{ess-sup}\,u - \int_{(0,1)} u\,dx - \frac{1}{2} \le w \le \text{ess-inf}\,u - \int_{(0,1)} u\,dx + \frac{1}{2}\Big\}. \tag{12}$$

2. Non representability of the lower-semicontinuous envelope

We now prove that there exists no g such that

$$\overline{F}(u) = \int_{\Omega\times\Omega} g(u(x) - u(y))\,dx dy. \tag{13}$$

Note that g can be assumed to be even, up to replacing $g(z)$ with $\frac{1}{2}(g(z) + g(-z))$.

We first describe $\overline{F}(u)$ more precisely in some "extreme" cases. In the first one the minimization does not involve constraint (7), so that $\overline{F}(u) = \frac{1}{2}$. To get this, we note that if

$$\text{ess-sup}\,u - \text{ess-inf}\,u \le \frac{1}{2} \tag{14}$$

then we can take $z = \int_{(0,1)} u\,dx - \frac{1}{2}$, and by (14) we have

$$\text{ess-sup}\,u \le \text{ess-inf}\,u + \frac{1}{2} \le z+1 \quad \text{and} \quad z \le \text{ess-sup}\,u - \frac{1}{2} \le \text{ess-inf}\,u,$$

and $\overline{F}(u) = \frac{1}{2}$ by formula (10). As a particular case of a function satisfying (14) we can take u a constant. In this case (13) would give

$$g(0) = \frac{1}{2}. \tag{15}$$

The other "extreme" case is when only one z is involved in the minimization in (10); which is the case when $\text{ess-sup}\,u - \text{ess-inf}\,u = 1$, so that $z = \text{ess-inf}\,u$ and $z+1 = \text{ess-sup}\,u$. The value of $\overline{F}(u)$ is then simply

$$\overline{F}(u) = \Big(\int_{(0,1)} u\,dx - \text{ess-inf}\,u\Big)^2 + \Big(\int_{(0,1)} u\,dx - \text{ess-sup}\,u\Big)^2.$$

This can be applied, for fixed $t \in (0,1)$, with u given by

$$u(x) = \begin{cases} 1 & \text{if } x \le t \\ 0 & \text{if } x > t, \end{cases}$$

for which $\overline{F}(u) = 2t^2 - 2t + 1$. If (13) held true then by (15) we would also have

$$\overline{F}(u) = \left(2t^2 - 2t + 1\right) g(0) + 2t(1-t) g(1) = \frac{1}{2}\left(2t^2 - 2t + 1\right) + 2t(1-t) g(1),$$

which would give

$$g(1) = \frac{2t^2 - 2t + 1}{4t(1-t)} = \frac{1}{4}\left(\frac{t}{1-t} + \frac{1-t}{t}\right).$$

Taking different values for $t \in (0,1)$ we get different values for $g(1)$, which is a contradiction.

3. Conclusions and remarks

Formula (10) shows that $\overline{F}(u)$ is obtained by functions u_j weakly converging to u and oscillating between two values z and $z+1$ maximizing the measure of the subset of points $(x,y) \in \Omega \times \Omega$ such that $u_j(x) = z$ and $u_j(y) = z+1$. This operation depends only on z, which satisfies some constraints due to the convergence of u_j to u; minimizing the outcome in z gives the optimal choice of u_j. Minimization in z is unconstrained if $\text{ess-sup}\, u - \text{ess-inf}\, u \le \frac{1}{2}$, while it is limited to a single z when $\text{ess-sup}\, u - \text{ess-inf}\, u = 1$. The dependence on the quantity $\text{ess-sup}\, u - \text{ess-inf}\, u$ highlights the nonlocality of the recovery sequences. An example of this fact is obtained by considering constants $u = c$, for which we have minimizing sequences oscillating between $c - \frac{1}{2}$ and $c + \frac{1}{2}$, while this is not true for piecewise-constant functions: if u takes only two values at distance 1 then a recovery sequence is u itself, without oscillations.

We remark that from this example we also obtain examples with finite integrand. Indeed, if f_n is a sequence of functions increasingly converging to f given by (2) and

$$F_n(u) := \int_{\Omega\times\Omega} f_n(u(x) - u(y))\, dx dy, \tag{16}$$

then the lower-semicontinuous envelopes $\overline{F}_n$ converge to $\overline{F}$. If there existed (convex) functions g_n such that

$$\overline{F}_n(u) := \int_{\Omega\times\Omega} g_n(u(x) - u(y))\, dx dy, \tag{17}$$

then this would hold also for $\overline{F}$.

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Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than his research organizations.

References

[1] J. C. Bellido and C. Mora-Corral, "Lower semicontinuity and relaxation via Young measures for nonlocal variational problems and applications to peridynamics", *SIAM J. Math. Anal.* **50** (2018), pp. 779–809.

[2] E. De Giorgi, G. Buttazzo and G. Dal Maso, "On the lower semicontinuity of certain integral functionals", *Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat.* **74** (1983), pp. 274–282.

[3] C. Kreisbeck and E. Zappale, "Loss of double-integral character during relaxation", *SIAM J. Math. Anal.* **53** (2021), pp. 351–385.

[4] C. Mora-Corral and A. Tellini, "Relaxation of a scalar nonlocal variational problem with a double-well potential", *Calc. Var. Partial Differ. Equ.* **59** (2020), article no. 67 (30 pages).

[5] P. Pedregal, "Weak lower semicontinuity and relaxation for a class of non-local functionals", *Rev. Mat. Complut.* **29** (2016), pp. 485–495.