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Derived equivalences of upper-triangular ring spectra via lax limits

Équivalences dérivées de spectres en anneaux triangulaires supérieurs via limites laxes

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Abstract. We extend a theorem of Ladkani concerning derived equivalences between upper-triangular matrix rings to ring spectra. Our result also extends an analogous theorem of Maycock for differential graded algebras. We illustrate the main result with certain canonical equivalences determined by a smooth or proper ring spectrum.

Résumé. Nous étendons un théorème de Ladkani concernant les équivalences dérivées entre les anneaux à matrice triangulaire supérieure aux spectres en anneaux. Notre résultat étend également un théorème analogue de Maycock pour les algèbres différentielles graduées. Nous illustrons le résultat principal par certaines équivalences canoniques déterminés par un spectre en anneaux lisse ou propre.

Keywords. Upper-triangular matrix ring, derived equivalences, reflection functors, ring spectrum.

Mots-clés. Anneaux de matrices triangulaires supérieures, équivalences dérivées, foncteurs de réflexion, spectre en anneaux.

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The purpose of this short article is to extend the following theorem of Ladkani [12] from ordinary rings to ring spectra in the sense of stable homotopy theory; we note that this theorem was extended to differential graded algebras by Maycock [16]. Recall that to rings R and S and an S-R-bimodule M one associates the upper-triangular matrix ring

$$\left(\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right) = \left\{ \left(\begin{smallmatrix} s & m \\ 0 & r \end{smallmatrix}\right) \,\middle|\, r \in R,\ s \in S,\ m \in M \right\}$$

with sum and product operations given the corresponding matrix operations. We denote the (triangulated) derived category of right modules over a ring R by $\mathrm{D}(\mathrm{Mod}(R))$ and recall than an object $X \in \mathrm{D}(\mathrm{Mod}(R))$ is *compact* if the functor

$$\mathrm{Hom}_R(X, -)\colon\ \mathrm{D}(\mathrm{Mod}(R)) \longrightarrow \mathrm{Ab}$$

preserves small coproducts.

Theorem 1 (Ladkani). *Let R and S be rings. Suppose given an S-R bimodule M such that M_R is compact as an object of* $\mathrm{D}(\mathrm{Mod}(R))$ *and an R-module T such that the functor*

$$-\bigotimes_{E}^{\mathbb{L}} T\colon \mathrm{D}(\mathrm{Mod}(E)) \xrightarrow{\sim} \mathrm{D}(\mathrm{Mod}(R))$$

is an equivalence of triangulated categories, where $E = \mathrm{Hom}_R(T,T)$ *is the ring of endomorphisms of T. Suppose, moreover, that* $\mathrm{Ext}_R^{>0}(M,T) = 0$*. Then, there is an equivalence of triangulated categories*

$$\mathrm{D}\big(\mathrm{Mod}\big(\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\big)\big) \simeq \mathrm{D}\big(\mathrm{Mod}\big(\begin{smallmatrix} E & \mathrm{Hom}_R(M,T) \\ 0 & S \end{smallmatrix}\big)\big).$$

As Ladkani explains in *loc. cit.*, interesting equivalences of derived categories are obtained from appropriate choices of R, S, M and T. The main focus of this article is to illustrate how formal properties of a higher-categorical upper-triangular gluing construction yield a simple and conceptual proof of (a vast generalisation of) the above theorem.

We use freely the theory of ∞-categories developed by Joyal, Lurie and others; our main references are [13–15]. Here we only recall that an ∞-category $\mathscr{C}$ is stable if it is pointed, admits finite colimits and the suspension functor $\Sigma\colon \mathscr{C} \longrightarrow \mathscr{C}$, $X \longmapsto 0 \amalg_X 0$, is an equivalence [14, Corollary 1.4.2.27]. The homotopy category of a stable ∞-category is additive (in the usual sense) and is canonically triangulated in the sense of Verdier [14, Theorem 1.1.2.14]. Working with ∞-categories rather than with triangulated categories permits us to construct the (homotopy) limit of a diagram of exact functors between stable ∞-categories, a construction that is not available in the realm of triangulated categories. We also mention that the gluing construction that we utilise below is used by Ladkani in [12] to glue (abelian) module categories; notwithstanding, our proof of the main theorem is different in the case of ordinary rings and of differential graded algebras in that it does not rely on explicit computations.

Let $\mathbf{k}$ be an $\mathbb{E}_\infty$-ring spectrum, for example the sphere spectrum $\mathbb{S}$ or the Eilenberg–Mac Lane spectrum of an ordinary commutative ring [14, Theorem 7.1.2.13]. The presentable stable ∞-category $\mathscr{D}(\mathbf{k})$ of $\mathbf{k}$-module spectra is a (closed) symmetric monoidal ∞-category [14, Proposition 7.1.2.7]. Below we work within the symmetric monoidal ∞-category $\mathrm{PrSt}_{\mathbf{k}}^{\mathrm{L}}$ of $\mathbf{k}$-linear presentable stable ∞-categories and $\mathbf{k}$-linear colimit-preserving functors between them [15, Variants D.1.5.1 and D.2.3.3]. Thus, an object of $\mathrm{PrSt}_{\mathbf{k}}^{\mathrm{L}}$ is a presentable (stable) ∞-category equipped with an action of $\mathscr{D}(\mathbf{k})$. The ∞-category $\mathrm{PrSt}_{\mathbf{k}}^{\mathrm{L}}$ admits small limits and these are preserved by the forgetful functor $\mathrm{PrSt}_{\mathbf{k}}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ to the ∞-category of presentable ∞-categories and colimit-preserving functors between them, see [15, Remark D.1.6.4] and [14, Corollary 4.2.3.3]. Limits of presentable stable ∞-categories along colimit-preserving functors can be computed using [13, Proposition 5.5.3.13 and Corollary 3.3.3.2] since the limit of a diagram of stable ∞-categories and exact functors is itself stable [14, Theorem 1.1.4.4], see also [14, Propositions 1.1.4.1 and 4.8.2.18].

Let $\mathscr{C}$ and $\mathscr{D}$ be $\mathbf{k}$-linear presentable stable ∞-categories and $F\colon \mathscr{C} \to \mathscr{D}$ a $\mathbf{k}$-linear colimit-preserving functor. Define $\mathscr{L}_*(F)$ via the pullback square

$$\begin{array}{ccc} \mathscr{L}_*(F) & \longrightarrow & \mathrm{Fun}\big(\Delta^1,\mathscr{D}\big) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 0^*} \\ \mathscr{C} & \xrightarrow[F]{} & \mathscr{D} \end{array}$$

in the ∞-category $\mathrm{PrSt}_{\mathbf{k}}^{\mathrm{L}}$; an object of the ∞-category $\mathscr{L}_*(F)$ is a pair $(c, f\colon F(c) \to d)$ where $c \in \mathscr{C}$ and $f\colon F(c) \to d$ is a morphism in $\mathscr{D}$. The above pullback is well defined since the ∞-category

$\operatorname{Fun}(\Delta^1,\mathscr{D})$ is presentable [13, Proposition 5.5.3.6] and stable [14, 1.1.3.1] and inherits a $\mathbf{k}$-linear structure from $\mathscr{D}$ via the equivalence of ∞-categories

$$\begin{aligned}\operatorname{Fun}(\Delta^1,\mathscr{D}) &\simeq \operatorname{Fun}\big((\Delta^1)^{\mathrm{op}},\mathscr{D}^{\mathrm{op}}\big)^{\mathrm{op}}\\ &\simeq \operatorname{LFun}\big(\operatorname{Fun}(\Delta^1,\mathscr{S}),\mathscr{D}^{\mathrm{op}}\big)^{\mathrm{op}}\\ &\simeq \operatorname{RFun}\big(\mathscr{D}^{\mathrm{op}},\operatorname{Fun}(\Delta^1,\mathscr{S})\big) \simeq \mathscr{D}\otimes\operatorname{Fun}(\Delta^1,\mathscr{S}).\end{aligned}$$

Above, $\mathscr{S}$ denotes the ∞-category of spaces, $\operatorname{LFun}(-,-)$ (resp. $\operatorname{RFun}(-,-)$) denotes the ∞-category of functors that admit a right adjoint (resp. a left adjoint), and the symbol $\otimes$ denotes Lurie's tensor product of presentable ∞-categories [14, Propositions 4.8.1.15 and 4.8.1.17] (see also [13, Theorem 5.1.5.6 and Proposition 5.2.6.2]). Similarly, the restriction functor

$$0^*\colon \operatorname{Fun}(\Delta^1,\mathscr{D}) \longrightarrow \operatorname{Fun}(\Delta^0,\mathscr{D}) \simeq \mathscr{D}$$

has a canonical $\mathbf{k}$-linear structure. When the right adjoint $G\colon \mathscr{D}\to\mathscr{C}$ of F, which exists by [13, Corollary 5.5.2.9], is also colimit-preserving we may also form the pullback square

$$\begin{array}{ccc}\mathscr{L}^*(G) & \longrightarrow & \operatorname{Fun}(\Delta^1,\mathscr{C})\\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 1^*}\\ \mathscr{D} & \xrightarrow[G]{} & \mathscr{C}\end{array}$$

in the ∞-category $\mathrm{PrSt}^{\mathrm{L}}_{\mathbf{k}}$ [15, Remark D.1.5.3]. There is a canonical equivalence of $\mathbf{k}$-linear presentable stable ∞-categories

$$\mathscr{L}_*(F) \xrightarrow{\sim} \mathscr{L}^*(G), \qquad (c, f\colon F(c)\to d) \longmapsto (d, \overline{f}\colon c\to G(d)), \tag{1}$$

stemming from the fact that both ∞-categories $\mathscr{L}_*(F)$ and $\mathscr{L}^*(G)$ are equivalent to the ∞-category of sections of the biCartesian fibration over Δ^1 classified by the adjunction $F\dashv G$, see [13, Lemma 5.4.7.15]. We also remind the reader of the equivalence of $\mathbf{k}$-linear presentable stable ∞-categories [5, Lemma 1.3]

$$\mathscr{L}^*(F) \xrightarrow{\sim} \mathscr{L}_*(F), \qquad (d, f\colon c\to F(d)) \longmapsto (d, F(d)\to \operatorname{cofib}(f)), \tag{2}$$

induced by the passage from a morphism to its cofibre, that we regard as a very general version of the Bernšteĭn–Gel′fand–Ponomarev reflection functors [1]. The gluing operation $F\mapsto \mathscr{L}_*(F)$ is an example of a lax limit [6] and is also considered in the setting of differential graded categories, see for example [11].

For a given $\mathbf{k}$-algebra spectrum R, that is an $\mathbb{E}_1$-algebra object of the symmetric monoidal ∞-category $\mathscr{D}(\mathbf{k})$, we denote the $\mathbf{k}$-linear stable ∞-category of (right) R-module spectra by $\mathscr{D}(R)$, see also [14, Remark. 7.1.3.7]. The underlying stable ∞-category of $\mathscr{D}(R)$ is compactly generated by the regular representation of R [15, Corollary D.7.6.3]. We identify the $\mathbf{k}$-linear stable ∞-category of *left* R-module spectra with $\mathscr{D}(R^{\mathrm{op}})$, where R^{op} denotes the opposite $\mathbf{k}$-algebra spectrum of R [14, Remark 4.1.1.7]. If M and N are R-module spectra, we denote by $\underline{\operatorname{Map}}_R(M,N)$ the $\mathbf{k}$-module spectrum of morphisms $M\to N$ [15, Example D.7.1.2].

Let R and S be $\mathbf{k}$-algebra spectra. We identify the ∞-category of S-R-bimodule spectra with the ∞-category $\mathscr{D}(S^{\mathrm{op}}\otimes_{\mathbf{k}} R)$ [14, Proposition 4.6.3.15]. The $\mathbf{k}$-linear variant of the Eilenberg–Watts Theorem [14, Proposition 7.1.2.4 and p. 738] yields an equivalence of $\mathbf{k}$-linear presentable stable ∞-categories

$$\mathscr{D}(S^{\mathrm{op}}\otimes_{\mathbf{k}} R) \xrightarrow{\sim} \operatorname{LFun}_{\mathbf{k}}(\mathscr{D}(S),\mathscr{D}(R)), \qquad M \longmapsto -\otimes_S M,$$

where $\operatorname{LFun}_{\mathbf{k}}(\mathscr{D}(S),\mathscr{D}(R))$ is the ∞-category of $\mathbf{k}$-linear colimit-preserving functors $\mathscr{D}(S)\to\mathscr{D}(R)$.

Given a bimodule spectrum $M\in\mathscr{D}(S^{\mathrm{op}}\otimes_{\mathbf{k}} R)$, we denote the right adjoint to the tensor product functor $-\otimes_S M$ by $\underline{\operatorname{Map}}_R(M,-)$. We also introduce the $\mathbf{k}$-linear presentable stable ∞-category

$$\mathscr{D}\left(\begin{smallmatrix} S & M\\ 0 & R\end{smallmatrix}\right) = \mathscr{L}_*(-\otimes_S M).$$

The notation $\mathscr{D}\left(\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right)$ is justified by the Recognition Theorem of Schwede and Shipley [15, Corollary D.7.6.3] (see also [14, Theorem 7.1.2.1]). Indeed, a standard argument using the recollement

$$\mathscr{D}(R) \underset{\overset{\twoheadleftarrow}{i_R}}{\overset{\overset{i_L}{\twoheadleftarrow}}{\xrightarrow{i}}} \mathscr{L}_*(-\otimes_S M) \underset{\overset{\hookleftarrow}{p_R}}{\overset{\overset{p_L}{\hookleftarrow}}{\xrightarrow{p}}} \mathscr{D}(S)$$

described in [5, Remark 1.4] shows that the object $X = i(R) \oplus p_L(S)$ is a compact generator of the stable ∞-category $\mathscr{L}_*(-\otimes_S M)$ whose $\mathbf{k}$-algebra spectrum of endomorphisms decomposes as the direct sum of $\mathbf{k}$-module spectra

$$\begin{aligned} S &\simeq \underline{\mathrm{Map}}\big(p_L(S), p_L(S)\big) & \underline{\mathrm{Map}}\big(i(R), p_L(S)\big) &\simeq M \\ 0 &\simeq \underline{\mathrm{Map}}\big(p_L(S), i(R)\big) & \underline{\mathrm{Map}}\,(i(R), i(R)) &\simeq R, \end{aligned}$$

since $i_R p_L(S) \simeq S \otimes_S M$. Upper-triangular ring spectra are considered for example in [18].

We are ready to state and prove the main result in this article.

Theorem 2. *Let R, S and E be $\mathbf{k}$-algebra spectra. Suppose given a bimodule spectrum $M \in \mathscr{D}\big(S^{\mathrm{op}} \otimes_{\mathbf{k}} R\big)$ such that the R-module spectrum $M_R = S \otimes_S M$ is compact and a bimodule spectrum $T \in \mathscr{D}\big(E^{\mathrm{op}} \otimes_{\mathbf{k}} R\big)$ such that the functor*

$$-\otimes_E T : \mathscr{D}(E) \xrightarrow{\sim} \mathscr{D}(R)$$

is an equivalence. Then, there is an equivalence of $\mathbf{k}$-linear presentable stable ∞-categories

$$\mathscr{D}\left(\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right) \simeq \mathscr{D}\left(\begin{smallmatrix} E & N \\ 0 & S \end{smallmatrix}\right),$$

where $N = \underline{\mathrm{Map}}_R(M, T)$.

Proof. The commutative square

$$\begin{array}{ccc} \mathscr{D}(E) & \xrightarrow{\underline{\mathrm{Map}}_R(M, -\otimes_E T)} & \mathscr{D}(S) \\ {\scriptstyle -\otimes_E T}\big\downarrow & & \big\| \\ \mathscr{D}(R) & \xrightarrow{\underline{\mathrm{Map}}_R(M, -)} & \mathscr{D}(S) \end{array}$$

in which the left vertical functor is an equivalence by assumption, induces an equivalence of $\mathbf{k}$-linear presentable stable ∞-categories

$$\mathscr{L}_*\Big(\underline{\mathrm{Map}}_R(M, -)\Big) \simeq \mathscr{L}_*\Big(\underline{\mathrm{Map}}_R(M, -\otimes_E T)\Big). \tag{3}$$

Since $\mathscr{D}(S)$ is generated under filtered colimits by the compact S-modules [14, Definition 7.2.4.1 and Proposition 7.2.4.2], the assumption that the R-module spectrum $M_R = S \otimes_S M$ is compact is equivalent to the requirement that the (exact) functor

$$\underline{\mathrm{Map}}_R(M, -) : \mathscr{D}(R) \longrightarrow \mathscr{D}(S)$$

preserves small colimits [14, Propositions 1.1.4.1 and 1.4.4.1]. Hence, in view of the Eilenberg–Watts Theorem, the $\mathbf{k}$-linear colimit-preserving functors

$$\underline{\mathrm{Map}}_R(M, -\otimes_E T) : \mathscr{D}(E) \longrightarrow \mathscr{D}(S) \quad \text{and} \quad -\otimes_E \underline{\mathrm{Map}}_R(M, T) : \mathscr{D}(E) \longrightarrow \mathscr{D}(S)$$

are equivalent. Consequently, there is an equivalence of $\mathbf{k}$-linear presentable stable ∞-categories

$$\mathscr{L}_*\Big(\underline{\mathrm{Map}}_R(M, -\otimes_E T)\Big) \simeq \mathscr{L}_*\Big(-\otimes_E \underline{\mathrm{Map}}_R(M, T)\Big). \tag{4}$$

We conclude the proof by considering the following composite of equivalences of **k**-linear presentable stable ∞-categories (recall that $N = \underline{\mathrm{Map}}_R(M,T)$):

$$\begin{aligned}\mathscr{D}\left(\begin{smallmatrix} S & M \\ 0 & R\end{smallmatrix}\right) &= \mathscr{L}_*(-\otimes_S M)\\ &\overset{(1)}{\simeq} \mathscr{L}^*\left(\underline{\mathrm{Map}}_R(M,-)\right)\\ &\overset{(2)}{\simeq} \mathscr{L}_*\left(\underline{\mathrm{Map}}_R(M,-)\right)\\ &\overset{(3)}{\simeq} \mathscr{L}_*\left(\underline{\mathrm{Map}}_R(M,-\otimes_E T)\right)\\ &\overset{(4)}{\simeq} \mathscr{L}_*\left(-\otimes_E \underline{\mathrm{Map}}_R(M,T)\right) = \mathscr{D}\left(\begin{smallmatrix} E & N \\ 0 & S\end{smallmatrix}\right).\end{aligned}$$

□

Remark 3. When **k** is the Eilenberg–Mac Lane spectrum of the ordinary ring of integer numbers, Ladkani's theorem is recovered from the previous theorem by considering the case where the underlying spectra of R, S, M and T are discrete, that is their stable homotopy groups vanish in non-zero degrees. The assumptions in Ladkani's theorem are sufficient to guarantee that the upper-triangular ring spectra in the statement in the previous theorem are both discrete. Ladkani's theorem then follows from the fact that the ∞-category of module spectra over a discrete ring spectrum A is equivalent to the derived ∞-category of modules over the ordinary ring $\pi_0(A)$, see [14, Remark 7.1.1.16]. Maycock's extension of Ladkani's theorem to differential graded algebras corresponds to the case where **k** is the Eilenberg–Mac Lane spectrum of an ordinary commutative ring, see [14, Proposition 7.1.4.6].

Example 4. Let $R = S = E$ be arbitrary **k**-algebra spectra and $M = T = R$ with its canonical R-bimodule structure. The functors $-\otimes_R R$ and $-\otimes_R \underline{\mathrm{Map}}_R(R,R)$ are both equivalent to the identity functor of $\mathscr{D}(R)$ and the equivalence in the main theorem reduces to the (non-trivial) equivalence of **k**-linear presentable stable ∞-categories

$$\mathscr{D}\left(\begin{smallmatrix} R & R \\ 0 & R\end{smallmatrix}\right) \simeq \mathrm{Fun}\left(\Delta^1,\mathscr{D}(R)\right) \xrightarrow{\sim} \mathrm{Fun}\left(\Delta^1,\mathscr{D}(R)\right) \simeq \mathscr{D}\left(\begin{smallmatrix} R & R \\ 0 & R\end{smallmatrix}\right)$$

given by the passage from a morphism in $\mathscr{D}(R)$ to its cofibre.

We conclude this article by describing certain canonical equivalences attached to an algebra spectrum (or, more generally, a morphism between such) that satisfies suitable finiteness/dualisability conditions. The bimodule spectra that arise play a central role in the study of right/left Calabi–Yau structures [7,10] and their relative variants [3,19], see [2,4,8,9,20–22]. Given a **k**-algebra spectrum A, we write $A^e = A \otimes_{\mathbf{k}} A^{\mathrm{op}}$ and recall that A can be viewed either as a right or as a left A^e-module spectrum [14, Construction 4.6.3.7 and Remark 4.6.3.8]. We also make implicit use of the canonical equivalences between the **k**-linear ∞-category of A-bimodule spectra and those of A^e-**k**-bimodule spectra and of **k**-A^e-bimodule spectra, see [14, Proposition 4.6.3.15] and the discussing succeeding it.

(i) Let A be a proper **k**-algebra spectrum, that is the underlying **k**-module spectrum of A is compact; equivalently, A is a right dualisable object of the ∞-category of A^e-**k**-bimodule spectra, see [14, Definition 4.6.4.2] and [15, Example D.7.4.2 and Remark D.7.4.3]. We write

$$DA = \underline{\mathrm{Map}}_{\mathbf{k}}(A,\mathbf{k})$$

for the **k**-linear dual of A. Setting $R = E = \mathbf{k}$, $S = A^e$, $M = A$ and $T = \mathbf{k}$, the main theorem affords an equivalence of **k**-linear presentable stable ∞-categories

$$\mathscr{D}\left(\begin{smallmatrix} A^e & A \\ 0 & \mathbf{k}\end{smallmatrix}\right) \xrightarrow{\sim} \mathscr{D}\left(\begin{smallmatrix} \mathbf{k} & DA \\ 0 & A^e\end{smallmatrix}\right)$$

between the derived ∞-category of the "one-point extension" of A^e by the diagonal A-bimodule spectrum and that of the "one-point coextension" of A^e by DA (this terminology originates in representation theory of algebras [17]).

(ii) Let A be a smooth $\mathbf{k}$-algebra spectrum, that is $A \in \mathscr{D}(A^e)$ is a compact object [15, Definition 11.3.2.1]; equivalently, A is a left dualisable object of the ∞-category of A^e-$\mathbf{k}$-bimodule spectra, see [14, Definition 4.6.4.13] and [15, Remark 11.3.2.2]. The A-bimodule spectrum

$$\Omega_A = \underline{\mathrm{Map}}_{A^e}\big(A, A^e\big)$$

is called the inverse dualising A-bimodule (not to be confused with the based-loops functor on $\mathscr{D}(A)$). Setting $R = E = A^e$, $S = \mathbf{k}$, $M = A$ and $T = A^e$, the main theorem yields an equivalence of $\mathbf{k}$-linear presentable stable ∞-categories

$$\mathscr{D}\begin{pmatrix} \mathbf{k} & A \\ 0 & A^e \end{pmatrix} \xrightarrow{\sim} \mathscr{D}\begin{pmatrix} A^e & \Omega_A \\ 0 & \mathbf{k} \end{pmatrix}.$$

(iii) Let A be a smooth and proper $\mathbf{k}$-algebra spectrum. In this case there are mutually-inverse equivalences of $\mathbf{k}$-linear presentable stable ∞-categories

$$- \otimes_A \Omega_A \colon \mathscr{D}(A) \overset{\sim}{\longleftrightarrow} \mathscr{D}(A) \, : - \otimes_A DA,$$

see [14, Proposition 4.6.4.20] where DA is called the Serre A-bimodule [14, Definition 4.6.4.5] and Ω_A is called the dual Serre A-bimodule [14, Definition 4.6.4.16] (the fact that DA and Ω_A are the right and left duals of A in the ∞-category of A^e-$\mathbf{k}$-bimodule spectra in the sense of [14, Definition 4.6.2.3] follows from [14, Proposition 4.6.2.1 and Remark 4.6.2.2]). Setting $R = E = S = A$, $M = A$ and $T = DA$ or $T = \Omega_A$, the main theorem provides equivalences of $\mathbf{k}$-linear presentable stable ∞-categories

$$\mathscr{D}\begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \xrightarrow{\sim} \mathscr{D}\begin{pmatrix} A & DA \\ 0 & A \end{pmatrix} \qquad \text{and} \qquad \mathscr{D}\begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \xrightarrow{\sim} \mathscr{D}\begin{pmatrix} A & \Omega_A \\ 0 & A \end{pmatrix},$$

where we use that $\underline{\mathrm{Map}}_A(A, DA) \simeq DA$ and $\underline{\mathrm{Map}}_A(A, \Omega_A) \simeq \Omega_A$ as A-bimodule spectra.

(iv) Let $f \colon B \to A$ be a morphism of $\mathbf{k}$-algebra spectra that is not necessarily unital. By the Eilenberg–Watts Theorem, the counit of the induced adjunction

$$- \otimes_B A \simeq f_! \colon \mathscr{D}(B) \longleftrightarrow \mathscr{D}(A) \, : f^*$$

can be interpreted as a morphism of A-bimodule spectra

$$\varepsilon \colon A \otimes_B A \longrightarrow A.$$

Suppose that A is smooth and that $f^*(A)$ is compact as a B-module spectrum, so that the source and target of the morphism ε are compact A-bimodule spectra and, consequently, so is its cofibre. The A-bimodule spectrum

$$\Omega_{A,B} = \underline{\mathrm{Map}}_{A^e}\big(\mathrm{cofib}(\varepsilon), A^e\big)$$

is called the relative inverse dualising A-bimodule [22]. Setting $R = E = A^e$, $S = \mathbf{k}$, $M = \mathrm{cofib}(\varepsilon)$ and $T = A^e$, the main theorem yields an equivalence of $\mathbf{k}$-linear presentable stable ∞-categories

$$\mathscr{D}\begin{pmatrix} \mathbf{k} & \mathrm{cofib}(\varepsilon) \\ 0 & A^e \end{pmatrix} \xrightarrow{\sim} \mathscr{D}\begin{pmatrix} A^e & \Omega_{A,B} \\ 0 & \mathbf{k} \end{pmatrix}$$

that specialises to the equivalence in (ii) when $B = 0$.

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References

[1] I. N. Bernšteĭn, I. M. Gel′fand & V. A. Ponomarev, "Coxeter functors, and Gabriel's theorem", *Usp. Mat. Nauk* **28** (1973), no. 2(170), p. 19-33.

[2] T. Bozec, D. Calaque & S. Scherotzke, "Relative critical loci and quiver moduli", to appear in *Ann. Sci. École Norm. Sup.*, https://arxiv.org/abs/2006.01069.

[3] C. Brav & T. Dyckerhoff, "Relative Calabi–Yau structures", *Compos. Math.* **155** (2019), no. 2, p. 372-412.

[4] ———, "Relative Calabi–Yau structures II: shifted Lagrangians in the moduli of objects", *Sel. Math., New Ser.* **27** (2021), no. 4, article no. 63 (45 pages).

[5] T. Dyckerhoff, G. Jasso & T. Walde, "Generalised BGP reflection functors via the Grothendieck construction", *Int. Math. Res. Not.* (2021), no. 20, p. 15733-15745.

[6] D. Gepner, R. Haugseng & T. Nikolaus, "Lax colimits and free fibrations in ∞-categories", *Doc. Math.* **22** (2017), p. 1225-1266.

[7] V. Ginzburg, "Calabi–Yau algebras", 2006, https://arxiv.org/abs/math/0612139.

[8] B. Keller, "Deformed Calabi–Yau completions", *J. Reine Angew. Math.* **654** (2011), p. 125-180, with an appendix by Michel Van den Bergh.

[9] B. Keller & Y. Wu, "Relative cluster categories and Higgs categories with infinite-dimensional morphism spaces", with an appendix Chris Fraser and Keller, Bernhard, 2023, https://arxiv.org/abs/2307.12279.

[10] M. Kontsevich & Y. Soibelman, "Notes on A_∞-algebras, A_∞-categories and non-commutative geometry", in *Homological mirror symmetry*, Lecture Notes in Physics, vol. 757, Springer, 2006, p. 153-219.

[11] A. Kuznetsov & V. A. Lunts, "Categorical resolutions of irrational singularities", *Int. Math. Res. Not.* (2015), no. 13, p. 4536-4625.

[12] S. Ladkani, "Derived equivalences of triangular matrix rings arising from extensions of tilting modules", *Algebr. Represent. Theory* **14** (2011), no. 1, p. 57-74.

[13] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, 2009, xviii+925 pages.

[14] ———, "Higher Algebra", 2017, available online at the author's webpage: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[15] ———, "Spectral Algebraic Geometry", 2018, available online at the author's website: https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

[16] D. Maycock, "Derived equivalences of upper triangular differential graded algebras", *Commun. Algebra* **39** (2011), no. 7, p. 2367-2387.

[17] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, vol. 1099, Springer, 1984, xiii+376 pages.

[18] V. Sosnilo, "Regularity of spectral stacks and discreteness of weight-hearts", *Q. J. Math.* **73** (2022), no. 1, p. 23-44.

[19] B. Toën, "Derived algebraic geometry", *EMS Surv. Math. Sci.* **1** (2014), no. 2, p. 153-240.

[20] Y. Wu, "Categorification of ice quiver mutation", *Math. Z.* **304** (2023), no. 1, article no. 11 (42 pages).

[21] ———, "Relative cluster categories and Higgs categories", *Adv. Math.* **424** (2023), article no. 109040 (112 pages).

[22] W.-K. Yeung, "Relative Calabi–Yau completions", 2016, https://arxiv.org/abs/1612.06352.