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Sobolev regularity of the canonical solutions to $\bar{\partial}$ on product domains

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Abstract. Let $\Omega$ be a product domain in $\mathbb{C}^n$, $n \geq 2$, where each slice has smooth boundary. We observe that the canonical solution operator for the $\bar{\partial}$ equation on $\Omega$ is bounded in $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+$, $1 < p < \infty$. This Sobolev regularity is sharp in view of Kerzman-type examples.

Keywords. canonical solution, $\bar{\partial}$ equation, Bergman projection, product domains, Sobolev regularity.

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1. Introduction

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 1$. According to Hörmander’s $L^2$ theory, given a $\bar{\partial}$-closed $(0, 1)$ form $f \in L^2(\Omega)$, there exists a unique $L^2$ function that is perpendicular to $\ker(\bar{\partial})$ and solves

$$\bar{\partial}u = f \quad \text{in} \quad \Omega.$$ 

This solution is called the canonical solution (of the $\bar{\partial}$ equation). The $L^2$-Sobolev regularity of the canonical solutions has been investigated through Kohn’s $\bar{\partial}$-Neumann approach for domains with nice regularity and geometry, such as convexity and/or finite type conditions.

The goal of the note is to give the $L^p$-Sobolev estimate of the canonical solutions on product domains. Here a product domain $\Omega$ in $\mathbb{C}^n$ is a Cartesian product $D_1 \times \cdots \times D_n$ of bounded planar domains $D_j$, $j = 1, \ldots, n$. In particular, $D_j$ need not be simply-connected. Then $\Omega$ is (weakly) pseudoconvex with at most Lipschitz boundary. The $L^p$-Sobolev regularity of the canonical solutions on product domains was already thoroughly understood through works of [5–7, 12, 15, 22, 25] and the references therein. In the Sobolev category, combined efforts in [2, 11, 18, 23] have given the existence of a bounded solution operator of $\bar{\partial}$ sending $W^{k+n-2,p}(\Omega)$ into $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+$, $1 < p < \infty$. Here $W^{k,p}(\Omega)$ is the Sobolev space consisting of functions whose weak derivatives on $\Omega$ up to order $k$ exist and belong to $L^p(\Omega)$. The main theorem is stated as follows.

**Theorem 1.** Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, $n \geq 2$, where each $D_j$ is a bounded domain in $\mathbb{C}$ with smooth boundary, $j = 1, \ldots, n$. Given a $\bar{\partial}$-closed $(0, 1)$ form $f \in W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+$, $1 < p < \infty$, the canonical solution $Tf$ of $\bar{\partial}u = f$ on $\Omega$ is in $W^{k,p}(\Omega)$. Moreover, there exists a constant $C$ dependent only on $\Omega$, $k$ and $p$ such that

$$\|Tf\|_{W^{k,p}(\Omega)} \leq C\|f\|_{W^{k,p}(\Omega)}.$$
The proof of Theorem 1 is essentially an observation on a representation formula of the canonical solutions by Chakrabarti–Shaw [2] for \( p = 2 \) and Li [15] for \( 1 < p \leq \infty \), according to which it boils down to the Sobolev estimates of the Bergman projection and canonical solution operators on planar domains. The \( L^p \) boundedness of the Bergman projection has been widely studied in several complex variables. See [14, 16, 17, 20, 24], etc. on some types of domains with sufficient smoothness and nice geometry. It is worth pointing out that the Bergman projection may fail to be \( L^p \) bounded over the full range \((1, \infty)\) of \( p \) on certain domains, such as those with rather rough boundaries like the Hartogs triangle ([3, 4], etc.). With an application of Spencer’s formula on planar domains, the Sobolev estimates of the Bergman projection and canonical solution operators are simply a consequence of a result of Jerison and Kenig in [10].

In Example 7, a datum \( f \) on the bidisc is constructed, such that \( f \in W^{k,q} \) for all \( 1 < q < p \), yet \( \partial u = f \) has no \( W^{k,p} \) solutions. This example indicates that the \( \partial \) problem does not gain Sobolev regularity on product domains in general, and thus the estimate in Theorem 1 is sharp.

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2. Bergman projection and canonical solutions on planar domains

Let \( D \) be a bounded domain in \( \mathbb{C} \) whose boundary \( bD \) is smooth, and \( g \) be the Green’s function on \( D \). In other words, at a fixed pole \( w \in D \),

\[
g(z, w) := -\frac{1}{2\pi} \sup \left\{ u(z) : u \in SH^{-}(\Delta D) \text{ and } \limsup_{\zeta \to -w} (u(\zeta) - \log |\zeta - w|) < \infty \right\}, \quad z \in D,
\]

where \( SH^{-}(\Delta D) \) is the collection of negative subharmonic functions on \( D \). It is known ([9] etc.) that \( g \) is symmetric on the two variables \( z \) and \( w \). Moreover, there exists a harmonic function \( h_w \) on \( D \) with \( h_w = \frac{1}{2\pi} \log |z - w| \) on \( bD \) such that

\[
g(\cdot, w) = -\frac{1}{2\pi} \log |\cdot - w| + h_w \quad \text{in} \quad D. \tag{1}
\]

In particular, \( h_w \in C^\infty(\Delta D) \) and

\[
g(z, w) = g(w, z) = 0, \quad z \in bD. \tag{2}
\]

Given \( f \in L^p(\Delta D), 1 < p < \infty \), define

\[
Gf := -4 \int_\Delta g(\cdot, w) f(w) \, dv_w \quad \text{in} \quad D. \tag{3}
\]

Here \( dv \) is the Lebesgue measure on \( \mathbb{C} \). Then \( Gf \) is the solution to the Dirichlet problem

\[
\begin{cases}
\Delta u = 4f, & \text{in } D; \\
u = 0, & \text{on } bD.
\end{cases} \tag{4}
\]

Denote by \( L^p_{\alpha}(\Delta D), \alpha \in \mathbb{R} \) the (fractional) Sobolev space following the notation in [10, p. 162] of Jerison and Kenig. By [10, Theorem 0.3], \( G \) is a bounded operator sending \( L^p_{\alpha - 2}(\Delta D) \) into \( L^p_{\alpha}(\Delta D) \), \( 1 < p < \infty, \alpha > \frac{1}{p} \). In particular, if \( f \in W^{k-1,p}(\Delta D), k \in \mathbb{Z}^+ \cup \{0\}, 1 < p < \infty \), then

\[
\|Gf\|_{W^{k+1,p}(\Delta D)} \lesssim \|f\|_{W^{k-1,p}(\Delta D)}. \tag{5}
\]

Here and throughout the rest of the paper, we say two quantities \( a \) and \( b \) to satisfy \( a \lesssim b \) if there exists a constant \( C \) dependent only possibly on the underlying domain, \( k \) and \( p \) such that \( a \leq Cb \).
The Bergman projection operator $P$ on a domain $\Omega$ is the orthogonal projection of $L^2(\Omega)$ onto the Bergman space $A^2(\Omega)$, the space of $L^2$ holomorphic functions on $\Omega$. Since $A^2(\Omega)$ is a reproducing kernel Hilbert space, there exists a function $k : \Omega \times \Omega \to \mathbb{C}$, called the Bergman kernel, such that for all $f \in L^2(\Omega)$,
\[
P f = \int_{\Omega} k(z, w) f(w) dv_w \quad \text{in} \quad \Omega.
\]
On a smooth planar domain $D$, the Bergman kernel $k$ is related to the Green's function $g$ by
\[
k(z, w) = -4\partial_z \overline{\partial}_w g(z, w), \quad z \neq w \in D.
\]
See [1, p. 180]. Clearly, $k(\cdot, w) \in C^\infty(D)$ by (1).

If $D$ is simply-connected, the Sobolev boundedness of the Bergman projection $P$ can be obtained by applying the known Sobolev regularity on the unit disc and the Riemann mapping theorem. On general smooth planar domains, Lanzani and Stein suggested an approach to estimate $P$ briefly in [13]. For completeness and convenience of the reader, the detail of their approach to the Sobolev regularity of $P$ is provided below.

**Theorem 2.** Let $D \subset \mathbb{C}$ be a bounded domain with $C^\infty$ boundary. Then the Bergman projection $P$ is (or, extends as) a bounded operator on $W^{k, p}(D)$, $k \in \mathbb{Z}^+ \cup \{0\}$, $1 < p < \infty$. Namely, for any $f \in W^{k, p}(D)$,
\[
\|P f\|_{W^{k, p}(D)} \lesssim \|f\|_{W^{k, p}(D)}.
\]

**Proof.** We shall need the following Spencer's formula: for any $f \in C^\infty(D)$,
\[
P f + \partial G \tilde{\partial} f = f \quad \text{in} \quad D,
\]
where $G$ is defined in (3), and for simplification with an abuse of notation, the $\partial$ and $\tilde{\partial}$ operators here and in the rest of the section represent the corresponding complex vector fields. The proof of (7) can be found, for instance, in [1, p. 73–75]. Employing a standard density argument and the estimate (5) for $G$, we can extend $P = I - \partial G \tilde{\partial}$ as a continuous operator on $W^{k, p}(D)$, $k \in \mathbb{Z}^+ \cup \{0\}$, $1 < p < \infty$.

By (5) the (extended) operator $P$ satisfies for all $f \in W^{k, p}(D)$,
\[
\|P f\|_{W^{k, p}(D)} \lesssim \|f\|_{W^{k, p}(D)} + \|G \tilde{\partial} f\|_{W^{k+1, p}(D)} \lesssim \|f\|_{W^{k, p}(D)} + \|\tilde{\partial} f\|_{W^{k+1, p}(D)} \lesssim \|f\|_{W^{k, p}(D)}.
\]
This completes the proof of the theorem. \hfill \square

Given $f \in L^p(D), 1 < p < \infty$, define
\[
T f := \partial G f \left( -4\partial \int_D g(z, w) f(w) dv_w \right) \quad \text{in} \quad D.
\]
We shall show below that $T$ is the canonical solution operator of $\tilde{\partial}$ on $D$ and improves the Sobolev regularity by order one.

**Theorem 3.** Let $D$ be a bounded domain in $\mathbb{C}$ with smooth boundary. For each $k \in \mathbb{Z}^+ \cup \{0\}$, $1 < p < \infty$, $T$ defined in (8) is the canonical solution operator of $\tilde{\partial}$ on $D$, and is a bounded operator sending $W^{k, p}(D)$ into $W^{k+1, p}(D)$. Namely, for any $f \in W^{k, p}(D)$,
\[
\|T f\|_{W^{k+1, p}(D)} \lesssim \|f\|_{W^{k, p}(D)}.
\]

**Proof.** First for $f \in L^2(D), T f \in L^2(D)$ following (5). Moreover, by (4) one has $\tilde{\partial} T f = \partial G f = f$ on $D$. Furthermore, for any $h \in A^2(D)$,
\[
\langle T f, h \rangle = \langle \tilde{\partial} A f, h \rangle = \langle T f - PT f, h \rangle = \langle T f - PT f, Ph \rangle = \langle PT f - PT f, h \rangle = 0,
\]
implying $T f \perp A^2(\Omega)$. Here in the first equality we used the fact that $\tilde{\partial} T f = f$ on $D$; in the second equality we used (7) with $f$ replaced by $T f$; in the third equality we used the fact that $Ph = h$ when $h \in A^2(D)$; in the fourth equality we used the projection properties of $P$, i.e., $P^* = P = P^2$. Thus
$T$ is the canonical solution operator of $\bar{\partial}$ on $D$. The Sobolev regularity of $T$ follows immediately from (5) and (8).

□

**Remark 4.**

(a) We can further make use of Theorem 3 and the Sobolev embedding theorem to conclude that the canonical solution operator $T$ sends $W^{k,\infty}(D)$ into $C^{k,\alpha}(D)$ for all $0 < \alpha < 1$ with

$$\|Tf\|_{C^{k,\alpha}(D)} \leq C\|f\|_{W^{k,\infty}(D)},$$

(b) Another well-known solution operator $\tilde{T}$ of $\bar{\partial}$ on $D$ is given in terms of the universal Cauchy kernel as follows.

$$\tilde{T}f := -\frac{1}{\pi} \int_D \frac{f(w)}{w-z} \, dw \quad \text{in} \quad D.$$ 

It was proved by Prats in [21] that $\tilde{T}$ enjoys a similar Sobolev regularity as $T$ (see also [19] for a much simpler proof using Calderón–Zygmund’s classical singular integral theory):

$$\|\tilde{T}f\|_{W^{k,1,p}(D)} \lesssim \|f\|_{W^{k,p}(D)}.$$

### 3. Canonical solutions on product domains

Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, $n \geq 2$, where each $D_j$ is a bounded planar domain with smooth boundary. Denote by $P_j$ the Bergman projection operator of $D_j$, $j = 1, \ldots, n$. Then the Bergman projection $P$ of $\Omega$ satisfies

$$P = P_1 \cdots P_n. \quad (9)$$

Let $T_j$ be the canonical solution operator on $D_j$ defined in (8), with $D$ replaced by $D_j$, $j = 1, \ldots, n$. Given a $\bar{\partial}$-closed $(0,1)$ form $f = \sum^n_{j=1} f_j d\bar{z}_j \in L^p(\Omega)$, it was shown in [2, Lemma 4.4] and [15, Theorem 2.5] (or, through a repeated application of (7) together with the $\bar{\partial}$-closedness of $f$) that

$$Tf = T_1 f_1 + T_2 P_1 f_2 + \cdots + T_n P_1 \cdots P_{n-1} f_n \quad (10)$$

is the canonical solution to $\bar{\partial}u = f$ on $\Omega$. Note that when $j \neq k$, the two operators $P_j$ and $T_k$ (or $P_k$) commute on $L^p(\Omega)$ due to Fubini’s theorem. The following proposition gives the Sobolev boundedness of $T_j$ and $P_j$ on $\Omega$.

**Proposition 5.** Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, where each $D_j$ is a bounded domain in $\mathbb{C}$ with smooth boundary, $j = 1, \ldots, n$. Then $T_j$ and $P_j$ are bounded operators in $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}$, $1 < p < \infty$. Namely, for all $f \in W^{k,p}(\Omega),$

$$\|T_j f\|_{W^{k,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}; \quad \|P_j f\|_{W^{k,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}.$$ 

**Proof.** For simplicity yet without loss of generality, assume $j = 1$ and $n = 2$. Denote by $\nabla_j$ the gradient in the $z_j$ variable. Since $\bar{\partial}_1 T_1 = id$ and $\bar{\partial}_1 P_1 = 0$, we only need to prove for all $k_1, k_2 \in \mathbb{Z}^+ \cup \{0\}$, $k_1 + k_2 = k$,

$$\|\bar{\partial}_1^{k_1} T_1 \nabla_2^{k_2} f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}; \quad \|\bar{\partial}_1^{k_1} P_1 \nabla_2^{k_2} f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}.$$ 

In fact, making use of Theorem 3 and Fubini’s theorem,

$$\|\bar{\partial}_1^{k_1} T_1 \nabla_2^{k_2} f\|_{L^p(\Omega)}^p = \int_{D_2} \left\|\bar{\partial}_1^{k_1} T_1 \left(\nabla_2^{k_2} f\right)(\cdot, w_2)\right\|_{L^p(D_1)}^p \, dw_2$$

$$\lesssim \sum_{m=0}^{k_1} \int_{D_2} \left\|\nabla_1^{m_1} \nabla_2^{k_2} f(\cdot, w_2)\right\|_{L^p(D_1)}^p \, dw_2 \lesssim \|f\|_{W^{k,p}(\Omega)}^p.$$ 

The estimate for $P_1$ is done similarly with an application of Theorem 2. □
In particular, the proposition states that $T_j$ does not lose Sobolev regularity. This estimate of $T_j$ is also the best that one can expect when $n \geq 2$. This is because $T_j$ only improves the regularity in the $z_j$ direction and has no smoothing effect on the rest of the variables.

**Proof of Theorem 1.** It is a direct consequence of Proposition 5 and (10). \hfill \Box

Proposition 5 and (9) also immediately give the following Sobolev regularity of the Bergman projection operator $P$ on general product domains. We mention that the Sobolev regularity of $P$ on the polydisc was due to [8, 11].

**Theorem 6.** Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n$, $n \geq 1$, where each $D_j$ is a bounded domain in $\mathbb{C}$ with smooth boundary, $j = 1, \ldots, n$. The Bergman projection $P$ is (or, extends as) a bounded operator in $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}$, $1 < p < \infty$. Namely, for any $f \in W^{k,p}(\Omega)$,

$$
\| Pf \|_{W^{k,p}(\Omega)} \lesssim \| f \|_{W^{k,p}(\Omega)}.
$$

Denote by $\Delta^2$ the bidisc in $\mathbb{C}^2$. The following Kerzman-type example demonstrates that the $\bar{\partial}$ problem in general does not improve the Sobolev regularity. In this sense the Sobolev estimate of the canonical solution operator in Theorem 1 is sharp.

**Example 7.** For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $1 < p < \infty$, consider $f = (z_2 - 1)^{\frac{k-p}{2}} d \bar{z}_1$ on $\Delta^2$ if $p \neq 2$, or $f = (z_2 - 1)^{k-1} \log(z_2 - 1) d \bar{z}_1$ on $\Delta^2$ if $p = 2$, $\frac{3}{2} \pi < \arg(z_2 - 1) < \frac{3}{2} \pi$. Then $f \in W^{k,q}(\Delta^2)$ for all $1 < q < p$, and is $\bar{\partial}$-closed on $\Delta^2$. However, there does not exist a solution $u \in W^{k,p}(\Delta^2)$ to $\bar{\partial} u = f$ on $\Delta^2$.

**Proof.** One can directly verify that $f \in W^{k,q}(\Delta^2)$ for all $1 < q < p$ and is $\bar{\partial}$-closed on $\Delta^2$. Suppose there exists some $u \in W^{k,p}(\Delta^2)$ satisfying $\bar{\partial} u = f$ on $\Delta^2$. Then $u = (z_2 - 1)^{\frac{k-p}{2}} z_1 + h \in W^{k,p}(\Delta^2)$ for some holomorphic function $h$ on $\Delta^2$. For each $(r, z_2) \in U := (0,1) \times \Delta \subset \mathbb{R}^3$, consider

$$
u(r, z_2) := \int_{|z_1| = r} u(z_1, z_2) d z_1.
$$

By Fubini’s theorem and Hölder’s inequality,

$$
\| \Delta^2 u \|_{L^p(U)}^p = \int_U \left| \int_{|z_1| = r} \Delta^2 u(z_1, z_2) d z_1 \right|^p d v(r, z_2) d r = \int_U \left| \int_0^1 r \int_0^{2\pi} |\Delta^2 u(r e^{i\theta}, z_2)| d \theta \right|^p d r d v(r, z_2) \lesssim \int_{|z_1| < 1} \int_0^1 \left| \int_0^{2\pi} |\Delta^2 u(r e^{i\theta}, z_2)| d \theta \right|^p d \theta d r d v(r, z_2) \leq \| u \|^p_{W^{k,p}(\Delta^2)} < \infty.
$$

Thus $\Delta^2 u \in L^p(U)$.

On the other hand, by Cauchy’s theorem, for each $(r, z_2) \in U$,

$$
\partial^k \nu(r, z_2) = C_{k,p} \int_{|z_1| = r} (z_2 - 1)^{-\frac{k-p}{2}} z_1 d z_1 = C_{k,p}(z_2 - 1)^{-\frac{k-p}{2}} \int_{|z_1| = r} \frac{r^2}{z_1} d z_1 = 2\pi C_{k,p} r^2 i(z_2 - 1)^{-\frac{k-p}{2}}
$$

for some non-zero constant $C_{k,p}$ depending only on $k$ and $p$. However, $r^2 (z_2 - 1)^{-\frac{k-p}{2}} \notin L^p(U)$. This is a contradiction! \hfill \Box

**References**


