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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 171-176

Online since: 7 March 2024

https://doi.org/10.5802/crmath.561

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www.centre-mersenne.org — e-ISSN : 1778-3569

Comptes Rendus. Mathématique
2024, Vol. 362, p. 171-176
https://doi.org/10.5802/crmath.561

Research article / *Article de recherche*
Complex analysis and geometry / *Analyse et géométrie complexes*

Sobolev regularity of the canonical solutions to $\bar\partial$ on product domains

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Abstract. Let Ω be a product domain in $\mathbb{C}^n$, $n \geq 2$, where each slice has smooth boundary. We observe that the canonical solution operator for the $\bar\partial$ equation on Ω is bounded in $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+, 1 < p < \infty$. This Sobolev regularity is sharp in view of Kerzman-type examples.

Keywords. canonical solution, $\bar\partial$ equation, Bergman projection, product domains, Sobolev regularity.

2020 Mathematics Subject Classification. 32W05, 32A25, 32A36.

Manuscript received 21 November 2022, revised 23 June 2023 and 1 September 2023, accepted 1 September 2023.

1. Introduction

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 1$. According to Hörmander's L^2 theory, given a $\bar\partial$-closed $(0,1)$ form $f \in L^2(\Omega)$, there exists a unique L^2 function that is perpendicular to $\ker(\bar\partial)$ and solves

$$\bar\partial u = f \quad \text{in} \quad \Omega.$$

This solution is called the *canonical solution* (of the $\bar\partial$ equation). The L^2-Sobolev regularity of the canonical solutions has been investigated through Kohn's $\bar\partial$-Neumann approach for domains with nice regularity and geometry, such as convexity and/or finite type conditions.

The goal of the note is to give the L^p-Sobolev estimate of the canonical solutions on product domains. Here a product domain Ω in $\mathbb{C}^n$ is a Cartesian product $D_1 \times \cdots \times D_n$ of bounded planar domains $D_j, j = 1, \ldots, n$. In particular, D_j need not be simply-connected. Then Ω is (weakly) pseudoconvex with at most Lipschitz boundary. The L^p regularity of the canonical solutions on product domains was already thoroughly understood through works of [5–7, 12, 15, 22, 25] and the references therein. In the Sobolev category, combined efforts in [2, 11, 18, 23] have given the existence of a bounded solution operator of $\bar\partial$ sending $W^{k+n-2,p}(\Omega)$ into $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+, 1 < p < \infty$. Here $W^{k,p}(\Omega)$ is the Sobolev space consisting of functions whose weak derivatives on Ω up to order k exist and belong to $L^p(\Omega)$. The main theorem is stated as follows.

Theorem 1. *Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n, n \geq 2$, where each D_j is a bounded domain in $\mathbb{C}$ with smooth boundary, $j = 1, \ldots, n$. Given a $\bar\partial$-closed $(0,1)$ form $f \in W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+, 1 < p < \infty$, the canonical solution Tf of $\bar\partial u = f$ on Ω is in $W^{k,p}(\Omega)$. Moreover, there exists a constant C dependent only on Ω, k and p such that*

$$\|Tf\|_{W^{k,p}(\Omega)} \leq C\|f\|_{W^{k,p}(\Omega)}.$$

The proof of Theorem 1 is essentially an observation on a representation formula of the canonical solutions by Chakrabarti–Shaw [2] for $p = 2$ and Li [15] for $1 < p \le \infty$, according to which it boils down to the Sobolev estimates of the Bergman projection and canonical solution operators on planar domains. The L^p boundedness of the Bergman projection has been widely studied in several complex variables. See [14, 16, 17, 20, 24], etc. on some types of domains with sufficient smoothness and nice geometry. It is worth pointing out that the Bergman projection may fail to be L^p bounded over the full range $(1,\infty)$ of p on certain domains, such as those with rather rough boundaries like the Hartogs triangle ([3, 4], etc.). With an application of Spencer's formula on planar domains, the Sobolev estimates of the Bergman projection and canonical solution operators are simply a consequence of a result of Jerison and Kenig in [10]. In Example 7, a datum f on the bidisc is constructed, such that $f \in W^{k,q}$ for all $1 < q < p$, yet $\bar\partial u = f$ has no $W^{k,p}$ solutions. This example indicates that the $\bar\partial$ problem does not gain Sobolev regularity on product domains in general, and thus the estimate in Theorem 1 is sharp.

Acknowledgement

The author thanks Professor Song-Ying Li for helpful comments, and John Treuer for pointing out a few typos. She also thanks the anonymous referee for providing valuable suggestions.

2. Bergman projection and canonical solutions on planar domains

Let D be a bounded domain in $\mathbb{C}$ whose boundary bD is smooth, and g be the Green's function on D. In other words, at a fixed pole $w \in D$,

$$g(z,w) := -\frac{1}{2\pi}\sup\left\{u(z) : u \in SH^-(D) \text{ and } \limsup_{\zeta\to w}(u(\zeta) - \log|\zeta - w|) < \infty\right\}, \quad z \in D,$$

where $SH^-(D)$ is the collection of negative subharmonic functions on D. It is known ([9] etc.) that g is symmetric on the two variables z and w. Moreover, there exists a harmonic function h_w on D with $h_w = \frac{1}{2\pi}\ln|\cdot - w|$ on bD such that

$$g(\cdot,w) = -\frac{1}{2\pi}\ln|\cdot - w| + h_w \quad \text{in} \quad D. \tag{1}$$

In particular, $h_w \in C^\infty(D)$ and

$$g(z,w) = g(w,z) = 0, \quad z \in bD. \tag{2}$$

Given $f \in L^p(D), 1 < p < \infty$, define

$$Gf := -4\int_D g(\cdot,w) f(w)\mathrm{d}v_w \quad \text{in} \quad D. \tag{3}$$

Here dv is the Lebesgue measure on $\mathbb{C}$. Then Gf is the solution to the Dirichlet problem

$$\begin{cases} \Delta u = 4f, & \text{in} \quad D; \\ u = 0, & \text{on} \quad bD. \end{cases} \tag{4}$$

Denote by $L^p_\alpha(D), \alpha \in \mathbb{R}$ the (fractional) Sobolev space following the notation in [10, p. 162] of Jerison and Kenig. By [10, Theorem 0.3], G is a bounded operator sending $L^p_{\alpha-2}(D)$ into $L^p_\alpha(D)$, $1 < p < \infty$, $\alpha > \frac{1}{p}$. In particular, if $f \in W^{k-1,p}(D)$, $k \in \mathbb{Z}^+ \cup \{0\}$, $1 < p < \infty$, then

$$\|Gf\|_{W^{k+1,p}(D)} \lesssim \|f\|_{W^{k-1,p}(D)}. \tag{5}$$

Here and throughout the rest of the paper, we say two quantities a and b to satisfy $a \lesssim b$ if there exists a constant C dependent only possibly on the underlying domain, k and p such that $a \le Cb$.

The Bergman projection operator P on a domain Ω is the orthogonal projection of $L^2(\Omega)$ onto the Bergman space $A^2(\Omega)$, the space of L^2 holomorphic functions on Ω. Since $A^2(\Omega)$ is a reproducing kernel Hilbert space, there exists a function $k:\Omega\times\Omega\to\mathbb{C}$, called the Bergman kernel, such that for all $f\in L^2(\Omega)$,

$$Pf=\int_\Omega k(\cdot,w)f(w)\mathrm{d}v_w \quad \text{in} \quad \Omega.$$

On a smooth planar domain D, the Bergman kernel k is related to the Green's function g by

$$k(z,w)=-4\partial_z\partial_{\overline{w}}g(z,w),\ \ z\neq w\in D. \tag{6}$$

See [1, p. 180]. Clearly, $k(\cdot,w)\in C^\infty(\overline{D})$ by (1).

If D is simply-connected, the Sobolev boundedness of the Bergman projection P can be obtained by applying the known Sobolev regularity on the unit disc and the Riemann mapping theorem. On general smooth planar domains, Lanzani and Stein suggested an approach to estimate P briefly in [13]. For completeness and convenience of the reader, the detail of their approach to the Sobolev regularity of P is provided below.

Theorem 2. *Let $D\subset\mathbb{C}$ be a bounded domain with C^∞ boundary. Then the Bergman projection P is (or, extends as) a bounded operator on $W^{k,p}(D)$, $k\in\mathbb{Z}^+\cup\{0\}$, $1<p<\infty$. Namely, for any $f\in W^{k,p}(D)$,*

$$\|Pf\|_{W^{k,p}(D)}\lesssim\|f\|_{W^{k,p}(D)}.$$

Proof. We shall need the following Spencer's formula: for any $f\in C^\infty(\overline{D})$,

$$Pf+\partial G\bar\partial f=f \quad \text{in} \quad D, \tag{7}$$

where G is defined in (3), and for simplification with an abuse of notation, the ∂ and $\bar\partial$ operators here and in the rest of the section represent the corresponding complex vector fields. The proof of (7) can be found, for instance, in [1, p. 73–75]. Employing a standard density argument and the estimate (5) for G, we can extend $P=I-\partial G\bar\partial$ as a continuous operator on $W^{k,p}(D)$, $k\in\mathbf{Z}^+\cup\{0\}$, $1<p<\infty$.

By (5) the (extended) operator P satisfies for all $f\in W^{k,p}(D)$,

$$\left\|Pf\right\|_{W^{k,p}(D)}\lesssim\|f\|_{W^{k,p}(D)}+\left\|G\bar\partial f\right\|_{W^{k+1,p}(D)}\lesssim\|f\|_{W^{k,p}(D)}+\left\|\bar\partial f\right\|_{W^{k-1,p}(D)}\lesssim\|f\|_{W^{k,p}(D)}.$$

This completes the proof of the theorem. □

Given $f\in L^p(D), 1<p<\infty$, define

$$Tf:=\partial Gf\left(=-4\partial\int_D g(\cdot,w)f(w)\mathrm{d}v_w\right) \quad \text{in} \quad D. \tag{8}$$

We shall show below that T is the canonical solution operator of $\bar\partial$ on D and improves the Sobolev regularity by order one.

Theorem 3. *Let D be a bounded domain in $\mathbb{C}$ with smooth boundary. For each $k\in\mathbb{Z}^+\cup\{0\}$, $1<p<\infty$, T defined in* (8) *is the canonical solution operator of $\bar\partial$ on D, and is a bounded operator sending $W^{k,p}(D)$ into $W^{k+1,p}(D)$. Namely, for any $f\in W^{k,p}(D)$,*

$$\|Tf\|_{W^{k+1,p}(D)}\lesssim\|f\|_{W^{k,p}(D)}.$$

Proof. First for $f\in L^2(D)$, $Tf\in L^2(D)$ following (5). Moreover, by (4) one has $\bar\partial Tf=\bar\partial\partial Gf=f$ on D. Furthermore, for any $h\in A^2(D)$,

$$\langle Tf,h\rangle=\langle T\bar\partial Tf,h\rangle=\langle Tf-PTf,h\rangle=\langle Tf-PTf,Ph\rangle=\langle PTf-PTf,h\rangle=0,$$

implying $Tf\perp A^2(\Omega)$. Here in the first equality we used the fact that $\bar\partial Tf=f$ on D; in the second equality we used (7) with f replaced by Tf; in the third equality we used the fact that $Ph=h$ when $h\in A^2(D)$; in the fourth equality we used the projection properties of P, i.e., $P^*=P=P^2$. Thus

T is the canonical solution operator of $\bar\partial$ on D. The Sobolev regularity of T follows immediately from (5) and (8). $\square$

Remark 4.

(a) We can further make use of Theorem 3 and the Sobolev embedding theorem to conclude that the canonical solution operator T sends $W^{k,\infty}(D)$ into $C^{k,\alpha}(D)$ for all $0<\alpha<1$ with

$$\|Tf\|_{C^{k,\alpha}(D)} \le C\|f\|_{W^{k,\infty}(D)},$$

(b) Another well-known solution operator $\widetilde{T}$ of $\bar\partial$ on D is given in terms of the universal Cauchy kernel as follows.

$$\widetilde{T}f := -\frac{1}{\pi}\int_D \frac{f(w)}{w-\cdot}\mathrm{d}v_w \quad \text{in} \quad D.$$

It was proved by Prats in [21] that $\widetilde{T}$ enjoys a similar Sobolev regularity as T (see also [19] for a much simpler proof using Caldrón–Zygmund's classical singular integral theory):

$$\|\widetilde{T}f\|_{W^{k+1,p}(D)} \lesssim \|f\|_{W^{k,p}(D)}.$$

3. Canonical solutions on product domains

Let $\Omega := D_1\times\cdots\times D_n \subset \mathbb{C}^n$, $n\ge 2$, where each D_j is a bounded planar domain with smooth boundary. Denote by P_j the Bergman projection operator of D_j, $j=1,\dots,n$. Then the Bergman projection P of Ω satisfies

$$P = P_1\cdots P_n. \tag{9}$$

Let T_j be the canonical solution operator on D_j defined in (8), with D replaced by D_j, $j=1,\dots,n$. Given a $\bar\partial$-closed $(0,1)$ form $f=\sum_{j=1}^n f_j \mathrm{d}\bar z_j \in L^p(\Omega)$, it was shown in [2, Lemma 4.4] and [15, Theorem 2.5] (or, through a repeated application of (7) together with the $\bar\partial$-closedness of f) that

$$Tf = T_1f_1 + T_2P_1f_2 + \cdots + T_nP_1\cdots P_{n-1}f_n \tag{10}$$

is the canonical solution to $\bar\partial u = f$ on Ω. Note that when $j\ne k$, the two operators P_j and T_k (or P_k) commute on $L^p(\Omega)$ due to Fubini's theorem. The following proposition gives the Sobolev boundedness of T_j and P_j on Ω.

Proposition 5. *Let $\Omega := D_1\times\cdots\times D_n\subset\mathbb{C}^n$, where each D_j is a bounded domain in $\mathbb{C}$ with smooth boundary, $j=1,\dots n$. Then T_j and P_j are bounded operators in $W^{k,p}(\Omega)$, $k\in\mathbb{Z}^+\cup\{0\}$, $1<p<\infty$. Namely, for all $f\in W^{k,p}(\Omega)$,*

$$\|T_jf\|_{W^{k,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)};\quad \|P_jf\|_{W^{k,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}.$$

Proof. For simplicity yet without loss of generality, assume $j=1$ and $n=2$. Denote by ∇_j the gradient in the z_j variable. Since $\bar\partial_1 T_1 = id$ and $\bar\partial_1 P_1 = 0$, we only need to prove for all $k_1,k_2\in\mathbb{Z}^+\cup\{0\}$, $k_1+k_2=k$,

$$\|\partial_1^{k_1}T_1\nabla_2^{k_2}f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)};\quad \|\partial_1^{k_1}P_1\nabla_2^{k_2}f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}.$$

In fact, making use of Theorem 3 and Fubini's theorem,

$$\begin{aligned}\|\partial_1^{k_1}T_1\nabla_2^{k_2}f\|^p_{L^p(\Omega)} &= \int_{D_2}\left\|\partial_1^{k_1}T_1\left(\nabla_2^{k_2}f\right)(\cdot,w_2)\right\|^p_{L^p(D_1)}\mathrm{d}v_{w_2}\\ &\lesssim \sum_{m_1=0}^{k_1}\int_{D_2}\left\|\nabla_1^{m_1}\nabla_2^{k_2}f(\cdot,w_2)\right\|^p_{L^p(D_1)}\mathrm{d}v_{w_2} \lesssim \|f\|^p_{W^{k,p}(\Omega)}.\end{aligned}$$

The estimate for P_1 is done similarly with an application of Theorem 2. $\square$

In particular, the proposition states that T_j does not lose Sobolev regularity. This estimate of T_j is also the best that one can expect when $n \geq 2$. This is because T_j only improves the regularity in the z_j direction and has no smoothing effect on the rest of the variables.

Proof of Theorem 1. It is a direct consequence of Proposition 5 and (10). □

Proposition 5 and (9) also immediately give the following Sobolev regularity of the Bergman projection operator P on general product domains. We mention that the Sobolev regularity of P on the polydisc was due to [8, 11].

Theorem 6. *Let $\Omega := D_1 \times \cdots \times D_n \subset \mathbb{C}^n, n \geq 1$, where each D_j is a bounded domain in $\mathbb{C}$ with smooth boundary, $j = 1, \ldots, n$. The Bergman projection P is (or, extends as) a bounded operator in $W^{k,p}(\Omega)$, $k \in \mathbb{Z}^+ \cup \{0\}$, $1 < p < \infty$. Namely, for any $f \in W^{k,p}(\Omega)$,*

$$\|Pf\|_{W^{k,p}(\Omega)} \lesssim \|f\|_{W^{k,p}(\Omega)}.$$

Denote by $\triangle^2$ the bidisc in $\mathbb{C}^2$. The following Kerzman-type example demonstrates that the $\bar\partial$ problem in general does not improve the Sobolev regularity. In this sense the Sobolev estimate of the canonical solution operator in Theorem 1 is sharp.

Example 7. For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $1 < p < \infty$, consider $f = (z_2 - 1)^{k - \frac{2}{p}} d\bar z_1$ on $\triangle^2$ if $p \neq 2$, or $f = (z_2 - 1)^{k-1} \log(z_2 - 1) d\bar z_1$ on $\triangle^2$ if $p = 2$, $\frac{1}{2}\pi < \arg(z_2 - 1) < \frac{3}{2}\pi$. Then $f \in W^{k,q}(\triangle^2)$ for all $1 < q < p$, and is $\bar\partial$-closed on $\triangle^2$. However, there does not exist a solution $u \in W^{k,p}(\triangle^2)$ to $\bar\partial u = f$ on $\triangle^2$.

Proof. One can directly verify that $f \in W^{k,q}(\triangle^2)$ for all $1 < q < p$ and is $\bar\partial$-closed on $\triangle^2$. Suppose there exists some $u \in W^{k,p}(\triangle^2)$ satisfying $\bar\partial u = f$ on $\triangle^2$. Then $u = (z_2 - 1)^{k - \frac{2}{p}} \bar z_1 + h \in W^{k,p}(\triangle^2)$ for some holomorphic function h on $\triangle^2$. For each $(r, z_2) \in U := (0,1) \times \triangle \subset \mathbb{R}^3$, consider

$$v(r, z_2) := \int_{|z_1| = r} u(z_1, z_2) dz_1.$$

By Fubini's theorem and Hölder's inequality,

$$\begin{aligned}\|\partial_2^k v\|^p_{L^p(U)} &= \int_U \left| \int_{|z_1|=r} \partial_2^k u(z_1, z_2) dz_1 \right|^p dv_{z_2} dr = \int_{|z_2|<1} \int_0^1 \left| r \int_0^{2\pi} |\partial_2^k u(re^{i\theta}, z_2)| d\theta \right|^p dr dv_{z_2} \\ &\lesssim \int_{|z_2|<1} \int_0^1 \int_0^{2\pi} |\partial_2^k u(re^{i\theta}, z_2)|^p d\theta r dr dv_{z_2} \leq \|u\|^p_{W^{k,p}(\triangle^2)} < \infty.\end{aligned}$$

Thus $\partial_2^k v \in L^p(U)$.

On the other hand, by Cauchy's theorem, for each $(r, z_2) \in U$,

$$\partial_2^k v(r, z_2) = C_{k,p} \int_{|z_1|=r} (z_2 - 1)^{-\frac{2}{p}} \bar z_1 dz_1 = C_{k,p} (z_2 - 1)^{-\frac{2}{p}} \int_{|z_1|=r} \frac{r^2}{z_1} dz_1 = 2\pi C_{k,p} r^2 i (z_2 - 1)^{-\frac{2}{p}}$$

for some non-zero constant $C_{k,p}$ depending only on k and p. However, $r^2 (z_2 - 1)^{-\frac{2}{p}} \notin L^p(U)$. This is a contradiction! □

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