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Characterization of Lipschitz Spaces via Commutators of Fractional Maximal Function on the p-Adic Variable Exponent Lebesgue Spaces

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Abstract. In this paper, the main aim is to give some characterizations of the boundedness of the maximal or nonlinear commutator of the p-adic fractional maximal operator $\mathscr{M}_{\alpha}^{p}$ with the symbols belong to the p-adic Lipschitz spaces in the context of the p-adic version of variable Lebesgue spaces, by which some new characterizations of the Lipschitz spaces and nonnegative Lipschitz functions are obtained in the p-adic field context. Meanwhile, Some equivalent relations between the p-adic Lipschitz norm and the p-adic variable Lebesgue norm are also given.

Keywords. p-adic field, Lipschitz function, fractional maximal function, variable exponent Lebesgue space.

2020 Mathematics Subject Classification. 42B35, 11E95, 26A16, 26A33, 47G10.

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1. Introduction and main results

During the last several decades, the p-adic analysis has attracted extensive attention due to its important applications in mathematical physics, science and technology, for instance, p-adic harmonic analysis, p-adic pseudo-differential equations, p-adic wavelet theory, etc. (see [2,19,26, 29]). Moreover, the theory of variable exponent function spaces has been intensely investigated in the past twenty years since some elementary properties were established by Kováčik and Rákosník in [21].

It is worthwhile to note that the fractional maximal operator plays an important role in real and harmonic analysis and applications, such as potential theory and partial differential equations (PDEs), since it is intimately related to the Riesz potential operator, which is a powerful tool in the study of the smooth function spaces (see [4, 6, 13]).

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On the other hand, there are two major reasons why the study of the commutators has got widespread attention. The first one is that the boundedness of commutators can produce some characterizations of function spaces [18, 24]. The other one is that the theory of commutators is intimately related to the regularity properties of the solutions of certain PDEs [5, 9, 12, 25].

Let T be the classical singular integral operator. The Coifman–Rochberg–Weiss type commutator $[b,T]$ generated by T and a suitable function b is defined by

$$[b,T]f = bT(f) - T(bf). \tag{1}$$

A well-known result shows that $[b,T]$ is bounded on $L^s(\mathbb{R}^n)$ for $1<s<\infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$ (the space of bounded mean oscillation functions). The sufficiency was provided by [10] and the necessity was obtained by [18]. Furthermore, [18] also established some characterizations of the Lipschitz space $\Lambda_\beta(\mathbb{R}^n)$ via commutator (1) and proved that $[b,T]$ is bounded from $L^s(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1<s<n/\beta$ and $1/s-1/q=\beta/n$ with $0<\beta<1$ if and only if $b\in\Lambda_\beta(\mathbb{R}^n)$ (see also [24]).

Denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of positive integers, integers, rational numbers and real numbers, respectively. For $\gamma\in\mathbb{Z}$ and a prime number p, let $\mathbb{Q}_p^n$ be a vector space over the p-adic field $\mathbb{Q}_p$, $B_\gamma(x)$ denote a p-adic ball with center $x\in\mathbb{Q}_p^n$ and radius p^γ (for the notations and notions, see Section 2 below).

Let $0\le\alpha<n$, for a locally integrable function f, the p-adic fractional maximal function of f is defined by

$$\mathcal{M}_\alpha^p(f)(x) = \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\alpha/n}} \int_{B_\gamma(x)} |f(y)|\mathrm{d}y,$$

where the supremum is taken over all p-adic balls $B_\gamma(x)\subset\mathbb{Q}_p^n$ and $|E|_h$ represents the Haar measure of a measurable set $E\subset\mathbb{Q}_p^n$. When $\alpha=0$, we simply write $\mathcal{M}^p$ instead of $\mathcal{M}_0^p$, which is the p-adic Hardy–Littlewood maximal function defined as

$$\mathcal{M}^p(f)(x) = \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y)|\mathrm{d}y.$$

The reader can refer to Stein [27] for the definition on the Euclidean case.

Similar to (1), we can define two different kinds of commutator of the fractional maximal function as follows.

Definition 1. *Let $0\le\alpha<n$ and b be a locally integrable function on $\mathbb{Q}_p^n$.*

(1) *The maximal commutator of $\mathcal{M}_\alpha^p$ with b is given by*

$$\mathcal{M}_{\alpha,b}^p(f)(x) = \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\alpha/n}} \int_{B_\gamma(x)} |b(x)-b(y)||f(y)|\mathrm{d}y,$$

where the supremum is taken over all p-adic balls $B_\gamma(x)\subset\mathbb{Q}_p^n$.

(2) *The nonlinear commutator $[b,\mathcal{M}_\alpha^p]$ generated by $\mathcal{M}_\alpha^p$ and b is defined by*

$$[b,\mathcal{M}_\alpha^p](f)(x) = b(x)\mathcal{M}_\alpha^p(f)(x) - \mathcal{M}_\alpha^p(bf)(x).$$

When $\alpha=0$, we simply denote by $[b,\mathcal{M}^p]=[b,\mathcal{M}_0^p]$ and $\mathcal{M}_b^p=\mathcal{M}_{0,b}^p$.

We call $[b,\mathcal{M}_\alpha^p]$ the nonlinear commutator because it is not even a sublinear operator, although the commutator $[b,T]$ is a linear one. It is worth noting that the nonlinear commutator $[b,\mathcal{M}_\alpha^p]$ and the maximal commutator $\mathcal{M}_{\alpha,b}^p$ essentially differ from each other. For example, $\mathcal{M}_{\alpha,b}^p$ is positive and sublinear, but $[b,\mathcal{M}_\alpha^p]$ is neither positive nor sublinear

Denote by M and M_α the classical Hardy–Littlewood maximal function and the fractional maximal function in $\mathbb{R}^n$ respectively. In fact, the nonlinear commutator $[b,M]$ and $[b,M_\alpha]$ have been studied by many authors in the Euclidean spaces, for instance, [1,3,14,23,32,33,35–38] etc. When the symbol b belongs to BMO($\mathbb{R}^n$), [3] studied the necessary and sufficient conditions for the boundedness of $[b,M]$ in $L^q(\mathbb{R}^n)$ for $1<q<\infty$. Zhang and Wu obtained similar results for the fractional maximal function in [35] and extended the mentioned results to variable exponent Lebesgue spaces in [36, 37]. When the symbol b belongs to Lipschitz spaces, [34, 38] gave the necessary and sufficient conditions for the boundedness of $[b,M_\alpha]$ on Orlicz spaces and variable Lebesgue spaces respectively. And recently, [31] considered some new characterizations of a variable version of Lipschitz spaces in terms of the boundedness of commutators of fractional maximal functions or fractional maximal commutators in the context of the variable Lebesgue spaces.

On the other hand, [15] gave the characterization of p-adic Lipschitz spaces in terms of the boundedness of commutators of maximal function $\mathcal{M}^p$ in the context of the p-adic Lebesgue spaces and Morrey spaces when the symbols b belong to p-adic Lipschitz spaces $\Lambda_\beta(\mathbb{Q}_p^n)$. And [8] proved the boundedness of the fractional maximal and the fractional integral operator in the p-adic variable exponent Lebesgue spaces.

Inspired by the above literature, we focus on the case of p-adic fields $\mathbb{Q}_p$, in some sense it can also be pointed out that our work was motivated by the standard harmonic analysis on the Euclidean space, the purpose of this paper is to study the boundedness of the p-adic fractional maximal commutator $\mathcal{M}^p_{\alpha,b}$ or the nonlinear commutator $[b,\mathcal{M}^p_\alpha]$ generated by p-adic fractional maximal function $\mathcal{M}^p_\alpha$ over p-adic variable exponent Lebesgue spaces, where the symbols b belong to the p-adic Lipschitz spaces, by which some new characterizations of the p-adic version of Lipschitz spaces are given.

Let $\alpha\ge 0$, for a fixed p-adic ball B_*, the fractional maximal function with respect to B_* of a locally integrable function f is given by

$$\mathcal{M}^p_{\alpha,B_*}(f)(x)=\sup_{\substack{\gamma\in\mathbb{Z}\\ B_\gamma(x)\subset B_*}}\frac{1}{|B_\gamma(x)|_h^{1-\alpha/n}}\int_{B_\gamma(x)}|f(y)|\mathrm{d}y,$$

where the supremum is taken over all the p-adic ball $B_\gamma(x)$ with $B_\gamma(x)\subset B_*$ for a fixed p-adic ball B_*. When $\alpha=0$, we simply write $\mathcal{M}^p_{B_*}$ instead of $\mathcal{M}^p_{0,B_*}$.

Our main results can be stated as follows, which are to study the boundedness of $\mathcal{M}^p_{\alpha,b}$ and $[b,\mathcal{M}^p_\alpha]$ in the context of p-adic variable exponent Lebesgue spaces when the symbol belongs to a p-adic version of Lipschitz spaces $\Lambda_\beta(\mathbb{Q}_p^n)$ (see Section 2 below). And some new characterizations of the Lipschitz spaces via such commutators are given.

Theorem 2. *Let $0<\beta<1$, $0<\alpha<\alpha+\beta<n$ and b be a locally integrable function on $\mathbb{Q}_p^n$. Then the following assertions are equivalent:*

(A.1) $b\in\Lambda_\beta(\mathbb{Q}_p^n)$ *and* $b\ge 0$.

(A.2) *The commutator $[b,\mathcal{M}^p_\alpha]$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$ for all $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\alpha+\beta}$ and $1/q(\cdot)=1/r(\cdot)-(\alpha+\beta)/n$.*

(A.3) *The commutator $[b,\mathcal{M}^p_\alpha]$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$ for some $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\alpha+\beta}$ and $1/q(\cdot)=1/r(\cdot)-(\alpha+\beta)/n$.*

(A.4) *There exists some $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\alpha+\beta}$ and $1/q(\cdot)=1/r(\cdot)-(\alpha+\beta)/n$, such that*

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}}\frac{1}{|B_\gamma(x)|_h^{\beta/n}}\frac{\left\|\left(b-|B_\gamma(x)|_h^{-\alpha/n}\mathcal{M}^p_{\alpha,B_\gamma(x)}(b)\right)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}<\infty. \tag{2}$$

(A.5) *For all* $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ *with* $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < \frac{n}{\alpha+\beta}$ *and* $1/q(\cdot) = 1/r(\cdot) - (\alpha+\beta)/n$, *such that* (2) *holds.*

For the case of $r(\cdot)$ and $q(\cdot)$ being constants, we have the following result from Theorem 2, which is new even for this case.

Corollary 3. *Let* $0 < \beta < 1$, $0 < \alpha < \alpha+\beta < n$ *and* b *be a locally integrable function on* $\mathbb{Q}_p^n$. *Then the following statements are equivalent:*

(C.1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ *and* $b \geq 0$.

(C.2) *The commutator* $[b, \mathscr{M}_\alpha^p]$ *is bounded from* $L^r(\mathbb{Q}_p^n)$ *to* $L^q(\mathbb{Q}_p^n)$ *for all* r, q *with* $1 < r < \frac{n}{\alpha+\beta}$ *and* $1/q = 1/r - (\alpha+\beta)/n$.

(C.3) *The commutator* $[b, \mathscr{M}_\alpha^p]$ *is bounded from* $L^r(\mathbb{Q}_p^n)$ *to* $L^q(\mathbb{Q}_p^n)$ *for some* r, q *with* $1 < r < \frac{n}{\alpha+\beta}$ *and* $1/q = 1/r - (\alpha+\beta)/n$.

(C.4) *There exists some* r, q *with* $1 < r < \frac{n}{\alpha+\beta}$ *and* $1/q = 1/r - (\alpha+\beta)/n$, *such that*

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\beta/n}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} \left| b(y) - |B_\gamma(x)|_h^{-\alpha/n} \mathscr{M}_{\alpha, B_\gamma(x)}^p(b)(y) \right|^q \mathrm{d}y \right)^{1/q} < \infty. \tag{3}$$

(C.5) *For all* r, q *with* $1 < r < \frac{n}{\alpha+\beta}$ *and* $1/q = 1/r - (\alpha+\beta)/n$, *such that* (3) *holds.*

Remark 4.

(i) For the case $\alpha = 0$, the partial results of Corollary 3 were given in [15, Theorem 4].

(ii) Moreover, it was proved in [15, Theorem 4], see also Lemma 22 below, that $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$ if and only if

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\beta/n}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} \left| b(y) - \mathscr{M}_{B_\gamma(x)}^p(b)(y) \right|^q \mathrm{d}y \right)^{1/q} < \infty. \tag{4}$$

Compared with (4), (3) gives a new characterization for nonnegative Lipschitz functions.

In particular, when $\alpha = 0$, the results are also true come from Theorem 2 and Corollary 3. Now we only give the case in the context of the p-adic version of variable exponent Lebesgue spaces, and it is new.

Corollary 5. *Let* $0 < \beta < 1$ *and* b *be a locally integrable function on* $\mathbb{Q}_p^n$. *Then the following statements are equivalent:*

(C.1) $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ *and* $b \geq 0$.

(C.2) *The commutator* $[b, \mathscr{M}^p]$ *is bounded from* $L^{r(\cdot)}(\mathbb{Q}_p^n)$ *to* $L^{q(\cdot)}(\mathbb{Q}_p^n)$ *for all* $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ *with* $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < \frac{n}{\beta}$ *and* $1/q(\cdot) = 1/r(\cdot) - \beta/n$.

(C.3) *The commutator* $[b, \mathscr{M}^p]$ *is bounded from* $L^{r(\cdot)}(\mathbb{Q}_p^n)$ *to* $L^{q(\cdot)}(\mathbb{Q}_p^n)$ *for some* $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ *with* $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < \frac{n}{\beta}$ *and* $1/q(\cdot) = 1/r(\cdot) - \beta/n$.

(C.4) *There exists some* $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ *with* $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < \frac{n}{\beta}$ *and* $1/q(\cdot) = 1/r(\cdot) - \beta/n$, *such that*

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\beta/n}} \frac{\left\| \left(b - \mathscr{M}_{B_\gamma(x)}^p(b)\right) \chi_{B_\gamma(x)} \right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} < \infty. \tag{5}$$

(C.5) *For all* $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ *with* $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < \frac{n}{\beta}$ *and* $1/q(\cdot) = 1/r(\cdot) - \beta/n$, *such that* (5) *holds.*

Theorem 6. *Let $0<\beta<1$, $0<\alpha<\alpha+\beta<n$ and b be a locally integrable function on $\mathbb{Q}_p^n$. Then the following assertions are equivalent:*

(B.1) $b\in\Lambda_\beta(\mathbb{Q}_p^n)$.

(B.2) *The commutator $\mathcal{M}_{\alpha,b}^p$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$ for all $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\alpha+\beta}$ and $1/q(\cdot)=1/r(\cdot)-(\alpha+\beta)/n$.*

(B.3) *The commutator $\mathcal{M}_{\alpha,b}^p$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$ for some $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\alpha+\beta}$ and $1/q(\cdot)=1/r(\cdot)-(\alpha+\beta)/n$.*

(B.4) *There exists some $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\alpha+\beta}$ and $1/q(\cdot)=1/r(\cdot)-(\alpha+\beta)/n$, such that*

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}}\frac{1}{|B_\gamma(x)|_h^{\beta/n}}\frac{\left\|\left(b-b_{B_\gamma(x)}\right)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}<\infty. \tag{6}$$

(B.5) *For all $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\alpha+\beta}$ and $1/q(\cdot)=1/r(\cdot)-(\alpha+\beta)/n$, such that* (6) *holds.*

When $r(\cdot)$ and $q(\cdot)$ are constants, we get the following result from Theorem 6.

Corollary 7. *Let $0<\beta<1$, $0<\alpha<\alpha+\beta<n$ and b be a locally integrable function on $\mathbb{Q}_p^n$. Then the following statements are equivalent:*

(C.1) $b\in\Lambda_\beta(\mathbb{Q}_p^n)$.

(C.2) *The commutator $\mathcal{M}_{\alpha,b}^p$ is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$ for all r,q with $1<r<\frac{n}{\alpha+\beta}$ and $1/q=1/r-(\alpha+\beta)/n$.*

(C.3) *The commutator $\mathcal{M}_{\alpha,b}^p$ is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$ for some r,q with $1<r<\frac{n}{\alpha+\beta}$ and $1/q=1/r-(\alpha+\beta)/n$.*

(C.4) *There exists some r,q with $1<r<\frac{n}{\alpha+\beta}$ and $1/q=1/r-(\alpha+\beta)/n$, such that*

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}}\frac{1}{|B_\gamma(x)|_h^{\beta/n}}\left(\frac{1}{|B_\gamma(x)|_h}\int_{B_\gamma(x)}\left|b(y)-b_{B_\gamma(x)}\right|^q\mathrm{d}y\right)^{1/q}<\infty. \tag{7}$$

(C.5) *For all r,q with $1<r<\frac{n}{\alpha+\beta}$ and $1/q=1/r-(\alpha+\beta)/n$, such that* (7) *holds.*

Remark 8.

(i) For the case $\alpha=0$, Corollary 7 is also holds, and the equivalence of (C.1), (C.2) and (C.3) was proved in [15, Theorem 1].

(ii) Moreover, the equivalence of (C.1), (C.4) and (C.5) is contained in Lemma 21 below.

Finally, we give the follows result, which is valid and new, from Theorem 6 with $\alpha=0$.

Corollary 9. *Let $0<\beta<1$ and b be a locally integrable function on $\mathbb{Q}_p^n$. Then the following statements are equivalent:*

(C.1) $b\in\Lambda_\beta(\mathbb{Q}_p^n)$.

(C.2) *The commutator $\mathcal{M}_b^p$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$ for all $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\beta}$ and $1/q(\cdot)=1/r(\cdot)-\beta/n$.*

(C.3) *The commutator $\mathcal{M}_b^p$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$ for some $r(\cdot),q(\cdot)\in\mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot)\in\mathscr{P}(\mathbb{Q}_p^n)$, $r_+<\frac{n}{\beta}$ and $1/q(\cdot)=1/r(\cdot)-\beta/n$.*

(C.4) *There exists some* $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ *with* $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < \frac{n}{\beta}$ *and* $1/q(\cdot) = 1/r(\cdot) - \beta/n$, *such that*

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\beta/n}} \frac{\left\|\left(b-b_{B_\gamma(x)}\right)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} < \infty. \tag{8}$$

(C.5) *For all* $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ *with* $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < \frac{n}{\beta}$ *and* $1/q(\cdot) = 1/r(\cdot) - \beta/n$, *such that* (8) *holds.*

Throughout this paper, the letter C always stands for a constant independent of the main parameters involved and whose value may differ from line to line. In addition, we give some notations. Here and hereafter $|E|_h$ will always denote the Haar measure of a measurable set E on $\mathbb{Q}_p^n$ and by χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{Q}_p^n$.

2. Preliminaries and lemmas

To prove the main results of this paper, we first recall some necessary notions and remarks.

2.1. *p-adic field* $\mathbb{Q}_p$

Firstly, we introduce some basic and necessary notations for the p-adic field.

Let $p \geq 2$ be a fixed prime number in $\mathbb{Z}$ and $G_p = \{0, 1, \dots, p-1\}$. For every non-zero rational number x, by the unique factorization theorem, there is a unique $\gamma = \gamma(x) \in \mathbb{Z}$, such that $x = p^\gamma \frac{m}{n}$, where $m, n \in \mathbb{Z}$ are not divisible by p (i.e. p is coprime to m, n). Define the mapping $|\cdot|_p : \mathbb{Q} \to \mathbb{R}^+$ as follows:

$$|x|_p = \begin{cases} p^{-\gamma} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The p-adic absolute value $|\cdot|_p$ is endowed with many properties of the usual real norm $|\cdot|$ with an additional non-Archimedean property (i.e., $\{|m|_p, m \in \mathbb{Z}\}$ is bounded)

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}.$$

In addition, $|\cdot|_p$ also satisfies the following properties:

(1) (positive definiteness) $|x|_p \geq 0$. Specially, $|x|_p = 0 \Longleftrightarrow x = 0$.
(2) (multiplicativity) $|xy|_p = |x|_p|y|_p$.
(3) (non-Archimedean triangle inequality) $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. The equality holds if and only if $|x|_p \neq |y|_p$.

Denote by $\mathbb{Q}_p$ the p-adic field which is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$.

From the standard p-adic analysis, any non-zero element $x \in \mathbb{Q}_p$ can be uniquely represented as a canonical series form

$$x = p^\gamma(a_0 + a_1 p + a_2 p^2 + \cdots) = p^\gamma \sum_{j=0}^{\infty} a_j p^j,$$

where $a_j \in G_p$ and $a_0 \neq 0$, and $\gamma = \gamma(x) \in \mathbb{Z}$ is called as the p-adic valuation of x. The series converges in the p-adic absolute value since the inequality $|a_j p^j|_p \leq p^{-j}$ holds for all $j \in \mathbb{N}$.

Moreover, the n-dimensional p-adic vector space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ $(n \geq 1)$, consists of all points $x = (x_1, \dots, x_n)$, where $x_i \in \mathbb{Q}_p$ $(i = 1, \dots, n)$, equipped with the following absolute value

$$|x|_p = \max_{1\leq j\leq n} |x_j|_p.$$

For $\gamma \in \mathbb{Z}$ and $a = (a_1, a_2, \dots, a_n) \in \mathbb{Q}_p^n$, we denote by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p \le p^\gamma\}$$

the closed ball with the center at a and radius p^γ and by

$$S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x-a|_p = p^\gamma\} = B_\gamma(a) \setminus B_{\gamma-1}(a)$$

the corresponding sphere. For $a = 0$, we write $B_\gamma(0) = B_\gamma$, and $S_\gamma(0) = S_\gamma$. Note that $B_\gamma(a) = \bigcup_{k\le\gamma} S_k(a)$ and $\mathbb{Q}_p^n \setminus \{0\} = \bigcup_{\gamma\in\mathbb{Z}} S_\gamma$. It is easy to see that the equalities

$$a_0 + B_\gamma = B_\gamma(a_0) \text{ and } a_0 + S_\gamma = S_\gamma(a_0) = B_\gamma(a_0) \setminus B_{\gamma-1}(a_0)$$

hold for all $a_0 \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$.

It follows from non-Archimedean triangle inequality that two balls $B_\gamma(x)$ and $B_{\gamma'}(y)$ either do not intersect or one of them is contained in the other, which differ from those of the Euclidean case. And note that in the second case under conditions $\gamma = \gamma'$ these balls are equal. The above properties can also be found in [20, Lemma 3.1].

Lemma 10. *Let $\gamma, \gamma' \in \mathbb{Z}$, $x, y \in \mathbb{Q}_p^n$. The p-adic balls have the following properties:*

(1) *If $\gamma \le \gamma'$, then either $B_\gamma(x) \cap B_{\gamma'}(y) = \emptyset$ or $B_\gamma(x) \subset B_{\gamma'}(y)$.*
(2) *$B_\gamma(x) = B_\gamma(y)$ if and only if $y \in B_\gamma(x)$.*

Since $\mathbb{Q}_p^n$ is a locally compact commutative group with respect to addition, there exists a unique Haar measure $\mathrm{d}x$ on $\mathbb{Q}_p^n$ (up to positive constant multiple) which is translation invariant (i.e., $\mathrm{d}(x+a) = \mathrm{d}x$), such that

$$\int_{B_0} \mathrm{d}x = |B_0|_h = 1,$$

where $|E|_h$ denotes the Haar measure of measurable subset E of $\mathbb{Q}_p^n$. Furthermore, from this integral theory, it is easy to obtain that

$$\int_{B_\gamma(a)} \mathrm{d}x = |B_\gamma(a)|_h = p^{n\gamma} \tag{9}$$

and

$$\int_{S_\gamma(a)} \mathrm{d}x = |S_\gamma(a)|_h = p^{n\gamma}(1-p^{-n}) = |B_\gamma(a)|_h - |B_{\gamma-1}(a)|_h$$

hold for all $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$.

For more information about the p-adic field, we refer readers to [28,30].

2.2. *p-adic function spaces*

In what follows, we say that a real-valued measurable function f defined on $\mathbb{Q}_p^n$ is in $L^q(\mathbb{Q}_p^n)$, $1 \le q \le \infty$, if it satisfies

$$\|f\|_{L^q(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(x)|^q \mathrm{d}x\right)^{1/q} < \infty, \quad 1 \le q < \infty \tag{10}$$

and denote by $L^\infty(\mathbb{Q}_p^n)$ the set of all measurable real-valued functions f on $\mathbb{Q}_p^n$ satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} = \operatorname*{ess\,sup}_{x\in\mathbb{Q}_p^n} |f(x)| < \infty.$$

Here, the integral in equation (10) is defined as follows:

$$\int_{\mathbb{Q}_p^n} |f(x)|^q \mathrm{d}x = \lim_{\gamma\to\infty} \int_{B_\gamma(0)} |f(x)|^q \mathrm{d}x = \lim_{\gamma\to\infty} \sum_{-\infty<k\le\gamma} \int_{S_k(0)} |f(x)|^q \mathrm{d}x,$$

if the limit exists.

Some often used computational principles are worth noting. In particular, if $f \in L^1(\mathbb{Q}_p^n)$, then

$$\int_{\mathbb{Q}_p^n} f(x)\mathrm{d}x = \sum_{\gamma=-\infty}^{+\infty} \int_{S_\gamma} f(x)\mathrm{d}x$$

and

$$\int_{\mathbb{Q}_p^n} f(tx)\mathrm{d}x = \frac{1}{|t|_p^n} \int_{\mathbb{Q}_p^n} f(x)\mathrm{d}x,$$

where $t \in \mathbb{Q}_p \setminus \{0\}$, $tx = (tx_1, \ldots, tx_n)$ and $\mathrm{d}(tx) = |t|_p^n \mathrm{d}x$.

We now introduce the notion of p-adic variable exponent Lebesgue spaces and give some properties needed in the sequel (see [7] for the respective proofs).

We say that a measurable function $q(\cdot)$ is a variable exponent if $q(\cdot) : \mathbb{Q}_p^n \to (0, \infty)$.

Definition 11. *Given a measurable function $q(\cdot)$ defined on $\mathbb{Q}_p^n$, we denote by*

$$q_- := \operatorname*{ess\,inf}_{x \in \mathbb{Q}_p^n} q(x), \quad q_+ := \operatorname*{ess\,sup}_{x \in \mathbb{Q}_p^n} q(x).$$

(1) $q'_- = \operatorname{ess\,inf}_{x \in \mathbb{Q}_p^n} q'(x) = \frac{q_+}{q_+ - 1}$, $q'_+ = \operatorname{ess\,sup}_{x \in \mathbb{Q}_p^n} q'(x) = \frac{q_-}{q_- - 1}$.

(2) *Denote by $\mathscr{P}_0(\mathbb{Q}_p^n)$ the set of all measurable functions $q(\cdot) : \mathbb{Q}_p^n \to (0, \infty)$ such that*

$$0 < q_- \le q(x) \le q_+ < \infty, \quad x \in \mathbb{Q}_p^n.$$

(3) *Denote by $\mathscr{P}_1(\mathbb{Q}_p^n)$ the set of all measurable functions $q(\cdot) : \mathbb{Q}_p^n \to [1, \infty)$ such that*

$$1 \le q_- \le q(x) \le q_+ < \infty, \quad x \in \mathbb{Q}_p^n.$$

(4) *Denote by $\mathscr{P}(\mathbb{Q}_p^n)$ the set of all measurable functions $q(\cdot) : \mathbb{Q}_p^n \to (1, \infty)$ such that*

$$1 < q_- \le q(x) \le q_+ < \infty, \quad x \in \mathbb{Q}_p^n.$$

(5) *The set $\mathscr{B}(\mathbb{Q}_p^n)$ consists of all measurable functions $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ satisfying that the Hardy–Littlewood maximal operator $\mathscr{M}^p$ is bounded on $L^{q(\cdot)}(\mathbb{Q}_p^n)$.*

Definition 12 (p-adic variable exponent Lebesgue spaces). *Let $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$. Define the p-adic variable exponent Lebesgue spaces $L^{q(\cdot)}(\mathbb{Q}_p^n)$ as follows*

$$L^{q(\cdot)}(\mathbb{Q}_p^n) = \{f \text{ is measurable function} : \mathscr{F}_q(f/\eta) < \infty \text{ for some constant } \eta > 0\},$$

where $\mathscr{F}_q(f) := \int_{\mathbb{Q}_p^n} |f(x)|^{q(x)}\mathrm{d}x$. The Lebesgue space $L^{q(\cdot)}(\mathbb{Q}_p^n)$ is a Banach function space with respect to the Luxemburg norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \inf\left\{\eta > 0 : \mathscr{F}_q(f/\eta) = \int_{\mathbb{Q}_p^n} \left(\frac{|f(x)|}{\eta}\right)^{q(x)} \mathrm{d}x \le 1\right\}.$$

Definition 13 (log-Hölder continuity [7]). *Let measurable function $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$.*

(1) *Denote by $\mathscr{C}_0^{\log}(\mathbb{Q}_p^n)$ the set of all $q(\cdot)$ which satisfies*

$$\gamma\big(q_-(B_\gamma(x)) - q_+(B_\gamma(x))\big) \le C$$

for all $\gamma \in \mathbb{Z}$ and any $x \in \mathbb{Q}_p^n$, where C denotes a universal positive constant.

(2) *The set $\mathscr{C}_\infty^{\log}(\mathbb{Q}_p^n)$ consists of all $q(\cdot)$ which satisfies*

$$|q(x) - q(y)| \le \frac{C}{\log_p(p + \min\{|x|_p, |y|_p\})}$$

for any $x, y \in \mathbb{Q}_p^n$, where C is a positive constant.

(3) *Denote by $\mathscr{C}^{\log}(\mathbb{Q}_p^n) = \mathscr{C}_0^{\log}(\mathbb{Q}_p^n) \cap \mathscr{C}_\infty^{\log}(\mathbb{Q}_p^n)$ the set of all global log-Hölder continuous functions $q(\cdot)$.*

In what follows, we denote $\mathscr{C}(\mathbb{Q}_p^n) \cap \mathscr{P}(\mathbb{Q}_p^n)$ by $\mathscr{P}^{\log}(\mathbb{Q}_p^n)$. And for a function b defined on $\mathbb{Q}_p^n$, we denote

$$b^-(x) := -\min\{b, 0\} = \begin{cases} 0, & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Obviously, $b(x) = b^+(x) - b^-(x)$.

Definition 14 (p-adic Lipschitz space).

(1) *Assume that $0 < \beta < 1$. The p-adic version of homogeneous Lipschitz spaces $\Lambda_\beta(\mathbb{Q}_p^n)$ is the set of all measurable functions f on $\mathbb{Q}_p^n$ with the finite norm*

$$\|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)} = \sup_{\substack{x,y\in\mathbb{Q}_p^n \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|_p^\beta}.$$

(2) *For $1 \leq q < \infty$ and $0 \leq \beta < 1$, the p-adic version of Lipschitz spaces $\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)$ (or Campanato spaces) is the set of all measurable functions f on $\mathbb{Q}_p^n$ such that*

$$\|f\|_{\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)} = \sup_{\substack{x\in\mathbb{Q}_p^n \\ \gamma\in\mathbb{Z}}} \frac{1}{|B_\gamma(x)|_h^{\beta/n}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} |f(y) - f_{B_\gamma(x)}|^q \mathrm{d}y \right)^{1/q} < \infty,$$

where $f_{B_\gamma(x)}$ denotes the average of f over $B_\gamma(x)$, i.e., $f_{B_\gamma(x)} = |B_\gamma(x)|_h^{-1} \int_{B_\gamma(x)} f(y)\mathrm{d}y$. In particular, when $q = 1$, we use $\mathrm{Lip}_\beta(\mathbb{Q}_p^n)$ as $\mathrm{Lip}_\beta^1(\mathbb{Q}_p^n)$.

Remark 15 (see [22]).

(1) When $0 < \beta < 1$, $\Lambda_\beta(\mathbb{Q}_p^n)$ is just the homogeneous Besov–Lipschitz space.
(2) Since $\Lambda_\beta(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ are Campanato space when $0 < \beta < 1$ and $\beta = 0$, respectively. Thus, in this sense, the space $\mathrm{BMO}(\mathbb{Q}_p^n)$ can be seen as a limit case of $\Lambda_\beta(\mathbb{Q}_p^n)$ as $\beta \to 0$.

2.3. *Auxiliary propositions and lemmas*

The first part of Lemma 16 may be found in [7, Theorem 5.2]. By elementary calculations, the second of Lemma 16 can be obtained as well.

Lemma 16. *Let $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$.*

(1) *If $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$, then $q(\cdot) \in \mathscr{B}(\mathbb{Q}_p^n)$.*
(2) *The following conditions are equivalent:*
   (i) *$q(\cdot) \in \mathscr{B}(\mathbb{Q}_p^n)$,*
   (ii) *$q'(\cdot) \in \mathscr{B}(\mathbb{Q}_p^n)$,*
   (iii) *$q(\cdot)/q_0 \in \mathscr{B}(\mathbb{Q}_p^n)$ for some $1 < q_0 < q_-$,*
   (iv) *$(q(\cdot)/q_0)' \in \mathscr{B}(\mathbb{Q}_p^n)$ for some $1 < q_0 < q_-$,*
   where r' stand for the conjugate exponent of r, viz., $1 = \frac{1}{r(\cdot)} + \frac{1}{r'(\cdot)}$.

Remark 17. If $q(\cdot) \in \mathscr{B}(\mathbb{Q}_p^n)$ and $s > 1$, then $sq(\cdot) \in \mathscr{B}(\mathbb{Q}_p^n)$ (for the Euclidean case see [11, Remark 2.13] for more details).

We now present the following results related to the Hölder's inequality. The part (1) is known as the Hölder's inequality on Lebesgue spaces over p-adic vector space $\mathbb{Q}_p^n$. And similar to the Euclidean case, the part (2) can be deduced by simple calculations (or see [7, Lemma 3.8]).

Lemma 18 (Generalized Hölder's inequality on $\mathbb{Q}_p^n$). *Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space.*

(1) *Suppose that $1 \le q \le \infty$ with $\frac{1}{q} + \frac{1}{q'} = 1$, and measurable functions $f \in L^q(\mathbb{Q}_p^n)$ and $g \in L^{q'}(\mathbb{Q}_p^n)$. Then there exists a positive constant C such that*

$$\int_{\mathbb{Q}_p^n} |f(x)g(x)|\mathrm{d}x \le C\|f\|_{L^q(\mathbb{Q}_p^n)}\|g\|_{L^{q'}(\mathbb{Q}_p^n)}.$$

(2) *Suppose that $q_1(\cdot), q_2(\cdot), r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ and $r(\cdot)$ satisfy $\frac{1}{r(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$ almost everywhere. Then there exists a positive constant C such that the inequality*

$$\|fg\|_{L^{r(\cdot)}(\mathbb{Q}_p^n)} \le C\|f\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\|g\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}$$

holds for all $f \in L^{q_1(\cdot)}(\mathbb{Q}_p^n)$ and $g \in L^{q_2(\cdot)}(\mathbb{Q}_p^n)$.

(3) *When $r(\cdot) = 1$ in (2) as mentioned above, we have $q_1(\cdot), q_2(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ and $\frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)} = 1$ almost everywhere. Then there exists a positive constant C such that the inequality*

$$\int_{\mathbb{Q}_p^n} |f(x)g(x)|\mathrm{d}x \le C\|f\|_{L^{q_1(\cdot)}(\mathbb{Q}_p^n)}\|g\|_{L^{q_2(\cdot)}(\mathbb{Q}_p^n)}$$

holds for all $f \in L^{q_1(\cdot)}(\mathbb{Q}_p^n)$ and $g \in L^{q_2(\cdot)}(\mathbb{Q}_p^n)$.

The following results for the characteristic function $\chi_{B_\gamma(x)}$ are required as well. By elementary calculations, the first part may be obtain from the p-adic integral theory (or refer to (9)). The second part may be founded in [8, Lemma 7], and the part (4) follows from Lemma 18 (2). Moreover, according to Lemma 16 and Lemma 19 (2), the third part can also be deduced by simple calculations. So, we omit the proofs.

Lemma 19 (Norms of characteristic functions). *Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space.*

(1) *If $1 \le q < \infty$. Then there exist a constant $C > 0$ such that*

$$\|\chi_{B_\gamma(x)}\|_{L^q(\mathbb{Q}_p^n)} = |B_\gamma(x)|_h^{1/q} = p^{n\gamma/q}.$$

(2) *If $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$. Then*

$$\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \le Cp^{n\gamma/q(x,\gamma)},$$

where

$$q(x,\gamma) = \begin{cases} q(x) & \text{if } \gamma < 0, \\ q(\infty) & \text{if } \gamma \ge 0. \end{cases}$$

(3) *If $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ and $q(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$. Then there exist a constant $C > 0$ such that*

$$\frac{1}{|B_\gamma(x)|_h}\left\|\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\left\|\chi_{B_\gamma(x)}\right\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)} < C$$

holds for all p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$.

(4) *Let $0 < \alpha < n$. If $q(\cdot), r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ with $r_+ < \frac{n}{\alpha}$ and $1/q(\cdot) = 1/r(\cdot) - \alpha/n$, then there exists a constant $C > 0$ such that*

$$\|\chi_{B_\gamma(x)}\|_{L^{r(\cdot)}(\mathbb{Q}_p^n)} \le C|B_\gamma(x)|_h^{\alpha/n}\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}$$

holds for all p-adic balls $B_\gamma(x) \subset \mathbb{Q}_p^n$.

Lemma 20 ([17]). *Suppose $f \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $0 < \beta < 1$, then for any $x, y \in \mathbb{Q}_p^n$, one has*

$$|f(x) - f(y)| \le |x-y|_p^\beta \|f\|_{\Lambda_\beta(\mathbb{Q}_p^n)}.$$

The following are some properties of p-adic Lipschitz spaces (see [15] for more details).

Lemma 21. *Let $0 < \beta < 1$ and $1 \le q < \infty$. If $f \in \mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)$.*

(1) *Then the norm $\|f\|_{\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)}$ is equivalent to the norm $\|f\|_{\mathrm{Lip}_\beta(\mathbb{Q}_p^n)}$.*
(2) *Then the homogeneous Lipschitz space $\Lambda_\beta(\mathbb{Q}_p^n)$ coincides with the space $\mathrm{Lip}_\beta^q(\mathbb{Q}_p^n)$.*

From the proof of [15, Theorem 4], we can obtain the following characterization of nonnegative Lipschitz functions.

Lemma 22. *Let $0 < \beta < 1$ and b be a locally integrable function on $\mathbb{Q}_p^n$. Then the following assertions are equivalent:*

(1) *$b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$.*
(2) *For all $1 \le s < \infty$, there exists a positive constant C such that*

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\beta/n}} \left(\frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} \left| b(y) - \mathcal{M}^p_{B_\gamma(x)}(b)(y) \right|^s \mathrm{d}y \right)^{1/s} \le C. \tag{11}$$

(3) *(11) holds for some $1 \le s < \infty$.*

Proof. Since the implication (2) $\Longrightarrow$ (3) follows readily, and the implication (3) $\Longrightarrow$ (1) was proved in [15, Theorem 4], we only need to prove (1) $\Longrightarrow$ (2).

If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$, then it follows from [15, Theorem 4] that (11) holds for all s with $n/(n-\beta) < s < \infty$. Applying Hölder's inequality, we see that (11) holds for $1 \le s \le n/(n-\beta)$ as well.

Hence, the implication (1) $\Longrightarrow$ (2) is proven. □

Now we recall the Hardy–Littlewood–Sobolev inequality for the fractional maximal function $\mathcal{M}_\alpha^p$ on p-adic vector space. The first part of Lemma 23 can be founded in [15, Lemma 8], and the second comes from [8, Theorem 4].

Lemma 23. *Let $0 < \alpha < n$ and $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space.*

(1) *If $1 < r < n/\alpha$ such that $1/q = 1/r - \alpha/n$, then $\mathcal{M}_\alpha^p$ is bounded from $L^r(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$.*
(2) *If $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < n/\alpha$ and $1/q(\cdot) = 1/r(\cdot) - \alpha/n$, then $\mathcal{M}_\alpha^p$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$.*

By Lemma 23 (1), if $0 < \alpha < n$, $1 < r < n/\alpha$ and $f \in L^r(\mathbb{Q}_p^n)$, then $\mathcal{M}_\alpha^p(f)(x) < \infty$ almost everywhere. A similar result is also valid in variable Lebesgue spaces. And the method of proof can refer to [34, Lemma 2.6], so we omit its proof.

Lemma 24. *Let $0 < \alpha < n$, $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$ and $1 < r_- \le r_+ < n/\alpha$. If $f \in L^{r(\cdot)}(\mathbb{Q}_p^n)$, then $\mathcal{M}_\alpha^p(f)(x) < \infty$ for almost everywhere $x \in \mathbb{Q}_p^n$.*

Now, we give the following pointwise estimate for $[b, \mathcal{M}_\alpha^p]$ when $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

Lemma 25. *Let $0 \le \alpha < n$, $0 < \beta < 1$, $0 < \alpha+\beta < n$ and f be a locally integrable function on $\mathbb{Q}_p^n$. If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \ge 0$, then, for any $x \in \mathbb{Q}_p^n$ such that $\mathcal{M}_\alpha^p(f)(x) < \infty$, we have*

$$\left| [b, \mathcal{M}_\alpha^p](f)(x) \right| \le \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \mathcal{M}^p_{\alpha+\beta}(f)(x).$$

Proof. For any fixed $x \in \mathbb{Q}_p^n$ such that $\mathscr{M}_\alpha^p(f)(x) < \infty$, if $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$, then

$$\begin{aligned}\left|[b, \mathscr{M}_\alpha^p](f)(x)\right| &= \left|b(x)\mathscr{M}_\alpha^p(f)(x) - \mathscr{M}_\alpha^p(bf)(x)\right| \\ &= \left| \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\alpha/n}} \int_{B_\gamma(x)} b(x)|f(y)|\mathrm{d}y - \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\alpha/n}} \int_{B_\gamma(x)} b(y)|f(y)|\mathrm{d}y \right| \\ &\leq \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\alpha/n}} \int_{B_\gamma(x)} \left|b(x)-b(y)\right||f(y)|\mathrm{d}y \\ &\leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha+\beta}{n}}} \int_{B_\gamma(x)} |f(y)|\mathrm{d}y \\ &\leq C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \mathscr{M}_{\alpha+\beta}^p(f)(x). \end{aligned}$$

□

Finally, we also need the following results. Similar to the proof of [35, Lemma 2.3], and referring to the course of the proof of [16, Theorem 1.4], using Lemma 10 and the properties of p-adic ball, through elementary calculations and derivations, the following assertions can be obtained. Hence, we omit the proofs.

Lemma 26. *Let b be a locally integrable function and $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space. For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$.*

(1) *If $0 \leq \alpha < n$, then for all $y \in B_\gamma(x)$, we have*

$$\mathscr{M}_\alpha^p(b\chi_{B_\gamma(x)})(y) = \mathscr{M}_{\alpha,B_\gamma(x)}^p(b)(y)$$

and

$$\mathscr{M}_\alpha^p(\chi_{B_\gamma(x)})(y) = \mathscr{M}_{\alpha,B_\gamma(x)}^p(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\alpha/n}.$$

(2) *Then for any $y \in B_\gamma(x)$, we have*

$$|b_{B_\gamma(x)}| \leq |B_\gamma(x)|_h^{-\alpha/n} \mathscr{M}_{\alpha,B_\gamma(x)}^p(b)(y).$$

(3) *Let $E = \{y \in B_\gamma(x) : b(y) \leq b_{B_\gamma(x)}\}$ and $F = B_\gamma(x) \setminus E = \{y \in B_\gamma(x) : b(y) > b_{B_\gamma(x)}\}$. Then the following equality is trivially true*

$$\int_E |b(y) - b_{B_\gamma(x)}|\mathrm{d}y = \int_F |b(y) - b_{B_\gamma(x)}|\mathrm{d}y.$$

3. Proofs of the main results

Now we give the proofs of the Theorem 2 and Theorem 6.

3.1. *Proof of Theorem 2*

To prove Theorem 2, we first prove the following lemma.

Lemma 27. *Let $0 < \alpha < n$, $0 < \beta < 1$ and b be a locally integrable function on $\mathbb{Q}_p^n$. If there exists a positive constant C such that*

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{\beta/n}} \frac{\left\|\left(b - |B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}_{\alpha,B_\gamma(x)}^p(b)\right)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \leq C \tag{12}$$

for some $q(\cdot) \in \mathscr{B}(\mathbb{Q}_p^n)$, then $b \in \Lambda_\beta(\mathbb{Q}_p^n)$.

Proof. Some ideas are taken from [3, 34, 35, 37]. Reasoning as the proof of [37, (4.4)], see also the proof of [34, Lemma 3.1], for any fixed p-adic ball $B_\gamma(x)\subset\mathbb{Q}_p^n$, we have (see Lemma 26 (2))

$$|b_{B_\gamma(x)}|\le|B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}^p_{\alpha,B_\gamma(x)}(b)(y),\qquad \forall\ y\in B_\gamma(x).$$

Let $E=\{y\in B_\gamma(x): b(y)\le b_{B_\gamma(x)}\}$ and $F=B_\gamma(x)\setminus E=\{y\in B_\gamma(x): b(y)>b_{B_\gamma(x)}\}$, then for any $y\in E\subset B_\gamma(x)$, we have $b(y)\le b_{B_\gamma(x)}\le|b_{B_\gamma(x)}|\le|B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}^p_{\alpha,B_\gamma(x)}(b)(y)$. It is clear that

$$|b(y)-b_{B_\gamma(x)}|\le\left|b(y)-|B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}^p_{\alpha,B_\gamma(x)}(b)(y)\right|,\qquad\forall\ y\in E.$$

Therefore, by using Lemma 26 (3), we get

$$\begin{aligned}\frac{1}{|B_\gamma(x)|_h^{\beta/n+1}}\int_{B_\gamma(x)}\left|b(y)-b_{B_\gamma(x)}\right|\mathrm{d}y&=\frac{1}{|B_\gamma(x)|_h^{\beta/n+1}}\int_{E\cup F}\left|b(y)-b_{B_\gamma(x)}\right|\mathrm{d}y\\&\le\frac{2}{|B_\gamma(x)|_h^{\beta/n+1}}\int_E\left|b(y)-|B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}^p_{\alpha,B_\gamma(x)}(b)(y)\right|\mathrm{d}y\\&\le\frac{2}{|B_\gamma(x)|_h^{\beta/n+1}}\int_{B_\gamma(x)}\left|b(y)-|B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}^p_{\alpha,B_\gamma(x)}(b)(y)\right|\mathrm{d}y.\end{aligned}$$

Using Lemma 18 (3) (generalized Hölder's inequality), (12) and Lemma 19 (3), we obtain

$$\begin{aligned}\frac{1}{|B_\gamma(x)|_h^{\beta/n+1}}\int_{B_\gamma(x)}\left|b(y)-b_{B_\gamma(x)}\right|\mathrm{d}y&\le\frac{2}{|B_\gamma(x)|_h^{\beta/n+1}}\int_{B_\gamma(x)}\left|b(y)-|B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}^p_{\alpha,B_\gamma(x)}(b)(y)\right|\mathrm{d}y\\&\le\frac{C}{|B_\gamma(x)|_h^{\beta/n+1}}\left\|\left(b-|B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}^p_{\alpha,B_\gamma(x)}(b)\right)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\\&\quad\times\|\chi_{B_\gamma(x)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\\&\le\frac{C}{|B_\gamma(x)|_h}\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_\gamma(x)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\\&\le C.\end{aligned}$$

So, the proof is completed by applying Lemma 21. □

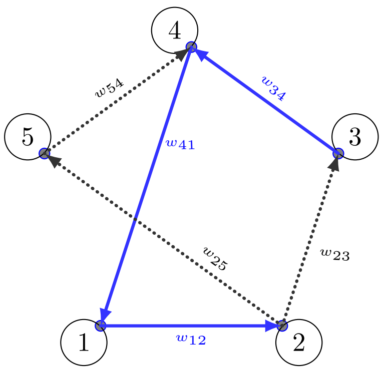

Figure 1. Proof structure of Theorem 2 where w_{ij} denotes $i\Longrightarrow j$

Proof of Theorem 2. Since the implications (A.2) $\Longrightarrow$ (A.3) and (A.5) $\Longrightarrow$ (A.4) follow readily, and (A.2) $\Longrightarrow$ (A.5) is similar to (A.3) $\Longrightarrow$ (A.4), we only need to prove (A.1) $\Longrightarrow$ (A.2), (A.3) $\Longrightarrow$ (A.4) and (A.4) $\Longrightarrow$ (A.1) (see Figure 1 for the proof structure).

(A.1) ⟹ (A.2). Let $b \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $b \geq 0$. We need to prove that $[b, \mathscr{M}_\alpha^p]$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$ for all $r(\cdot), q(\cdot) \in \mathscr{C}^{\log}(\mathbb{Q}_p^n)$ with $r(\cdot) \in \mathscr{P}(\mathbb{Q}_p^n)$, $r_+ < \frac{n}{\alpha+\beta}$ and $1/q(\cdot) = 1/r(\cdot) - (\alpha+\beta)/n$. For such $r(\cdot)$ and any $f \in L^{r(\cdot)}(\mathbb{Q}_p^n)$, it follows from Lemma 24 that $\mathscr{M}_\alpha^p(f)(x) < \infty$ for almost everywhere $x \in \mathbb{Q}_p^n$. By Lemma 25, we have

$$\left|[b, \mathscr{M}_\alpha^p](f)(x)\right| \leq \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \mathscr{M}_{\alpha+\beta}^p(f)(x).$$

Then, statement (A.2) follows from Lemma 23 (2).

(A.3) ⟹ (A.4). For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$ and any $y \in B_\gamma(x)$, it follows from Lemma 26 (1) that

$$\mathscr{M}_\alpha^p(b\chi_{B_\gamma(x)})(y) = \mathscr{M}_{\alpha,B_\gamma(x)}^p(b)(y) \text{ and } \mathscr{M}_\alpha^p(\chi_{B_\gamma(x)})(y) = \mathscr{M}_{\alpha,B_\gamma(x)}^p(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\alpha/n}.$$

Then, for any $y \in B_\gamma(x)$, we have

$$\begin{aligned} b(y) - |B_\gamma(x)|_h^{-\alpha/n} \mathscr{M}_{\alpha,B_\gamma(x)}^p(b)(y) &= |B_\gamma(x)|_h^{-\alpha/n}\big(b(y)|B_\gamma(x)|_h^{\alpha/n} - \mathscr{M}_{\alpha,B_\gamma(x)}^p(b)(y)\big) \\ &= |B_\gamma(x)|_h^{-\alpha/n}\big(b(y)\mathscr{M}_\alpha^p(\chi_{B_\gamma(x)})(y) - \mathscr{M}_\alpha^p(b\chi_{B_\gamma(x)})(y)\big) \\ &= |B_\gamma(x)|_h^{-\alpha/n}[b, \mathscr{M}_\alpha^p](\chi_{B_\gamma(x)})(y). \end{aligned}$$

Thus, for any $y \in \mathbb{Q}_p^n$, we get

$$\big(b(y) - |B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}_{\alpha,B_\gamma(x)}^p(b)(y)\big)\chi_{B_\gamma(x)}(y) = |B_\gamma(x)|_h^{-\alpha/n}[b, \mathscr{M}_\alpha^p](\chi_{B_\gamma(x)})(y)\chi_{B_\gamma(x)}(y).$$

By using assertion (A.3) and Lemma 19 (4) (norms of characteristic functions), we have

$$\begin{aligned} \left\|\big(b - |B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}_{\alpha,B_\gamma(x)}^p(b)\big)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} &\leq |B_\gamma(x)|_h^{-\alpha/n}\left\|[b, \mathscr{M}_\alpha^p](\chi_{B_\gamma(x)})\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \\ &\leq C|B_\gamma(x)|_h^{-\alpha/n}\left\|\chi_{B_\gamma(x)}\right\|_{L^{r(\cdot)}(\mathbb{Q}_p^n)} \\ &\leq C|B_\gamma(x)|_h^{\beta/n}\left\|\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}, \end{aligned}$$

which gives (2) since $B_\gamma(x)$ is arbitrary and C is independent of $B_\gamma(x)$.

(A.4) ⟹ (A.1). By Lemma 22, it suffices to prove

$$\sup_{\substack{\gamma\in\mathbb{Z} \\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1+\beta/n}} \int_{B_\gamma(x)} \left|b(y) - \mathscr{M}_{B_\gamma(x)}^p(b)(y)\right| \mathrm{d}y < \infty. \tag{13}$$

For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, we have

$$\begin{aligned} &\frac{1}{|B_\gamma(x)|_h^{1+\beta/n}} \int_{B_\gamma(x)} \left|b(y) - \mathscr{M}_{B_\gamma(x)}^p(b)(y)\right| \mathrm{d}y \\ &\quad\leq \frac{1}{|B_\gamma(x)|_h^{1+\beta/n}} \int_{B_\gamma(x)} \left|b(y) - |B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}_{\alpha,B_\gamma(x)}^p(b)(y)\right| \mathrm{d}y \\ &\qquad + \frac{1}{|B_\gamma(x)|_h^{1+\beta/n}} \int_{B_\gamma(x)} \left||B_\gamma(x)|_h^{-\alpha/n}\mathscr{M}_{\alpha,B_\gamma(x)}^p(b)(y) - \mathscr{M}_{B_\gamma(x)}^p(b)(y)\right| \mathrm{d}y \\ &\quad:= I_1 + I_2. \end{aligned} \tag{14}$$

For I_1, by applying statement (A.4), Lemma 18 (3) (generalized Hölder's inequality) and Lemma 19 (3), we get

$$\begin{aligned} I_1 &= \frac{1}{|B_\gamma(x)|_h^{1+\beta/n}} \int_{B_\gamma(x)} \left| b(y) - |B_\gamma(x)|_h^{-\alpha/n} \mathcal{M}^p_{\alpha,B_\gamma(x)}(b)(y) \right| \mathrm{d}y \\ &\le \frac{C}{|B_\gamma(x)|_h^{\beta/n+1}} \left\| \left(b - |B_\gamma(x)|_h^{-\alpha/n} \mathcal{M}^p_{\alpha,B_\gamma(x)}(b)\right)\chi_{B_\gamma(x)} \right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_\gamma(x)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)} \\ &\le \frac{C}{|B_\gamma(x)|_h^{\beta/n}} \frac{\left\| \left(b - |B_\gamma(x)|_h^{-\alpha/n} \mathcal{M}^p_{\alpha,B_\gamma(x)}(b)\right)\chi_{B_\gamma(x)} \right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \\ &\le C, \end{aligned}$$

where the constant C is independent of $B_\gamma(x)$.

Now we consider I_2. For all $y \in B_\gamma(x)$, it follows from Lemma 26 that

$$\mathcal{M}^p_\alpha(\chi_{B_\gamma(x)})(y) = |B_\gamma(x)|_h^{\alpha/n} \text{ and } \mathcal{M}^p_\alpha(b\chi_{B_\gamma(x)})(y) = \mathcal{M}^p_{\alpha,B_\gamma(x)}(b)(y),$$

and

$$\mathcal{M}^p(\chi_{B_\gamma(x)})(y) = \chi_{B_\gamma(x)}(y) = 1 \text{ and } \mathcal{M}^p(b\chi_{B_\gamma(x)})(y) = \mathcal{M}^p_{B_\gamma(x)}(b)(y).$$

Then, for any $y \in B_\gamma(x)$, we get

$$\begin{aligned} &\left| |B_\gamma(x)|_h^{-\alpha/n} \mathcal{M}^p_{\alpha,B_\gamma(x)}(b)(y) - \mathcal{M}^p_{B_\gamma(x)}(b)(y) \right| \\ &\quad \le |B_\gamma(x)|_h^{-\alpha/n} \left| \mathcal{M}^p_{\alpha,B_\gamma(x)}(b)(y) - |B_\gamma(x)|_h^{\alpha/n}|b(y)| \right| + \left| |b(y)| - \mathcal{M}^p_{B_\gamma(x)}(b)(y) \right| \\ &\quad \le |B_\gamma(x)|_h^{-\alpha/n} \left| \mathcal{M}^p_\alpha(b\chi_{B_\gamma(x)})(y) - |b(y)|\mathcal{M}^p_\alpha(\chi_{B_\gamma(x)})(y) \right| \\ &\qquad + \left| |b(y)|\mathcal{M}^p(\chi_{B_\gamma(x)})(y) - \mathcal{M}^p(b\chi_{B_\gamma(x)})(y) \right| \\ &\quad \le |B_\gamma(x)|_h^{-\alpha/n} \left| [|b|, \mathcal{M}^p_\alpha](\chi_{B_\gamma(x)})(y) \right| + \left| [|b|, \mathcal{M}^p](\chi_{B_\gamma(x)})(y) \right|. \end{aligned} \tag{15}$$

Since $q(\cdot) \in \mathscr{B}(\mathbb{Q}_p^n)$ follows at once from Lemma 16 and statement (A.4). Then statement (A.4) along with Lemma 27 gives $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, which implies $|b| \in \Lambda_\beta(\mathbb{Q}_p^n)$. Thus, we can apply Lemma 25 to $[|b|, \mathcal{M}^p_\alpha]$ and $[|b|, \mathcal{M}^p]$ due to $|b| \in \Lambda_\beta(\mathbb{Q}_p^n)$ and $|b| \ge 0$.

By using Lemma 25 and Lemma 26 (1), for any $y \in B_\gamma(x)$, we have

$$\left| [|b|, \mathcal{M}^p_\alpha](\chi_{B_\gamma(x)})(y) \right| \le \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \mathcal{M}^p_{\alpha+\beta}(\chi_{B_\gamma(x)})(y) \le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} |B_\gamma(x)|_h^{(\alpha+\beta)/n}$$

and

$$\left| [|b|, \mathcal{M}^p](\chi_{B_\gamma(x)})(y) \right| \le \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \mathcal{M}^p_\beta(\chi_{B_\gamma(x)})(y) \le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} |B_\gamma(x)|_h^{\beta/n}.$$

Hence, it follows from (15) that

$$\begin{aligned} I_2 &= \frac{1}{|B_\gamma(x)|_h^{1+\beta/n}} \int_{B_\gamma(x)} \left| |B_\gamma(x)|_h^{-\alpha/n} \mathcal{M}^p_{\alpha,B_\gamma(x)}(b)(y) - \mathcal{M}^p_{B_\gamma(x)}(b)(y) \right| \mathrm{d}y \\ &\le \frac{C}{|B_\gamma(x)|_h^{1+(\alpha+\beta)/n}} \int_{B_\gamma(x)} \left| [|b|, \mathcal{M}^p_\alpha](\chi_{B_\gamma(x)})(y) \right| \mathrm{d}y + \frac{C}{|B_\gamma(x)|_h^{1+\beta/n}} \int_{B_\gamma(x)} \left| [|b|, \mathcal{M}^p](\chi_{B_\gamma(x)})(y) \right| \mathrm{d}y \\ &\le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)}. \end{aligned}$$

Putting the above estimates for I_1 and I_2 into (14), we obtain (13).

This completes the proof of Theorem 2. □

3.2. *Proof of Theorem 6*

Proof of Theorem 6. Similar to prove Theorem 2, we only need to prove the implications (B.1) $\Longrightarrow$ (B.2), (B.3) $\Longrightarrow$ (B.4) and (B.4) $\Longrightarrow$ (B.1) (the proof structure is also shown in Figure 1).

(B.1) $\Longrightarrow$ (B.2). If $b \in \Lambda_\beta(\mathbb{Q}_p^n)$, then for any p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$, using Lemma 20, we derive

$$\begin{aligned}
\mathscr{M}^p_{\alpha,b}(f)(x) &= \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\alpha/n}} \int_{B_\gamma(x)} |b(x)-b(y)||f(y)|\mathrm{d}y \\
&\le \|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\alpha/n}} \int_{B_\gamma(x)} |x-y|_p^\beta |f(y)|\mathrm{d}y \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \sup_{\substack{\gamma\in\mathbb{Z}\\ x\in\mathbb{Q}_p^n}} \frac{1}{|B_\gamma(x)|_h^{1-\frac{\alpha+\beta}{n}}} \int_{B_\gamma(x)} |f(y)|\mathrm{d}y \\
&\le C\|b\|_{\Lambda_\beta(\mathbb{Q}_p^n)} \mathscr{M}^p_{\alpha+\beta}(f)(x).
\end{aligned}$$

This, together with Lemma 23 (2), shows that $\mathscr{M}^p_{\alpha,b}$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$.

(B.3) $\Longrightarrow$ (B.4). For any fixed p-adic ball $B_\gamma(x) \subset \mathbb{Q}_p^n$. By using Lemma 10, for all $y \in B_\gamma(x)$, we have

$$\begin{aligned}
|b(y)-b_{B_\gamma(x)}| &\le \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} \big|b(y)-b(z)\big|\mathrm{d}z \\
&= \frac{1}{|B_\gamma(x)|_h} \int_{B_\gamma(x)} \big|b(y)-b(z)\big|\chi_{B_\gamma(x)}(z)\mathrm{d}z \\
&\le \frac{1}{|B_\gamma(x)|_h^{\alpha/n}} \mathscr{M}^p_{\alpha,b}(\chi_{B_\gamma(x)})(y).
\end{aligned}$$

Then, for all $y \in \mathbb{Q}_p^n$, we get

$$\big|\big(b(y)-b_{B_\gamma(x)}\big)\chi_{B_\gamma(x)}(y)\big| \le |B_\gamma(x)|_h^{-\alpha/n} \mathscr{M}^p_{\alpha,b}(\chi_{B_\gamma(x)})(y).$$

Since $\mathscr{M}^p_{\alpha,b}$ is bounded from $L^{r(\cdot)}(\mathbb{Q}_p^n)$ to $L^{q(\cdot)}(\mathbb{Q}_p^n)$, using Lemma 19 (4), we have

$$\begin{aligned}
\big\|\big(b-b_{B_\gamma(x)}\big)\chi_{B_\gamma(x)}\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} &\le |B_\gamma(x)|_h^{-\alpha/n} \big\|\mathscr{M}^p_{\alpha,b}(\chi_{B_\gamma(x)})\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \\
&\le C|B_\gamma(x)|_h^{-\alpha/n} \big\|\chi_{B_\gamma(x)}\big\|_{L^{r(\cdot)}(\mathbb{Q}_p^n)} \\
&\le C|B_\gamma(x)|_h^{\beta/n} \|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)},
\end{aligned}$$

which implies (6) since $B_\gamma(x)$ is arbitrary and C is independent of $B_\gamma(x)$.

(B.4) $\Longrightarrow$ (B.1). For any p-adic ball $B_\gamma(x)\subset\mathbb{Q}_p^n$, by using Lemma 18(3) (generalized Hölder's inequality), assertion (B.4) and Lemma 19(3), we obtain

$$\begin{aligned}\frac{1}{|B_\gamma(x)|_h^{1+\beta/n}}\int_{B_\gamma(x)}\left|b(y)-b_{B_\gamma(x)}\right|\mathrm{d}y &= \frac{1}{|B_\gamma(x)|_h^{1+\beta/n}}\int_{B_\gamma(x)}\left|b(y)-b_{B_\gamma(x)}\right|\chi_{B_\gamma(x)}(y)\mathrm{d}y\\ &\le \frac{C}{|B_\gamma(x)|_h^{1+\beta/n}}\left\|\left(b-b_{B_\gamma(x)}\right)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_\gamma(x)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\\ &= \frac{C}{|B_\gamma(x)|_h^{\beta/n}}\frac{\left\|\left(b-b_{B_\gamma(x)}\right)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}\\ &\quad\times\frac{1}{|B_\gamma(x)|_h}\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_\gamma(x)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\\ &\le \frac{C}{|B_\gamma(x)|_h^{\beta/n}}\frac{\left\|\left(b-b_{B_\gamma(x)}\right)\chi_{B_\gamma(x)}\right\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_\gamma(x)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}\\ &\le C.\end{aligned}$$

This shows that $b\in\Lambda_\beta(\mathbb{Q}_p^n)$ by Lemma 21 and Definition 14 since the constant C is independent of $B_\gamma(x)$.

The proof of Theorem 6 is finished. □

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References

[1] M. Agcayazi, A. Gogatishvili, K. Koca, R. Mustafayev, "A note on maximal commutators and commutators of maximal functions", *J. Math. Soc. Japan* **67** (2015), no. 2, p. 581-593.

[2] S. Albeverio, A. Y. Khrennikov, V. M. Shelkovich, "Harmonic analysis in the p-adic Lizorkin spaces: fractional operators, pseudo-differential equations, p-adic wavelets, Tauberian theorems", *J. Fourier Anal. Appl.* **12** (2006), no. 4, p. 393-425.

[3] J. Bastero, M. Milman, F. J. Ruiz, "Commutators for the maximal and sharp functions", *Proc. Am. Math. Soc.* **128** (2000), no. 11, p. 3329-3334.

[4] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, 2007.

[5] M. Bramanti, C. Cerutti, *Commutators of singular integrals and fractional integrals on homogeneous spaces*, Contemporary Mathematics, vol. 189, American Mathematical Society, 1995, 81-94 pages.

[6] E. Carneiro, J. Madrid, "Derivative bounds for fractional maximal functions", *Trans. Am. Math. Soc.* **369** (2017), no. 6, p. 4063-4092.

[7] L. F. Chacón-Cortés, H. Rafeiro, "Variable exponent Lebesgue spaces and Hardy–Littlewood maximal function on p-adic numbers", *p-Adic Numbers Ultrametric Anal. Appl.* **12** (2020), no. 2, p. 90-111.

[8] ———, "Fractional operators in p-adic variable exponent Lebesgue spaces and application to p-adic derivative", *J. Funct. Spaces* **2021** (2021), article no. 3096701 (9 pages).

[9] F. Chiarenza, M. Frasca, P. Longo, "$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients", *Trans. Am. Math. Soc.* **336** (1993), no. 2, p. 841-853.

[10] R. R. Coifman, R. Rochberg, G. Weiss, "Factorization theorems for Hardy spaces in several variables", *Ann. Math.* **103** (1976), p. 611-635.

[11] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, C. Pérez, "The boundedness of classical operators on variable L^p spaces", *Ann. Acad. Sci. Fenn., Math.* **31** (2006), no. 1, p. 239-264.

[12] G. Di Fazio, M. A. Ragusa, "Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients", *J. Funct. Anal.* **112** (1993), no. 2, p. 241-256.

[13] G. B. Folland, E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, 1982.

[14] V. S. Guliyev, F. Deringoz, S. G. Hasanov, "Fractional maximal function and its commutators on Orlicz spaces", *Anal. Math. Phys.* **9** (2017), no. 1, p. 165-179.

[15] Q. He, X. Li, "Characterization of Lipschitz spaces via commutators of maximal function on the p-adic vector space", *J. Math.* **2022** (2022), article no. 7430272 (15 pages).

[16] ———, "Necessary and sufficient conditions for boundedness of commutators of maximal function on the p-adic vector spaces", *AIMS Math.* **8** (2023), no. 6, p. 14064-14085.

[17] A. Hussain, N. Sarfraz, I. Khan, A. Alsubie, N. N. Hamadneh, "The boundedness of commutators of rough p-adic fractional Hardy type operators on Herz-type spaces", *J. Inequal. Appl.* **2021** (2021), article no. 123 (13 pages).

[18] S. Janson, "Mean oscillation and commutators of singular integral operators", *Ark. Mat.* **16** (1978), no. 1, p. 263-270.

[19] A. Y. Khrennikov, V. M. Shelkovich, "Non-Haar p-adic wavelets and their application to pseudo-differential operators and equations", *Appl. Comput. Harmon. Anal.* **28** (2010), no. 1, p. 1-23.

[20] Y.-C. Kim, "L^q-estimates of maximal operators on the p-adic vector space", *Commun. Korean Math. Soc.* **24** (2009), no. 3, p. 367-379.

[21] O. Kováčik, J. Rákosník, "On spaces $L^{p(x)}$ and $W^{k,p(x)}$", *Czech. Math. J.* **41** (1991), no. 4, p. 592-618.

[22] L. Li, C. Xiong, J. Zhou, "Lipschitz estimates for commutators of p-adic fractional Hardy operators", *Commun. Math. Res.* **31** (2015), no. 2, p. 131-140.

[23] M. Milman, T. Schonbek, "Second order estimates in interpolation theory and applications", *Proc. Am. Math. Soc.* **110** (1990), no. 4, p. 961-969.

[24] M. Paluszyński, "Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss", *Indiana Univ. Math. J.* **44** (1995), no. 1, p. 1-17.

[25] M. A. Ragusa, "Cauchy–Dirichlet problem associated to divergence form parabolic equations", *Commun. Contemp. Math.* **6** (2004), no. 3, p. 377-393.

[26] V. M. Shelkovich, M. Skopina, "p-adic Haar multiresolution analysis and pseudo-differential operators", *J. Fourier Anal. Appl.* **15** (2009), no. 3, p. 366-393.

[27] E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, 1993.

[28] M. H. Taibleson, *Fourier Analysis on Local Fields*, Mathematical Notes, vol. 15, Princeton University Press, 1975.

[29] A. Torresblanca-Badillo, A. A. Albarracín-Mantilla, "Some further classes of pseudo-differential operators in the p-adic context and their applications", *J. Pseudo-Differ. Oper. Appl.* **14** (2023), no. 2, article no. 24 (23 pages).

[30] V. S. Vladimirov, I. V. Volovich, E. I. Zelenov, *p-adic Analysis and Mathematical Physics*, World Scientific, 1994.

[31] X. Yang, Z. Yang, B. Li, "Characterization of Lipschitz space via the commutators of fractional maximal functions on variable lebesgue spaces", *Potential Anal.* **60** (2024), p. 703-720.

[32] P. Zhang, "Characterization of Lipschitz spaces via commutators of the Hardy–Littlewood maximal function", *C. R. Math. Acad. Sci. Paris* **355** (2017), no. 3, p. 336-344.

[33] ———, "Characterization of boundedness of some commutators of maximal functions in terms of Lipschitz spaces", *Anal. Math. Phys.* **9** (2019), no. 3, p. 1411-1427.

[34] P. Zhang, Z. Si, J. Wu, "Some notes on commutators of the fractional maximal function on variable Lebesgue spaces", *J. Inequal. Appl.* **2019** (2019), article no. 9 (17 pages).

[35] P. Zhang, J. Wu, "Commutators of the fractional maximal functions", *Acta Math. Sin.* **52** (2009), no. 6, p. 1235-1238.

[36] ———, "Commutators for the maximal functions on Lebesgue spaces with variable exponent", *Math. Inequal. Appl.* **17** (2014), no. 4, p. 1375-1386.

[37] ———, "Commutators of the fractional maximal function on variable exponent Lebesgue spaces", *Czech. Math. J.* **64** (2014), no. 1, p. 183-197.

[38] P. Zhang, J. Wu, J. Sun, "Commutators of some maximal functions with Lipschitz function on Orlicz spaces", *Mediterr. J. Math.* **15** (2018), no. 6, article no. 216 (13 pages).