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Lucio Boccardo, Juan Casado-Diaz and Luigi Orsina

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Dirichlet problems with skew-symmetric drift terms

Problèmes de Dirichlet avec des termes de drift asymétriques

Lucio Boccardo [a], **Juan Casado-Diaz** [b] **and Luigi Orsina** [*,c]

[a] Istituto Lombardo & Sapienza Università di Roma, Italy

[b] Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Spain

[c] Dipartimento di Matematica, Sapienza Università di Roma, Italy

E-mails: boccardo@mat.uniroma1.it, jcasadod@us.es, orsina@mat.uniroma1.it

Abstract. We prove existence of finite energy solutions for a linear Dirichlet problem with a drift and a convection term of the form $AE(x)\nabla u + \operatorname{div}(uE(x))$, with $A > 0$ and E in $(L^r(\Omega))^N$. The result is obtained using a nonlinear function of u as test function, in order to "cancel" this term.

Résumé. Nous prouvons l'existence de solutions d'énergie finie pour un problème de Dirichlet linéaire avec un terme de la forme $AE(x)\nabla u + \operatorname{div}(uE(x))$, où $A > 0$ et E est dans $(L^r(\Omega))^N$. Le résultat est obtenu en utilisant une fonction non linéaire de u comme fonction test, afin d'"annuler" ce terme.

Keywords. Singular drift, Dirichlet problems, nonlinear test functions.

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1. Introduction

In [8] and [9] is studied the Dirichlet problem

$$u \in W_0^{1,2}(\Omega): \ -\Delta u + E(x)\cdot\nabla u + \operatorname{div}(uE(x)) = f(x),$$

where

$$\Omega \text{ is a bounded domain of } \mathbb{R}^N, \text{ and the drift } E \text{ belongs to } (L^2(\Omega))^N,$$

and

$$f(x) \in L^{\frac{2N}{N+2}}(\Omega).$$

Remark that if u belongs to $W_0^{1,2}(\Omega)$, then the terms $E(x)\cdot\nabla u$ and $\operatorname{div}(uE(x))$ just belong to $L^1(\Omega)$ and $W^{-1,\frac{N}{N-1}}(\Omega)$ (or $W^{-1,p}(\Omega)$, $\forall\ p<2$ if $N=2$) respectively. The key point for the existence of finite energy solutions is that the map

$$P: u \to E(x)\cdot\nabla u + \operatorname{div}(uE(x))$$

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* Corresponding author.

is skew-symmetric and thus, it satisfies the "cancellation" property

$$\langle Pu, u\rangle = 0, \tag{1}$$

in the sense of distributions, if u is smooth enough.

The present paper deals with a more general framework, given by

$$u \in W_0^{1,q}(\Omega): -\mathrm{div}(M(x)\nabla u) + A E(x)\cdot\nabla u + \mathrm{div}(u E(x)) = f(x), \tag{2}$$

with

$$A > 0, \tag{3}$$

$M : \Omega \to \mathbb{R}^{N^2}$ a measurable matrix such that there exist $\alpha, \beta > 0$, satisfying

$$\alpha|\xi|^2 \le M(x)\xi\cdot\xi, \qquad |M(x)| \le \beta, \quad \text{a.e. in } \Omega, \ \ \forall\, \xi \in \mathbb{R}^N, \tag{4}$$

$$E \in (L^r(\Omega))^N : \begin{cases} r = 2 & \text{if } A \ge 1, \\ r = \frac{N(A+1)}{A(N-1)+1} & \text{if } 0 < A < 1,\ N \ge 3, \\ 2 < r < 1 + \frac{1}{A} & \text{if } 0 < A < 1,\ N = 2, \end{cases} \tag{5}$$

and

$$f \in L^m(\Omega), \quad m > 1 \ \text{ if } N = 2, \quad m = \frac{N(A+1)}{N+2A} \ \text{ if } N \ge 3. \tag{6}$$

Note that $r \ge 2$, so that $r' \le 2$; and also $r' > 1$.

We are going to prove the existence of a distributional solution u of (2) in the Sobolev space $W_0^{1,r'}(\Omega)$, with $r' = \frac{r}{r-1}$, the Hölder conjugate exponent of r. In particular, u is in $W_0^{1,2}(\Omega)$ if $A \ge 1$.

The proof of the result is based on the use of a nonlinear function of u as test function in (2), in such way that a cancellation property similar to (1) still holds.

We complete this introduction with some references about the existence of solutions for linear elliptic equations with a first order term whose coefficients have poor summability.

For a measurable matrix function $M : \Omega \to \mathbb{R}^{N^2}$, which satisfies (4),

$$E, F \in \begin{cases} (L^N(\Omega))^N & \text{if } N > 2, \\ (L^p(\Omega))^N,\ p > 2 & \text{if } N = 2, \end{cases} \qquad a \in \begin{cases} L^{\frac{N}{2}}(\Omega) & \text{if } N > 2, \\ L^q(\Omega),\ q > 1 & \text{if } N = 2, \end{cases} \tag{7}$$

such that

$$-\mathrm{div}(E(x)) + a(x) \ge 0 \ \text{ in } \Omega, \tag{8}$$

it has been proved in [11, 12] that the weak maximum principle holds for the equation

$$\mathscr{L} u = f \ \text{ in } \Omega,$$

with

$$\mathscr{L} u = -\mathrm{div}\big(M(x)\nabla u - u E(x)) + F(x)\cdot\nabla u + a(x)\,u. \tag{9}$$

Thus, the corresponding Dirichlet problem has at most one solution in $W_0^{1,2}(\Omega)$, for every $f \in W^{-1,2}(\Omega)$. By the Fredholm theory, for every $f \in W^{-1,2}(\Omega)$ there exists a unique solution u for problems

$$u \in W_0^{1,2}(\Omega), \quad \mathscr{L}(u) = f \ \text{ in } \Omega, \qquad u \in W_0^{1,2}(\Omega), \quad \mathscr{L}^*(u) = f \ \text{ in } \Omega, \tag{10}$$

where

$$\mathscr{L}^*(u) = -\mathrm{div}\big(M(x)^t\nabla u + u F(x)) - E(x)\cdot\nabla u + a(x)\,u,$$

is the adjoint operator of $\mathscr{L}$. We emphasize that these problems are not coercive.

If E or F do not satisfy (7), the operators $u \mapsto -\mathrm{div}(u E(x))$ or $u \mapsto F(x)\cdot\nabla u$ do not apply $W_0^{1,2}(\Omega)$ into $W^{-1,2}(\Omega)$, but some existence results have still been proved.

In [3] and [4], are considered the cases $F = 0$ and $E = 0$ respectively, and a Stampacchia–Calderón–Zygmund theory for finite or infinite energy solutions (depending on the summability of $f(x)$) is proved. For example, if $E \in (L^2(\Omega))^N$, $F = 0$, $f \in L^1(\Omega)$, it is proved the existence of a solution for the problem

$$\mathscr{L} u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{11}$$

in a very weak sense: the solution u is such that $\log(1+|u|)$ belongs to $W_0^{1,2}(\Omega)$ and is an "entropy solution" of the equation (for the theory of entropy, or renormalized, solutions, see e.g. [1], [7], [10]) in the sense that:

$$\int_\Omega \Big[M(x)\nabla u\cdot\nabla T_k(u-\varphi) - u\,E(x)\cdot\nabla T_k(u-\varphi)\Big] \le \int f(x)\,T_k(u-\varphi),$$

for every φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, and for every $k>0$. Here

$$T_k(s) = \begin{cases} -k & \text{if } s<-k,\\ s & \text{if } -k\le s\le k,\\ k & \text{if } s>k, \end{cases} \tag{12}$$

is the usual truncation function at levels $\pm k$. In [6], adding a zero order term greater than a positive constant, it has been proved that the above function u is also in $L^1(\Omega)$. A duality argument then shows that problem (11) has a solution in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ if $E=0$, F is in $(L^2(\Omega))^N$ and a is greater than a positive constant.

2. Proof of the main result

We devote this section to the proof of an existence result for problem (2). It will be based on the introduction of an approximate problem and then the use of a nonlinear test function, which will provide the estimates needed to pass to the limit.

Theorem 1. *Assume that $M:\Omega\to\mathbb{R}^{N^2}$ is a measurable matrix which satisfies* (4)*, that $A>0$, and that E belongs to $(L^r(\Omega))^N$, with r defined by* (5)*. Then, for every $f\in L^m(\Omega)$, with m given by* (6)*, there exists a distributional solution u of* (2) *in $W_0^{1,r'}(\Omega)$. Moreover, u satisfies*

$$|u|^{\frac{A-1}{2}}u\in W_0^{1,2}(\Omega),\quad u\in\begin{cases} L^{\frac{N(A+1)}{N-2}}(\Omega) & \text{if } N\ge 3,\\ L^p(\Omega),\ \forall\, p\in[1,\infty) & \text{if } N=2. \end{cases} \tag{13}$$

Remark 2. We point out a regularizing effect of the problem: for $A\ge1$, u belongs to $W_0^{1,2}(\Omega)$, even if the term $A\,E(x)\cdot\nabla u$ only belongs to $L^1(\Omega)$.

Remark 3. Defining for $\mu,\lambda>0$, $\widehat{E}(x)=\lambda E(x)$ and $A=\mu\lambda$, we deduce from Theorem 1 the existence of a distributional solution for problem

$$u\in W_0^{1,r'}(\Omega):\ -\mathrm{div}(M(x)\nabla u)+\lambda\,E(x)\cdot\nabla u+\mu\,\mathrm{div}(u\,E(x))=f(x).$$

Proof of Theorem 1. Let n in $\mathbb{N}$: the starting point is the nonlinear Dirichlet problem: $u_n\in W_0^{1,2}(\Omega)$:

$$\begin{aligned}\int_\Omega M(x)\nabla u_n\cdot\nabla v-\int_\Omega \frac{u_n}{1+\frac1n|u_n|}\,\frac{E_n(x)}{1+\frac1n|\nabla u_n|}\cdot\nabla v\\ +A\int_\Omega \frac{1}{1+\frac1n|u_n|}\,\frac{E_n(x)}{1+\frac1n|\nabla u_n|}\cdot\nabla u_n\,v=\int_\Omega f_n(x)\,v,\qquad \forall\, v\in W_0^{1,2}(\Omega),\end{aligned} \tag{14}$$

where

$$E_n(x)=\frac{E(x)}{1+\frac1n|E(x)|},\quad f_n(x)=\frac{f(x)}{1+\frac1n|f(x)|}.$$

The existence of u_n is a consequence of the use of the Schauder fixed point theorem (see also [2]), since all the terms are bounded. Note that u_n is the solution of the Dirichlet problem

$$u_n\in W_0^{1,2}(\Omega):\ -\mathrm{div}(M(x)\,\nabla u_n)=-\mathrm{div}(G_n(x))+g_n(x),$$

where
$$G_n(x)=\frac{u_n}{1+\frac{1}{n}|u_n|}\,\frac{E_n(x)}{1+\frac{1}{n}|\nabla u_n|},\quad g_n(x)=f_n(x)-A\,\frac{E_n(x)}{1+\frac{1}{n}|u_n|}\cdot\frac{\nabla u_n}{1+\frac{1}{n}|\nabla u_n|}.$$
Since
$$|G_n(x)|\le n^2,\qquad |g_n(x)|\le n^2,$$
from Lax–Milgram theorem and from a result by Stampacchia (see [12], Théorème 4.1), it follows that there exists $C>0$ such that
$$\|u_n\|_{W_0^{1,2}(\Omega)}+\|u_n\|_{L^\infty(\Omega)}\le C\,n^2.$$
Since every u_n is a bounded function, it is possible to use a nonlinear composition of u_n as test function in (14).

In the following, we recall that the Sobolev exponent 2^* is $2N/(N-2)$, if $N>2$. For $N=2$, we define 2^* as a positive number bigger than 2 to be chosen later.

Step 1. In this step, we assume $A\ge 1$ and we will prove the existence of finite energy solutions.

We use $|u_n|^{A-1}u_n$ as test function in (14), and we have
$$A\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,|u_n|^{A-1}-A\int_\Omega\frac{u_n}{1+\frac{1}{n}|u_n|}\,\frac{E_n(x)}{1+\frac{1}{n}|\nabla u_n|}\cdot\nabla u_n\,|u_n|^{A-1}$$
$$+A\int_\Omega\frac{u_n}{1+\frac{1}{n}|u_n|}\,\frac{E_n(x)}{1+\frac{1}{n}|\nabla u_n|}\cdot\nabla u_n\,|u_n|^{A-1}=\int_\Omega f_n(x)\,|u_n|^{A-1}u_n. \tag{15}$$
Thus, after cancellation of equal terms, we have that
$$A\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,|u_n|^{A-1}=\int_\Omega f_n(x)\,|u_n|^{A-1}u_n,$$
which, by (4) and Sobolev's inequality, implies that
$$\frac{4A\alpha}{(A+1)^2}\left[\int_\Omega|u_n|^{\frac{2^*}{2}(A+1)}\right]^{\frac{2}{2^*}}\le\mathscr{S}\,A\,\alpha\int_\Omega|\nabla u_n|^2|u_n|^{A-1}\le\mathscr{S}\,\|f\|_{L^m(\Omega)}\left[\int_\Omega|u_n|^{Am'}\right]^{\frac{1}{m'}}. \tag{16}$$
If $N>3$, the condition $\frac{2^*}{2}(A+1)=Am'$ holds if $m=\frac{N(A+1)}{N+2A}$ (which is our assumption); thus, from (16) we obtain that
$$\frac{4A\alpha}{(A+1)^2}\|u_n\|_{L^{\frac{2^*}{2}(A+1)}(\Omega)}\le\mathscr{S}\,\|f\|_{L^m(\Omega)}. \tag{17}$$
If $N=2$, taking into account that 2^* can be chosen arbitrarily large, we deduce that m can be chosen any number bigger than one in order to have again (17).

Thus, we proved that
$$\text{the sequence } \{u_n\} \text{ is bounded in } L^{\frac{2^*}{2}(A+1)}(\Omega). \tag{18}$$
Using this result in (16) yields that
$$\text{the sequence } \{|u_n|^{\frac{A-1}{2}}u_n\} \text{ is bounded in } W_0^{1,2}(\Omega). \tag{19}$$
We now use $T_1(u_n)$ as test function in (14) and we have (thanks to Young's inequality)
$$\begin{aligned}
&\alpha\int_\Omega|\nabla T_1(u_n)|^2\\
&\le\int_\Omega|u_n||E(x)||\nabla T_1(u_n)|+A\int_\Omega|E(x)||\nabla u_n|+\int_\Omega|f(x)|\\
&\le(A+1)\int_\Omega|E(x)||\nabla T_1(u_n)|+A\int_\Omega|E(x)||\nabla T_1(u_n)|+\frac{2A}{A+1}\int_\Omega|E(x)||\nabla(|u_n|^{\frac{A-1}{2}}u_n)|+\int_\Omega|f(x)|\\
&\le\frac{\alpha}{2}\int_\Omega|\nabla T_1(u_n)|^2+\int_\Omega|f(x)|+C\left(\int_\Omega|E(x)|^2+\int_\Omega\left|\nabla(|u_n|^{\frac{A-1}{2}}u_n)\right|^2\right)
\end{aligned}$$

which proves that the sequence $\{T_1(u_n)\}$ is bounded in $W_0^{1,2}(\Omega)$. Combined with (19), we get that

$$\text{the sequence } \{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega). \tag{20}$$

The reflexivity of $W_0^{1,2}(\Omega)$ and Rellich theorem then imply the existence of a subsequence $\{u_{n_j}\}$ and a function u in $W_0^{1,2}(\Omega)$, with $|u|^{\frac{A-1}{2}} u$ in $W_0^{1,2}(\Omega)$, such that

$$\begin{cases} u_{n_j} \rightharpoonup u & \text{in } W_0^{1,2}(\Omega) \\ |u_{n_j}|^{\frac{A-1}{2}} u_{n_j} \rightharpoonup |u|^{\frac{A-1}{2}} u & \text{in } W_0^{1,2}(\Omega), \\ u_{n_j} \to u & \text{in } L^\rho(\Omega),\ 1 \le \rho < \dfrac{2^*}{2}(A+1). \end{cases} \tag{21}$$

Note that the boundedness of $\{u_n\}$ in $W_0^{1,2}(\Omega)$ implies that $\|\frac{1}{n}\nabla u_n\|_{(L^2(\Omega))^N} \to 0$ which implies (up to a subsequence) that $\frac{1}{n}\nabla u_n(x) \to 0$ a.e. in Ω. Thus we can pass to the limit in (14) to prove that u is a weak solution of (2); that is u belongs to $W_0^{1,2}(\Omega)$ and is such that

$$\int_\Omega M(x)\nabla u\cdot\nabla v - \int_\Omega u\,[E(x)\cdot\nabla v] + A\int_\Omega [E(x)\cdot\nabla u]\,v = \int_\Omega f\,v, \qquad \forall\, v \in C_0^1(\Omega). \tag{22}$$

Step 2. In this step, we assume $0 < A < 1$ and we will prove the existence of infinite energy solutions.

Since $0 < A < 1$, we need to modify our test function, and use $[(h+|u_n|)^A - h^A]\operatorname{sign}(u_n)$, with $h > 0$. Then we have

$$\begin{aligned} A\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,(h+|u_n|)^{A-1} - A\int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\,\frac{E_n(x)}{1+\frac{1}{n}|\nabla u_n|}\cdot\nabla u_n (h+|u_n|)^{A-1} \\ + A\int_\Omega \frac{1}{1+\frac{1}{n}|u_n|}\,\frac{E_n(x)}{1+\frac{1}{n}|\nabla u_n|}\cdot\nabla u_n\,[(h+|u_n|)^A - h^A]\operatorname{sign}(u_n) \le \int_\Omega |f|(h+|u_n|)^A. \end{aligned}$$

Since

$$(h+|u_n|)^A\operatorname{sign}(u_n) = (h+|u_n|)^{A-1}\big(h\operatorname{sign}(u_n) + u_n\big),$$

the above identity implies that

$$\begin{aligned} A\int_\Omega M(x)\nabla u_n\cdot\nabla u_n\,(h+|u_n|)^{A-1} + A\int_\Omega \frac{1}{1+\frac{1}{n}|u_n|}\,\frac{E_n}{1+\frac{1}{n}|\nabla u_n|}\cdot\nabla u_n\,[(h+|u_n|)^{A-1}h - h^A]\operatorname{sign}(u_n) \\ \le \int_\Omega |f(x)|(h+|u_n|)^A. \end{aligned}$$

Since $0 < A < 1$, one has that

$$(h+|u_n|)^{A-1}h \le h^A,$$

so that we can use Lebesgue theorem to pass to the limit as h tends to zero in the second term. Using also the monotone convergence theorem in the first one and the Lebesgue theorem in the third one, we get

$$A\int_\Omega \frac{M(x)\nabla u_n\cdot\nabla u_n}{|u_n|^{1-A}} \le \int_\Omega |f(x)||u_n|^A. \tag{23}$$

Thus, the inequalities in (16) still hold. As above, this proves (18) and (19). For $1 \le r' < 2$, Hölder's inequality also gives

$$\int_\Omega |\nabla u_n|^{r'} = \int_\Omega \frac{|\nabla u_n|^{r'}}{|u_n|^{\frac{r'(1-A)}{2}}}|u_n|^{\frac{r'(1-A)}{2}} \le \left(\int_\Omega \frac{|\nabla u_n|^2}{|u_n|^{1-A}}\right)^{\frac{r'}{2}} \left(\int_\Omega |u_n|^{\frac{r'(1-A)}{2-r'}}\right)^{2-\frac{r'}{2}}.$$

Since $\frac{r'(1-A)}{2-r'} = \frac{2^*}{2}(1+A)$, in the case $N \geq 3$, from (18) and (19), it follows that the sequence $\{u_n\}$ is bounded in $W_0^{1,r'}(\Omega)$. The same result is true in the case $N = 2$ if we define $2^* > 2$ (recall that $1 + A < r' < 2$ by (5)) by

$$2^* = 2\frac{r'(1-A)}{(2-r')(1+A)}.$$

Thus it is possible to pass to the limit as in the first step to get that u is a solution of (2). □

We now prove that, under the assumption that $f(x) \geq 0$, the solution u is not only positive, but cannot be zero in a set of positive measure.

Proposition 4. *If $f(x) \geq 0$, then $u(x) \geq 0$. Moreover, if $f(x)$ is not identically zero, then $u(x)$ can be zero at most in a set of zero measure.*

Proof. We give the proof in the case $A \geq 1$; the case $0 < A < 1$ can be proved modifying the test function, as in Step 2 of the proof of Theorem 1. Choosing $v = |u_n|^{A-1} u_n^-$ as test function in (14) we obtain identity (15) with u_n^- instead of u_n. Thus, one can cancel two equal terms to obtain that

$$-A\int_\Omega M(x)|u_n|^{A-1}\nabla u_n^- \cdot \nabla u_n^- = A\int_\Omega M(x)|u_n|^{A-1}\nabla u_n \cdot \nabla u_n^- = \int_\Omega f_n(x)\, u_n^- |u_n|^{A-1}.$$

Since the right hand side is positive, and the left hand side is negative, one has that $u_n^- = 0$, so that $u_n \geq 0$. Recalling that u is the limit of the sequence $\{u_n\}$, we have proved that $u(x) \geq 0$.

The second statement can be proved exactly as in [5, Theorem 3.1 and Theorem 4.1]. □

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