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On the factorised subgroups of products of cyclic and non-cyclic finite p-groups

Sur les sous-groupes factorisés des produits de p-groupes finis cycliques et non cycliques

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Abstract. Let p be an odd prime and let $G = AB$ be a finite p-group that is the product of a cyclic subgroup A and a non-cyclic subgroup B. Suppose in addition that the nilpotency class of B is less than $\frac{p}{2}$. We denote by $\mho_i(B)$ the subgroup of B generated by the p^i-th powers of elements of B, that is $\mho_i(B) = \langle b^{p^i} \mid b \in B\rangle$. In this article we show that, for all values of i, the set $A\mho_i(B)$ is a subgroup of G. We also present some applications of this result.

Résumé. Soient p un nombre premier impair et $G = AB$ un p-groupe fini qui est le produit des sous-groupes A et B, tels que A soit un sous-groupe cyclique et B soit un sous-groupe non cyclique. Supposons également que la classe de nilpotence de B soit inférieure à $\frac{p}{2}$. On note $\mho_i(B)$ le sous-groupe de B engendré par les puissances p^i des éléments de B, alors $\mho_i(B) = \langle b^{p^i} \mid b \in B\rangle$. Dans cet article nous montrons que, pour chaque valeur du nombre i, l'ensemble $A\mho_i(B)$ est sous-groupe du groupe G. Nous présentons également quelques applications de ce résultat.

Keywords. factorised groups, products of groups, finite p-groups.

Mots-clés. groupes factorisés, produit de groupes, p-groupes finis.

2020 Mathematics Subject Classification. 20D40, 20D15.

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1. Introduction

We say that the group G is the product of the subgroups A and B if G is equal to the set product of A and B, that is $G = \{ab \mid a \in A, b \in B\}$. Such groups are also referred to as factorised groups. For a factorised group $G = AB$, is seems natural to ask whether we can identify subgroups of the "factors" A and B whose set product with each other forms a subgroup of G. By a result from elementary Group Theory (see, for instance, [4, I 2.12 Hilfssatz]), this is equivalent to determining subgroups $A_1 \leqslant A$ and $B_1 \leqslant B$ such that their set products satisfy $A_1B_1 = B_1A_1$. Despite an extensive literature on factorised groups, as documented in Amberg, Franciosi and de Giovanni [1] and Ballester-Bolinches, Esteban-Romero and Assad [2], results in this direction remain scarce, even when both factors are abelian. The most notable result goes back to that of Huppert (see [3, Satz 3] or [2, Corollary 3.1.9]), which states that if the finite p-group $G = AB$ is the product of two cyclic subgroups A and B, then G is the *totally permutable* product of A and

B, that is $A_1B_1 \leqslant G$ for each $A_1 \leqslant A$ and $B_1 \leqslant B$. Since A and B are cyclic p-groups, this can be restated as $\Omega_s(A)\Omega_t(B) \leqslant G$ for all values of s and t, where the characteristic subgroup $\Omega_i(W)$ of the finite p-group W is defined by $\Omega_i(W) = \langle w \in W \mid w^{p^i} = 1\rangle$. In general, we cannot expect that $G = AB$ will be a totally permutable product if either of the subgroups A or B is non-cyclic. However, if the prime p is odd, then in the case where $A \cap B = 1$, A is cyclic and the nilpotency class $c(B)$ of B satisfies $c(B) < \frac{p}{2}$, it has been shown in [5, Theorem 2.6 (i)] that $A\Omega_i(B) \leqslant G$ for all values of i. Since A is cyclic, it is then straightforward to deduce that $\Omega_s(A)\Omega_t(B) \leqslant G$, for all values of s and t. This can be viewed as a partial analogue to Huppert's result for products of cyclic subgroups. It is notable that Example 2.8 of [5] further shows that $A\Omega_i(B)$ is not always a subgroup of G when $A \cap B \neq 1$. This places a limit on the extent to which Huppert's result can be directly generalised.

In this paper we present some results that can be considered as dual to those contained in [5]. Our main result is Theorem 9, which states that if $G = AB$ is a finite p-group for subgroups A and B such that A is cyclic and $c(B) < \frac{p}{2}$ then, for all i, $A\mho_i(B) \leqslant G$. Here $\mho_i(B)$ denotes the characteristic subgroup of B generated by the p^i-th powers of elements of B, that is $\mho_i(B) = \langle b^{p^i} \mid b \in B\rangle$. We apply Theorem 9 in Theorem 10 to provide results that are dual to [5, Theorems 2.6 and 2.9]. This leads to a new derivation of established results concerning the structure of products of cyclic and non-cyclic finite p-groups (see [5, Theorems 2.9 and 4.1]). Namely we show in Corollary 11 that if p is an odd prime and $G = AB$ is a finite p-group, where A is a cyclic subgroup and B is a subgroup such that $c(B) < \frac{p}{2}$ and $\exp(B) = p^k$, then $\Omega_k(A)B \trianglelefteq G$ and $d(G) \leq 1 + k + d(B)$.

We denote the n^{th} term of the derived series of a group G by $G^{(n)}$. Thus $G^{(0)} = G$, $G^{(1)} = G'$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geq 1$. We denote the derived length of a soluble group G by $d(G)$. The i^{th} term of the lower (or descending) central series of G will be denoted by $K_i(G)$, that is $K_1(G) = G$, $K_2(G) = G'$ and $K_{i+1}(G) = [K_i(G), G]$ for $i \geq 2$. If G is nilpotent then $c(G)$ will denote the class of G. We note in particular that if $c(G) < s$, then $K_s(G) = 1$. The normal closure of the subgroup U in G is denoted by U^G, so that $U^G = \langle U^g \mid g \in G\rangle$. We remark that if $N \trianglelefteq G$, then $(UN/N)^{G/N} = U^G N/N$. We finally denote the cyclic group of order p^n by C_{p^n}.

2. Results and Proofs

We begin with some elementary lemmas that will find application in the proofs of Theorems 9 and 10.

Lemma 1. *Let G be a finite p-group and let $N \trianglelefteq G$. Let U be a subgroup of G. Then $\mho_i(UN/N) = \mho_i(U)N/N$.*

Proof. By definition we have

$$\mho_i(UN/N) = \langle (uN)^{p^i} \mid u \in U\rangle N/N = \langle u^{p^i} N \mid u \in U\rangle N/N = \langle u^{p^i} \mid u \in U\rangle N/N.$$

It then follows that $\mho_i(UN/N) = \mho_i(U)N/N$. □

Lemma 2. *Let G be a group such that $G = AB$, where A and B are subgroups of G. Suppose that $B_1 \trianglelefteq B$ is such that $AB_1 \leqslant G$. Then $B_1^G = B_1^{AB_1} = A_1B_1$, where $A_1 = A \cap B_1^G$.*

Proof. Since $B_1 \trianglelefteq B$, we see that

$$B_1^G = \langle B_1^g \mid g \in G\rangle = \langle B_1^{ba} \mid b \in B,\ a \in A\rangle = \langle B_1^a \mid a \in A\rangle \leqslant B_1^{AB_1} \leqslant AB_1.$$

But $B_1 \leqslant B_1^{AB_1} \leqslant B_1^G$, so we have

$$B_1^G = B_1^{AB_1} = (A \cap B_1^G)B_1 = A_1B_1,$$

where $A_1 = A \cap B_1^G$. □

Lemma 3. *Let G be a group and let $N \trianglelefteq G$. Suppose that H and K are subgroups of G with $N \leqslant H \cap K$ such that $(H/N)(K/N) \leqslant G/N$. Then $HK \leqslant G$. Moreover, if $(H/N)(K/N) \trianglelefteq G/N$, then $HK \trianglelefteq G$.*

Proof. We let $h \in H$ and $k \in K$. Since $(H/N)(K/N) \leqslant G/N$, we have $(H/N)(K/N) = (K/N)(H/N)$. Hence there exist $h_1 \in H$ and $k_1 \in K$ such that $(hN)(kN) = (k_1N)(h_1N)$. Equivalently, we see that $hkN = k_1h_1N$. Thus there exists $x \in N$ such that $hk = k_1h_1x$. But $N \leqslant H \cap K$, so $h_1x \in H$. It follows that

$$hk = k_1(h_1x) \in KH.$$

Hence $HK \subseteq KH$. Since $(K/N)(H/N) = (H/N)(K/N)$, we similarly see that $KH \subseteq HK$. We conclude that $HK = KH$, so $HK \leqslant G$.

We now make the additional assumption that $(H/N)(K/N) \trianglelefteq G/N$. We let $h \in H$ and $k \in K$. If $g \in G$ then, by normality, we have $((hN)(kN))^{gN} = (hkN)^{gN} = (hk)^g N \in (H/N)(K/N)$. Hence there exist $h_1 \in H$ and $k_1 \in K$ such that $(hk)^g N = (h_1N)(k_1N) = h_1k_1N$. It follows that there exists $x \in N \leqslant H \cap K$ such that $(hk)^g = h_1k_1x$. Since $x \in K$, we have $k_1x \in K$, so $(hk)^g = h_1(k_1x) \in HK$. We thus conclude that $HK \trianglelefteq G$. □

Our next lemma is a consequence of the Hall-Petrescu identity (see [4, III 9.4 Satz]).

Lemma 4. *Let p be a prime and let G be a finite p-group such that* $\mathrm{c}(G) < p$. *Suppose, in addition, that* $\exp(G') \leq p$. *Then for $g_1, g_2 \in G$, we have $(g_1g_2)^p = g_1^p g_2^p$.*

Proof. Since $\mathrm{c}(G) < p$, we have $K_p(G) = 1$. We further have $K_i(G) \leqslant G'$ for $i \geq 2$. Hence $\exp(K_i(G)) \leq p$ for $i \geq 2$. By the Hall-Petrescu identity, there exist $c_2, \dots, c_p$, with $c_s \in K_s(G)$ for $s = 2, \dots, p$, such that

$$(g_1g_2)^p = g_1^p g_2^p c_2^{\binom{p}{2}} \cdots c_{p-1}^{\binom{p}{p-1}} c_p.$$

We note that $c_p \in K_p(G) = 1$. Since $\exp(K_s(G)) \leq p$ for $s \geq 2$, we further see that $c_s^p = 1$ for $s \geq 2$. But p is a divisor of $\binom{p}{s}$ for $s = 2, \dots, p-1$. Hence $c_s^{\binom{p}{s}} = 1$ for $s = 2, \dots, p-1$. It follows that $(g_1g_2)^p = g_1^p g_2^p$. □

Corollary 5. *Let p be an odd prime and let G be a finite p-group such that* $\mathrm{c}(G) < p$ *and* $\exp(G) = p^2$. *Let $y \in G$ be such that* $\mathrm{o}(y) = p^2$ *and suppose that there exists $W \leqslant G$ such that $|G : W| = p$ and* $\exp(W) = p$. *Then* $\mho_1(G) = \langle y^p \rangle$.

Proof. By definition, we have $\langle y^p \rangle \leqslant \mho_1(G)$. Since $\exp(W) = p$, we see that $y \notin W$. Now $|G : W| = p$ so, by comparison of orders, we have $G = W\langle y \rangle$. In addition, we have $W \trianglelefteq G$ and $G/W \cong C_p$. Hence $G' \leqslant W$, so $\exp(G') \leq p$. We let $g \in G$. Then there exist a suitable value α and an element $w \in W$ such that $g = wy^\alpha$. By Lemma 4, we have $g^p = (wy^\alpha)^p = w^p(y^\alpha)^p = w^p(y^p)^\alpha$. But $\exp(W) = p$, so $w^p = 1$. Hence $g^p = (y^p)^\alpha \in \langle y^p \rangle$, so $\mho_1(G) \leqslant \langle y^p \rangle$. □

The following theorem is a restatement of results of Philip Hall concerning regular p-groups (see [4, III 10.2 Satz and 10.5 Hauptsatz]).

Theorem 6. *Let p be an odd prime and let G be a finite p-group such that* $\mathrm{c}(G) < p$. *Then, for all i,*

(i) $\Omega_i(G) = \{g \in G \mid g^{p^i} = 1\}$. *In particular* $\exp(\Omega_i(G)) \leq p^i$;

(ii) $\mho_i(G) = \{g^{p^i} \mid g \in G\}$.

We next prove a consequence of Theorem 6.

Corollary 7. *Let p be an odd prime and let G be a finite p-group such that* $\mathrm{c}(G) < p$. *Then, for all i, we have* $\mho_{i+1}(G) = \mho_1(\mho_i(G))$.

Proof. By Theorem 6 (ii), we see that $\mho_{i+1}(G) = \{g^{p^{i+1}} \mid g \in G\} = \{(g^{p^i})^p \mid g \in G\} = \{u^p \mid u \in \{g^{p^i} \mid g \in G\}\} = \{u^p \mid u \in \mho_i(G)\} = \mho_1(\mho_i(G))$. □

We make extensive use of the next result, which deals with p-groups that are the product of a cyclic subgroup and a subgroup that has exponent p and class less than $\frac{p}{2}$. The first part is a restatement of [5, Lemma 2.2].

Lemma 8. *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic,* $\exp(B) = p$ *and* $c(B) < \frac{p}{2}$. *Then:*

(i) $\Omega_1(A)B \trianglelefteq G$;

(ii) $\exp(B^G) = p$.

Proof. We note that the proof of (i) is given in [5, Lemma 2.2]. For (ii), we see that the result is trivial if $B \trianglelefteq G$. Letting $A = \langle x\rangle$, we can thus assume that $B^x \neq B$. In particular A is non-trivial, so $\Omega_1(A) \cong C_p$. By (i), we have $\Omega_1(A)B \trianglelefteq G$. Hence, by comparison of orders, we have $B^G = \Omega_1(A)B = BB^x$. We further see that $|B^G : B| = |B^G : B^x| = |\Omega_1(A)| = p$. In particular, B and B^x are normal in B^G. It follows that $c(B^G) \le c(B)+c(B^x) < \frac{p}{2}+\frac{p}{2} = p$. Now $B^G = BB^x$, so $B^G = \Omega_1(B^G)$. By Theorem 6, we then have $\exp(B^G) \le p$. But $\exp(B) = p$, so we conclude that $\exp(B^G) = p$. □

We now proceed to our main result.

Theorem 9. *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic and* $c(B) < \frac{p}{2}$. *Then, for all i, we have $A\mho_i(B) \leqslant G$.*

Proof. The bulk of our proof is devoted to showing that $A\mho_1(B) \leqslant G$. For this, we use induction on $|G|$. If $G = A$, then the result is trivial. Hence we can assume that A is a proper subgroup of G. We can further assume that $B \ntrianglelefteq G$, as otherwise $\mho_1(B) \trianglelefteq G$ and we trivially have $A\mho_1(B) \leqslant G$. Since $|G : B| \le p$ is then excluded, we have $|A| \ge p^2$. In particular, we see that $\Omega_1(A)$ is a proper subgroup of A with $\Omega_1(A) \cong C_p$.

By a result of Morigi (see [6, Lemma 1] or [2, Lemma 3.3.8]), there exists a non-trivial normal subgroup $W \trianglelefteq G$ such that either $W \leqslant A$ or $W \leqslant B$. Since G is a finite p-group, we then have either $A\cap Z(G) \neq 1$ or $B\cap Z(G) \neq 1$. We assume first that $A\cap Z(G) \neq 1$. By minimality, it follows that $\Omega_1(A) \leqslant Z(G)$. By induction, we see that $(A/\Omega_1(A))\mho_1(B\Omega_1(A)/\Omega_1(A)) \leqslant G/\Omega_1(A)$. By Lemma 1, we further have $\mho_1(B\Omega_1(A)/\Omega_1(A)) = \mho_1(B)\Omega_1(A)/\Omega_1(A)$. Hence

$$A/\Omega_1(A)(\mho_1(B)\Omega_1(A)/\Omega_1(A)) \leqslant G/\Omega_1(A).$$

We now apply Lemma 3 to see that $A\mho_1(B)\Omega_1(A) \leqslant G$. Since $\Omega_1(A) \leqslant Z(G)$, we have

$$A\mho_1(B)\Omega_1(A) = A\Omega_1(A)\mho_1(B) = A\mho_1(B),$$

so $A\mho_1(B) \leqslant G$. We can therefore assume that $A\cap Z(G) = 1$ and thus $B\cap Z(G) \neq 1$.

We let $z \in B\cap Z(G)$ be such that $\langle z\rangle \cong C_p$. By induction, we have $(A\langle z\rangle/\langle z\rangle)(\mho_1(B/\langle z\rangle) \leqslant G/\langle z\rangle$. By Lemma 1, we see that $\mho_1(B/\langle z\rangle) = \mho_1(B\langle z\rangle/\langle z\rangle) = \mho_1(B)\langle z\rangle/\langle z\rangle$. Thus $(A\langle z\rangle/\langle z\rangle)(\mho_1(B)\langle z\rangle/\langle z\rangle) \leqslant G/\langle z\rangle$. By Lemma 3, we then have $A\langle z\rangle\mho_1(B)\langle z\rangle \leqslant G$. But $A\langle z\rangle\mho_1(B)\langle z\rangle = A\mho_1(B)\langle z\rangle$. Hence

$$A\mho_1(B)\langle z\rangle \leqslant G.$$

Since A is a proper subgroup of G, we let M be a maximal subgroup of G such that $A \leqslant M$. Then $M = A(B\cap M)$. We let $B_1 = B\cap M$, so that $M = AB_1$. Since $A \leqslant M$, we have $A\cap B = A\cap B\cap M = A\cap B_1$. Hence

$$|G| = \frac{|A||B|}{|A\cap B|} = p|M| = p\frac{|A||B_1|}{|A\cap B_1|} = p\frac{|A||B_1|}{|A\cap B|}.$$

It follows that

$$|B : B_1| = \frac{|B|}{|B_1|} = p.$$

We consider the case where $\mho_1(B_1) \neq 1$. By induction, we have $A\mho_1(B_1) \leqslant AB_1 \leqslant G$. Now $\mho_1(B_1)$ is characteristic in B_1 and $|B : B_1| = p$, so $B_1 \trianglelefteq B$. Hence $\mho_1(B_1) \trianglelefteq B$. Applying Lemma 2, we have $\mho_1(B_1)^G = (A\cap\mho_1(B_1)^G)\mho_1(B_1)$. We let $N = \mho_1(B_1)^G$. Since $\mho_1(B_1) \neq 1$, we see that N is a non-trivial normal subgroup of G. By our induction hypothesis and Lemma 1, we then have

$$(AN/N)\mho_1(BN/N) = (AN/N)(\mho_1(B)N/N) \leqslant G/N.$$

Hence, by Lemma 3, we have $AN\mho_1(B)N = AN\mho_1(B) \leqslant G$. But $N = \mho_1(B_1)^G = (A\cap\mho_1(B_1)^G)\mho_1(B_1)$, so

$$AN\mho_1(B) = A(A\cap\mho_1(B_1)^G)\mho_1(B_1)\mho_1(B) = A\mho_1(B).$$

Thus, in the case where $\mho_1(B_1) \neq 1$, we conclude that $A\mho_1(B) \leqslant G$.

We now assume that $\mho_1(B_1) = 1$. Thus $\exp(B_1) \le p$. Since $|B : B_1| = p$, it follows that $\exp(B) \le p^2$. If $\exp(B) = p$, then $\mho_1(B) = 1$ and the result is trivial. Hence we can assume that there exists $y \in B$ such that $o(y) = p^2$. But $|B : B_1| = p$ and $c(B) < \frac{p}{2} < p$, so we can apply Corollary 5 to see that $\mho_1(B) = \langle y^p\rangle \cong C_p$. In particular, we have $\mho_1(B) \leqslant Z(B)$. From the above, there exists $z \in B\cap Z(G)$ with $\langle z\rangle \cong C_p$ such that $A\mho_1(B)\langle z\rangle \leqslant G$. But $\mho_1(B) \leqslant Z(B)$, so $\mho_1(B)\langle z\rangle \leqslant Z(B)$. Hence we can apply Lemma 2 to see that $(\mho_1(B)\langle z\rangle)^G = (\mho_1(B)\langle z\rangle)^{A\mho_1(B)\langle z\rangle}$. Since $\mho_1(B) \cong \langle z\rangle \cong C_p$, we see that $\mho_1(B)\langle z\rangle$ is elementary abelian. In particular, we have $\exp(\mho_1(B)\langle z\rangle) = p$ and $c(\mho_1(B)\langle z\rangle) = 1 < \frac{p}{2}$. We can thus apply Lemma 8 (i) to see that $\Omega_1(A)\mho_1(B)\langle z\rangle \trianglelefteq A\mho_1(B)\langle z\rangle$. Hence

$$(\mho_1(B)\langle z\rangle)^G = (\mho_1(B)\langle z\rangle)^{A\mho_1(B)\langle z\rangle} \leqslant \Omega_1(A)\mho_1(B)\langle z\rangle.$$

We can assume that B is a proper subgroup of G, since otherwise $\mho_1(B) \trianglelefteq G$ and it trivially follows that $A\mho_1(B) \leqslant G$. We let M_1 be a maximal subgroup of G such that $B \leqslant M_1$. Then $|G : M_1| = p$ and $M_1 = (A\cap M_1)B$. We let $A_1 = A\cap M_1$. As above, we have $|A : A_1| = p$. Since $|A| \ge p^2$, we see that $\Omega_1(A) \leqslant A_1$. Hence $\Omega_1(A) = \Omega_1(A_1) \cong C_p$. By induction, we have $A_1\mho_1(B) \leqslant A_1B \leqslant G$. We can thus apply Lemma 2 and Lemma 8 (i) to see that

$$\mho_1(B)^{A_1B} = \mho_1(B)^{A_1\mho_1(B)} \leqslant \Omega_1(A)\mho_1(B) \trianglelefteq A_1\mho_1(B).$$

Bearing in mind that $\Omega_1(A) \cong C_p$, we see by comparison of orders that either $\mho_1(B)^{A_1B} = \Omega_1(A)\mho_1(B)$ or $\mho_1(B)^{A_1B} = \mho_1(B)$.

If $\mho_1(B)^{A_1B} = \Omega_1(A)\mho_1(B)$, then $\Omega_1(A)\mho_1(B) \trianglelefteq A_1B$. In particular $\Omega_1(A)\mho_1(B)B = \Omega_1(A)B \leqslant A_1B$, so $\Omega_1(A)B$ is a subgroup of G. We can then apply Lemma 2 to see that

$$\Omega_1(A)^G = \Omega_1(A)^{\Omega_1(A)B} \leqslant \Omega_1(A)\mho_1(B) \trianglelefteq A_1B.$$

But $\mho_1(B) \cong C_p$, so either $\Omega_1(A)^G = \Omega_1(A)$ or $\Omega_1(A)^G = \Omega_1(A)\mho_1(B)$. In the former case, we have $\Omega_1(A) \trianglelefteq G$. But $\Omega_1(A) \cong C_p$, so $1 \neq \Omega_1(A) \leqslant A\cap Z(G)$, which has been excluded. If $\Omega_1(A)^G = \Omega_1(A)\mho_1(B)$ then $\Omega_1(A)\mho_1(B) \trianglelefteq G$, so $A\mho_1(B) = A\Omega_1(A)\mho_1(B) \leqslant G$, and we are done. We can thus assume that

$$\mho_1(B)^{A_1B} = \mho_1(B) \trianglelefteq A_1B.$$

Since $\mho_1(B) \cong C_p$, it follows that $\mho_1(B) \leqslant Z(A_1B)$. But $A_1B = M_1 \trianglelefteq G$, so we further have $\mho_1(B)^G \leqslant Z(A_1B) \trianglelefteq G$. Now, for z as above, we have $\langle z\rangle \leqslant B\cap Z(G) \leqslant Z(A_1B)$. We therefore see that

$$(\mho_1(B)\langle z\rangle)^G = \mho_1(B)^G\langle z\rangle \leqslant Z(A_1B).$$

We recall from the above that $(\mho_1(B)\langle z\rangle)^G \leqslant \Omega_1(A)\mho_1(B)\langle z\rangle$. Since $\Omega_1(A) \cong C_p$, we have either $\Omega_1(A) \leqslant (\mho_1(B)\langle z\rangle)^G \leqslant Z(A_1B)$ or $\Omega_1(A)\cap(\mho_1(B)\langle z\rangle)^G = 1$. In the former case we see that B centralises $\Omega_1(A)$. But then $G = AB \leqslant C_G(\Omega_1(A))$, so $\Omega_1(A) \leqslant Z(G)$. Since this is excluded, we have $\Omega_1(A)\cap(\mho_1(B)\langle z\rangle)^G = 1$. By comparison of orders, it follows that

$$(\mho_1(B)\langle z\rangle)^G = \mho_1(B)\langle z\rangle \trianglelefteq G.$$

We let $N_1 = \mho_1(B)\langle z\rangle$ and have $N_1 \leqslant B$. Hence $G/N_1 = (AN_1/N_1)(B/N_1)$. If $A\cap N_1 \neq 1$ then, by minimality, we have $\Omega_1(A) \leqslant N_1 = (\mho_1(B)\langle z\rangle)^G$. But this has been excluded, so $A\cap N_1 = 1$. Hence $\Omega_1(AN_1/N_1) = \Omega_1(A)N_1/N_1 \cong \Omega_1(A) \cong C_p$. We further note that $c(B/N_1) \le c(B) < \frac{p}{2}$. Since

$\exp(B) = p^2$ and $\mho_1(B) \leqslant N_1$, we also see that $\exp(B/N_1) = p$. We can thus apply Lemma 8 (i) to see that

$$\Omega_1(AN_1/N_1)(B/N_1) = (\Omega_1(A)N_1/N_1)(B/N_1) \trianglelefteq G/N_1.$$

By Lemma 3, we then have $\Omega_1(A)N_1B = \Omega_1(A)B \trianglelefteq G$.

We let $A = \langle x \rangle$. We can assume that $B^x \neq B$ as otherwise $B \trianglelefteq G$, which has been excluded. Since $\Omega_1(A) \cong C_p$, we then have $\Omega_1(A) \cap B = 1$ and see that $|\Omega_1(A)B : B| = |\Omega_1(A)B : B^x| = |\Omega_1(A)| = p$. Thus both B and B^x are normal in $\Omega_1(A)B$. By comparison of orders, we further see that $\Omega_1(A)B = BB^x$. Since $\mathrm{c}(B^x) = \mathrm{c}(B) < \frac{p}{2}$, it follows that

$$\mathrm{c}(\Omega_1(A)B) = \mathrm{c}(BB^x) \leq \mathrm{c}(B) + \mathrm{c}(B^x) < \frac{p}{2} + \frac{p}{2} = p.$$

From the above there exists $B_1 \leqslant B$, with $|B : B_1| = p$ and $\exp(B_1) = p$, such that $AB_1 \leqslant G$. Since $\Omega_1(A) \cap B = 1$, we see by minimality that $A \cap B = 1$. In particular, we have $A \cap B_1 = 1$. Since A is non-trivial, it follows that B_1 is a proper subgroup of AB_1. Hence B_1 is a proper subgroup of $N_{AB_1}(B_1)$. But $N_{AB_1}(B_1) = (A \cap N_{AB_1}(B_1))B_1$, so $A \cap N_{AB_1}(B_1)$ is non-trivial. It follows that $\Omega_1(A) \leqslant A \cap N_{AB_1}(B_1)$. In particular, we see that $\Omega_1(A)B_1 \leqslant G$. Since $|B : B_1| = p$, we further see that

$$|\Omega_1(A)B : \Omega_1(A)B_1| = |B : B_1| = p.$$

Now $\exp(B_1) = p$ and $\Omega_1(A) \cong C_p$, so $\Omega_1(A)B_1$ is generated by elements of order p. Thus $\Omega_1(A)B_1 = \Omega_1(\Omega_1(A)B_1)$. But $\mathrm{c}(\Omega_1(A)B_1) \leq \mathrm{c}(\Omega_1(A)B) < p$. Hence we can apply Theorem 6 to see that

$$\exp(\Omega_1(A)B_1) = p.$$

As shown above, there exists $y \in B$ such that $\mathrm{o}(y) = p^2$ and $\mho_1(B) = \langle y^p \rangle$. It follows, in particular, that $\exp(\Omega_1(A)B) \geq p^2$. But $|\Omega_1(A)B : \Omega_1(A)B_1| = p$, so $\Omega_1(A)B/\Omega_1(A)B_1 \cong C_p$. Since $\exp(\Omega_1(A)B_1) = p$, we further see that $\exp(\Omega_1(A)B) \leq p^2$. Hence

$$\exp(\Omega_1(A)B) = p^2.$$

We can therefore apply Corollary 5 to see that

$$\mho_1(\Omega_1(A)B) = \langle y^p \rangle = \mho_1(B).$$

Since $\Omega_1(A)B \trianglelefteq G$, we then have $\mho_1(B) \trianglelefteq G$. We thus conclude that $A\mho_1(B) \leqslant G$.

Having established that $A\mho_1(B) \leqslant G$, we use induction on i to show that $A\mho_i(B) \leqslant G$ for all i. We assume that the result holds for $i = s$, where $s \geq 1$. Since $\mathrm{c}(B) < \frac{p}{2} < p$, we see, by Corollary 7, that $\mho_{s+1}(B) = \mho_1(\mho_s(B))$. By induction, we can assume that $A\mho_s(B) \leqslant G$. Applying the result we have proven for $i = 1$, we then have $A\mho_1(\mho_s(B)) \leqslant A\mho_s(B) \leqslant G$. Since $\mho_1(\mho_s(B)) = \mho_{s+1}(B)$, it follows that $A\mho_{s+1}(B) \leqslant G$. □

We use Theorem 9 to prove the following result, which can be considered as an analogue to [5, Theorem 2.6].

Theorem 10. *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic,* $\mathrm{c}(B) < \frac{p}{2}$ *and* $\exp(B) = p^k$*, where $k \geq 1$. Then, for all i such that* $1 \leq i \leq k$*, we have:*

(i) $A\mho_{k-i}(B) \leqslant G$;
(ii) $\mho_{k-i}(B)^G \leqslant \Omega_i(A)\mho_{k-i}(B) \leqslant G$;
(iii) $\exp(\mho_{k-i}(B)^G) = p^i$.

Proof. We see that (i) holds by Theorem 9. For (ii) and (iii), we first deal with the case where $i = 1$. Since $\mathrm{c}(B) < \frac{p}{2} < p$, we see by Theorem 6 that $\mho_{k-1}(B) = \{b^{p^{k-1}} \mid b \in B\}$. Since $\exp(B) = p^k$, it follows that $\exp(\mho_{k-1}(B)) = p$. By (i), we have $A\mho_{k-1}(B) \leqslant G$. Hence we can apply Lemma 2 to see that

$$\mho_{k-1}(B)^G = \mho_{k-1}(B)^{A\mho_{k-1}(B)}.$$

Now $c(\mho_{k-1}(B)) \le c(B) < \frac{p}{2}$. Hence, by Lemma 8 (i), we have $\Omega_1(A)\mho_{k-1}(B) \trianglelefteq A\mho_{k-1}(B)$. It follows that

$$\mho_{k-1}(B)^G = \mho_{k-1}(B)^{A\mho_{k-1}(B)} \leqslant \Omega_1(A)\mho_{k-1}(B).$$

Thus (ii) holds for $i = 1$. By Lemma 8(ii), we further see that $\exp(\mho_{k-1}(B)^G) = \exp(\mho_{k-1}(B)^{A\mho_{k-1}(B)}) = p$, so (iii) also holds for $i = 1$.

We now assume that $k \ge 2$ and further assume that (ii) and (iii) hold for $i = s$, where $1 \le s < k$. Thus $\mho_{k-s}(B)^G \leqslant \Omega_s(A)\mho_{k-s}(B)$ and $\exp(\mho_{k-s}(B)^G) = p^s$. By Lemma 2, we have $\mho_{k-s}(B)^G = \mho_{k-s}(B)^{A\mho_{k-s}(B)} = A_1\mho_{k-s}(B)$, where $A_1 = A \cap \mho_{k-s}(B)^G$. Since $\exp(A_1) \le \exp((\mho_{k-s}(B))^G) = p^s$, we see that $A_1 = \Omega_t(A)$ for some $t \le s$.

We let $N = \mho_{k-s}(B)^G$. Then $A \cap N = \Omega_t(A) \leqslant \Omega_s(A)$. If $\Omega_s(A) = A$, then $\Omega_{s+1}(A) = A$ and $\Omega_1(AN/N) \leqslant AN/N = \Omega_{s+1}(A)N/N$. If $\Omega_s(A)$ is a proper subgroup of A, then $\Omega_{s+1}(A) \cong C_{p^{s+1}}$. Since $\exp(N) = p^s$, we see that $\Omega_{s+1}(A) \not\leqslant N$. By minimality, we then have $\Omega_1(AN/N) \leqslant \Omega_{s+1}(A)N/N$. Thus, in every case, we have

$$\Omega_1(AN/N) \leqslant \Omega_{s+1}(A)N/N.$$

By Lemma 1, we have $\mho_{k-s-1}(BN/N) = \mho_{k-s-1}(B)N/N$. Since $c(B) \le \frac{p}{2} < p$ we see, by Corollary 7, that $\mho_1(\mho_{k-s-1}(B)) = \mho_{k-s}(B) \leqslant N$. Hence $\exp(\mho_{k-s-1}(B)N/N) \le p$. But, by our inductive assumption, we have $\exp(N) = \exp(\mho_{k-s}(B)^G) = p^s$. In addition, $\exp(B) = p^k$, so $\mho_{k-s-1}(B)$ contains elements of order p^{s+1}. Hence $\exp(\mho_{k-s-1}(B)N/N) = p$. Since $A\mho_{k-s-1}(B) \leqslant G$ and $N = \Omega_t(A)\mho_{k-s}(B) \leqslant A\mho_{k-s-1}(B)$, we see that

$$(AN/N)(\mho_{k-s-1}(B)N/N) = A\mho_{k-s-1}(B)/N \leqslant G/N.$$

But $\exp(\mho_{k-s-1}(B)N/N) = p$ and $c(\mho_{k-s-1}(B)N/N) \le c(B) < \frac{p}{2}$. Hence, we can apply Lemma 8 (i) to see that

$$\Omega_1(AN/N)(\mho_{k-s-1}(B)N/N) \trianglelefteq A\mho_{k-s-1}(B)/N.$$

By Lemma 8(ii), we further have $\exp((\mho_{k-s-1}(B)N/N)^{A\mho_{k-s-1}(B)/N}) = p$. We note that $(\mho_{k-s-1}(B)N/N)^{A\mho_{k-s-1}(B)/N} = \mho_{k-s-1}(B)^{A\mho_{k-s-1}(B)}N/N$. In addition, we see by Lemma 2 that $\mho_{k-s-1}(B)^{A\mho_{k-s-1}(B)} = \mho_{k-s-1}(B)^G$. It then follows that

$$\exp(\mho_{k-s-1}(B)^G N/N) = \exp((\mho_{k-s-1}(B)N/N)^{A\mho_{k-s-1}(B)/N}) = p.$$

We let $W/N = \Omega_1(AN/N)$. Then $N \leqslant W \leqslant AN$, so $W = (A \cap W)N$. From the above, we have $W \leqslant \Omega_{s+1}(A)N$. Since A is cyclic, we see that if $A \cap W \not\leqslant \Omega_{s+1}(A)$ then $\Omega_{s+1}(A)$ is a proper subgroup of $A \cap W$. It follows that $\Omega_{s+2}(A) \leqslant A \cap W$, where $\Omega_{s+2}(A) \cong C_{p^{s+2}}$. Hence $\Omega_{s+2}(A) \leqslant W \leqslant \Omega_{s+1}(A)N$, so $\Omega_{s+2}(A) = \Omega_{s+1}(A)(\Omega_{s+2}(A) \cap N)$. But, for t as above, we have $\Omega_{s+2}(A) \cap N \leqslant A \cap N \leqslant \Omega_t(A)$. Since $t \le s$, the contradiction $\Omega_{s+2}(A) \leqslant \Omega_{s+1}(A)\Omega_t(A) = \Omega_{s+1}(A) \cong C_{p^{s+1}}$ then arises. We can thus assume that $A \cap W = \Omega_m(A)$ for some m such that $t \le m \le s+1$. Hence

$$\Omega_1(AN/N)(\mho_{k-s-1}(B)N/N) = (\Omega_m(A)N/N)(\mho_{k-s-1}(B)N/N) \trianglelefteq A\mho_{k-s-1}(B)/N.$$

By Lemma 3, it follows that $\Omega_m(A)N\mho_{k-s-1}(B)N \trianglelefteq A\mho_{k-s-1}(B)$. But $N = \Omega_t(A)\mho_{k-s}(B)$ and $\Omega_m(A)N\mho_{k-s-1}(B)N = \Omega_m(A)N\mho_{k-s-1}(B)$. Hence

$$\Omega_m(A)\Omega_t(A)\mho_{k-s}(B)\mho_{k-s-1}(B) = \Omega_m(A)\mho_{k-s-1}(B) \trianglelefteq A\mho_{k-s-1}(B).$$

We can now apply Lemma 2 to see that

$$\mho_{k-s-1}(B)^G = \mho_{k-s-1}(B)^{A\mho_{k-s-1}(B)} \leqslant \Omega_m(A)\mho_{k-s-1}(B).$$

Since A is cyclic and $A\mho_{k-s-1}(B) \leqslant G$, we see that $\Omega_j(A)\mho_{k-s-1}(B) \leqslant G$ for all j. Now $m \le s+1$, so

$$\mho_{k-s-1}(B)^G \leqslant \Omega_m(A)\mho_{k-s-1}(B) \leqslant \Omega_{s+1}(A)\mho_{k-s-1}(B) \leqslant G.$$

We thus conclude that (ii) holds for $i = s+1$.

From the above, we have $\exp(\mho_{k-s-1}(B)^G N/N)) = p$. But $N = \mho_{k-s}(B)^G$ and, by our inductive assumption, we have $\exp(\mho_{k-s}(B)^G) = p^s$. Hence $\exp(\mho_{k-s-1}(B)^G) \le p^{s+1}$. Now $\exp(B) = p^k$ so there exists $b \in B$ such that $\mathrm{o}(b) = p^k$. Since $s+1 \le k$, we see that $\mathrm{o}(b^{p^{k-s-1}}) = p^{s+1}$. Thus $b^{p^{k-s-1}}$ is an element of order p^{s+1} in $\mho_{k-s-1}(B)$. Hence $\exp(\mho_{k-s-1}(B)^G) \ge p^{s+1}$.We conclude that $\exp(\mho_{k-s-1}(B)^G) = p^{s+1}$, so (iii) also holds for $i = s+1$. □

In our final result we use Theorem 10 to provide an alternative derivation of two results concerning the structure of products of cyclic p-groups with p-groups of class less than $\frac{p}{2}$ (see [5, Theorems 2.9 and 4.1]).

Corollary 11. *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic,* $\mathrm{c}(B) < \frac{p}{2}$ *and* $\exp(B) = p^k$*, where $k \ge 1$. Then:*

(i) $\Omega_k(A)B \trianglelefteq G$;

(ii) $\mathrm{d}(G) \le 1+k+\mathrm{d}(B)$.

Proof. We let $i = k$ in Theorem 10 (iii) and see that $\exp(B^G) = p^k$. Now $B \leqslant B^G$, so $B^G = (A \cap B^G)B$. We have $A \cap B^G = \Omega_t(A)$, for a suitable t. Since $\exp(B^G) = p^k$, we can assume that $t \le k$. Now $G/B^G = G/\Omega_t(A)B$ is isomorphic to a subgroup of A, so G/B^G is cyclic. Since $B^G = \Omega_t(A)B \leqslant \Omega_k(A)B$, we then see that $\Omega_k(A)B/B^G \trianglelefteq G$. It follows that $\Omega_k(A)B \trianglelefteq G$, so (i) is established.

For (ii), we note that $G/\Omega_k(A)B$ is isomorphic to a factor group of the cyclic group A. Hence $G' \leqslant \Omega_k(A)B$. Since A is cyclic, we see that $\Omega_1(A)B \leqslant \cdots \leqslant \Omega_k(A)B \leqslant G$. For $i = 1,\ldots,k$, we have $|\Omega_i(A)B : \Omega_{i-1}(A)B| \le |\Omega_i(A) : \Omega_{i-1}(A)| \le p$, so $\Omega_{i-1}(A)B \trianglelefteq \Omega_i(A)B$. We further see that $\Omega_i(A)B/\Omega_{i-1}(A)B$ is isomorphic to a factor group of the cyclic group $\Omega_i(A)/\Omega_{i-1}(A)$. Hence $(\Omega_i(A)B)' \leqslant \Omega_{i-1}(A)B$ for $i = 1,\ldots,k$, so $G^{(1+k)} \leqslant B$. It then follows that $G^{(1+k+\mathrm{d}(B))} = 1$, in accordance with (ii). □

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