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Norm inflation for the derivative nonlinear Schrödinger equation

Inflation de la norme pour l'équation de Schrödinger non linéaire dérivée

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Abstract. In this note, we study the ill-posedness problem for the derivative nonlinear Schrödinger equation (DNLS) in the one-dimensional setting. More precisely, by using a ternary-quinary tree expansion of the Duhamel formula we prove norm inflation in Sobolev spaces below the (scaling) critical regularity for the gauged DNLS. This ill-posedness result is sharp since DNLS is known to be globally well-posed in $L^2(\mathbb{R})$ [16]. The main novelty of our approach is to control the derivative loss from the cubic nonlinearity by the quintic nonlinearity with carefully chosen initial data.

Résumé. Dans cette note, nous étudions le caractère mal posé de l'équation de Schrödinger avec perte de dérivée dans la non-linéarité (DNLS) en une dimension d'espace. Plus précisément, en utilisant un développement ternaire-quinaire de la formule de Duhamel, nous prouvons l'inflation de la norme des solutions dans les espaces de Sobolev en dessous de la régularité critique pour l'équation DNLS gaugée. Ce résultat est optimal puisque l'équation DNLS est connue pour être globalement en régularité positive [16]. La principale nouveauté de notre approche est de contrôler la perte de dérivée de la non-linéarité cubique par la non-linéarité quintique avec des données initiales soigneusement choisies.

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1. Setup of the problem

1.1. *The derivative nonlinear Schrödinger equation*

We consider the derivative nonlinear Schrödinger equation (DNLS) defined on $\mathbb{R}$:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x\left(|u|^2 u\right) \\ u|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{R}^2, \tag{1}$$

where $u = u(t,x)$ is a complex-valued function. The equation (1) was derived in the plasma physics literature [24] and has been extensively studied from a theoretical perspective. It is known that (1) is completely integrable and admits infinitely many conservation laws [21]. See [22, 25] and reference therein for recent developments.

If $u(x,t)$ solves (1) on $\mathbb{R}$, then, for any $\lambda > 0$,

$$u^\lambda := \lambda^{\frac{1}{2}} u\left(\lambda^2 t, \lambda x\right)$$

also solves (1) with scaled initial data $u_0^\lambda := \lambda^{\frac{1}{2}} u_0(\lambda x)$. This scaling invariance heuristically suggests that the critical Sobolev regularity of DNLS (3) is given by $s_{\text{crit}} := 0$.

Therefore, it is natural to conjecture the following:

Conjecture 1. *The DNLS equation* (1) *is well-posed in* $H^s(\mathbb{R})$ *for* $s \geq 0$*, and ill-posed for* $s < 0$.

Regarding well-posedness, Conjecture 1 has seen some recent progress culminating in the breakthrough work [16] which proves global well-posedness of (1) in $L^2(\mathbb{R})$. We also mention the following works on the well-posedness theory for (1) [1, 17, 18, 22, 30–32]. On the other hand, Biagioni and Linares [3] showed a mild form of ill-posedness for (1): they showed that the data-to-solution map fails to be uniformly continuous (strictly) below $H^{\frac{1}{2}}(\mathbb{R})$.

In the study of (1), the gauge transform plays an important role. Define the nonlinear map $\mathcal{G} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ by

$$\mathcal{G} f(x) := e^{-i\int_{-\infty}^x |f(y)|^2 dy} f(x). \tag{2}$$

Then, a smooth function u satisfies (1) if $v(t) = \mathcal{G}(u(t))$ satisfies

$$\begin{cases} i\partial_t v + \partial_x^2 v = -i v^2 \partial_x \overline{v} - \frac{1}{2}|v|^4 v \\ v|_{t=0} = \phi, \end{cases} \tag{3}$$

with $\phi = \mathcal{G}(u_0)$. The inverse gauge transform $\mathcal{G}^{-1}$ is given by

$$\mathcal{G}^{-1} f(x) := e^{i\int_{-\infty}^x |f(y)|^2 dy} f(x).$$

Since one can show that both $\mathcal{G}$ and $\mathcal{G}^{-1}$ are continuous functions in relevant topologies (i.e. from $C\big([0,T]; L^2(\mathbb{R})^1\big)$ to itself, for any $T > 0$, see [19] for a proof in the periodic setting), one can transfer any well-posedness result for (1) in $L^2(\mathbb{R})$ to a well-posedness result for (3) in $L^2(\mathbb{R})$, and vice-versa.

The aim of this note is to study the ill-posedness problem of (3) in Sobolev spaces H^s. One way of showing ill-posedness is to show the discontinuity of the solution map, which can be done through *n*orm inflation. More precisely, given $s \in \mathbb{R}$, we say that (3) exhibits norm inflation in $H^s(\mathbb{R})$ if for any $\varepsilon > 0$ and $\phi \in H^s(\mathbb{R})$, there exists a solution v to (3) on $\mathbb{R}$ and $t \in (0, \varepsilon)$ such that

$$\| v(0) - \phi \|_{H^s(\mathbb{R})} < \varepsilon \quad \text{and} \quad \| v(t) \|_{H^s(\mathbb{R})} > \varepsilon^{-1}. \tag{4}$$

Note that if (3) exhibits norm inflation in $H^s(\mathbb{R})$, then it not possible to define a continuous data-to-solution map. We invite the reader to consult [5–8, 10–12, 15, 20, 23, 27–29, 33] and references therein for more information on the norm inflation phenomena.

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[1] Here, $\mathcal{G}$ (and $\mathcal{G}^{-1}$) is viewed as a transformation acting on time-dependent functions simply by writing $\mathcal{G}(u)(t) = \mathcal{G}(u(t))$.

The main purpose of this note is to show norm inflation of the gauged DNLS equation (3). Our main result is the following:

Theorem 2. *Suppose* $s < 0$*. Fix* $\phi \in H^s(\mathbb{R})$*. Then, given any* $\varepsilon > 0$*, there exist a global solution* v_ε *to the gauged equations* (3) *on* $\mathbb{R}$ *and* $t \in (0, \varepsilon)$ *such that* (4) *holds.*

To the best of our best knowledge, Theorem 2 is the first ill-posedness result of (3) in terms of discontinuity of the flow map. We prove Theorem 2 via a Fourier analytic method. In [20] Iwabuchi–Ogawa, showed norm inflation (4) with $\phi = 0$ for quadratic nonlinear Schrödinger equations. Building upon the work of Bejenaru and Tao [2], their approach is based on a Picard iteration scheme to show norm inflation using the high-to-low energy transfer in the second Picard iterate. It turns out that this method is widely applicable and works particularly well with power type nonlinearities [8, 15, 23, 27–29]; yielding norm inflation for almost optimal Sobolev exponents in most cases. Oh [27] removed the constraint $\phi = 0$ and proved norm inflation with general initial data for the cubic nonlinear Schrödinger equation by introducing a ternary tree expansion of the Duhamel formula.

The main difficulty when one tries to apply these methods to (3) comes from the presence of derivative in (3). Okamoto [29] pointed out that Iwabuchi-Ogawa's argument is applicable for some dispersive equations where a derivative appears in the nonlinearity. In particular, he proved norm inflation for the Kawahara equation:

$$\partial_t u - \partial_x^5 u + \partial_x \left(u^2\right) = 0,$$

in $H^s(\mathbb{R})$ with $s < -2$, which is sharp in the sense that the well-posedness is known in H^s when $s > -2$.

In [29], the strong dispersion of the Kawahara equation plays a crucial rule in absorbing the derivative loss along the Picard iteration. Unfortunately, the dispersion of (3) is not strong enough to handle the derivative loss. In order to achieve norm inflation below the critical scaling exponent $s < 0$, we need to further exploit the cubic-quintic structure of the nonlinearity. Since the derivative loss in (3) only appears in the cubic part of the nonlinearity, we observe that by carefully choosing our initial data, the cubic part can be controlled by the quintic component of the nonlinearity. Therefore, the problem then essentially reduces to showing norm inflation for the quintic nonlinear Schrödinger equation below $L^2(\mathbb{R})$, which is already known in the literature, see [23].

Another difficulty comes from the nonlinearity, which consists of a derivative cubic term and a non-derivative quintic term. This non-homogeneous structure makes the series expansion of a solution complicated, see (9) below. In running a Picard iteration scheme, we can approximate the solution v to (3) by multilinear terms depending on the initial data ϕ. However, in view of the ternary-quinary structure of the nonlinearity in (3), it becomes cumbersome to write out these terms "by hand"; see again (9). In order to track these multilinear expressions, we introduce a notion of ternary-quinary trees generalizing the ternary trees in [27]. We note that Kishimoto [23] had to deal with this issue with a polynomial nonlinearity (consisting of several terms) without any derivative loss.

1.2. *Further remarks*

We conclude this section with several remarks.

Remark 3. Since (1) is globally well-posed in $L^2(\mathbb{R})$ [16], norm inflation cannot occur in $H^s(\mathbb{R})$ for $s \geq 0$ in view of the continuity of the Gauge transform (2) from $C([0,T]; L^2(\mathbb{R}))$ to itself, for any $T > 0$. Therefore, Theorem 2 is sharp.

Remark 4. Regarding the periodic setting, Theorem 2 still holds, i.e. we can prove norm inflation for the gauged DNLS on $\mathbb{T}$ in $H^s(\mathbb{T})$ for $s<0$. (Note that the gauged equation (3) is modified on the torus, see [13, 18].) The proof is a minor modification of the real line case. We also note that in the setting of Fourier-Lebesgue spaces, (1) (posed on $\mathbb{T}$) was shown to be locally well-posed in [13] in the whole subcritical regime (i.e. in Fourier–Lebesgue spaces that scale like $H^s(\mathbb{T})$, for any $s>0$).

Remark 5. Using a ternary tree expansion as in [27], we can also prove norm inflation for (1) in $H^s(\mathcal{M})$ for $\mathcal{M}=\mathbb{R}$ or $\mathbb{T}$ for $s<-1$. The $-1\le s<0$ case in Conjecture 1 remains however open.

Remark 6. On the torus, another notion of probabilisitic criticality was developed in [14] giving rise to a probabilistic critical exponent s_p. More precisely, let $\{g_n\}_{n\in\mathbb{Z}}$ be an i.i.d. family of standard complex Gaussian random variables and define for any $s\in\mathbb{R}$, the function

$$u_0^s(\omega) := \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^{s+\frac{1}{2}}} e^{inx}.$$

Then the exponent s_p is defined to be the smallest $s\in\mathbb{R}$ such that the second Picard iterate of (1) (after some appropriate frequency truncation) with initial data given by u_0^s stays bounded in $H^s(\mathbb{T})$; see [14] for more details. A computation shows that for (1) (posed on $\mathbb{T}$), we have $s_p=s_c=0$. Hence, it is unlikely that any random data theory (see for instance [4, 14, 26]) would allow us to go beyond the $L^2(\mathbb{T})$ well-posedness threshold.

The rest of the paper is organized as follows. We introduce the notion of ternary-quinary trees and establish multilinear estimates in Section 2. In Section 3, we prove Theorem 2.

2. Preliminary analysis

2.1. *Power series expansion indexed by trees*

We define two Duhamel integral operators $\mathcal{J}$ and $\mathcal{K}$ by

$$\begin{aligned} \mathcal{J}[v_1,v_2,v_3](t) &:= -i\int_0^t S(t-t')v_1(t')v_2(t')\partial_x\overline{v}_3(t')\,dt', \\ \mathcal{K}[v_1,v_2,v_3,v_4,v_5](t) &:= -\frac{1}{2}\int_0^t S(t-t')v_1(t')\overline{v}_2(t')v_3(t')\overline{v}_4(t')v_5(t')\,dt', \end{aligned} \tag{5}$$

where $S(t):=e^{it\partial_x^2}$ denotes the linear propagator associated with (3). We also use the following shorthand notations:

$$\begin{aligned} \mathcal{J}^3[v] &:= \mathcal{J}[v,v,v], \\ \mathcal{K}^5[v] &:= \mathcal{K}[v,v,v,v,v]. \end{aligned} \tag{6}$$

In what follows, a solution v to (3) with $v|_{t=0}=\phi$ is a distribution which satisfies the Duhamel formulation

$$v(t) = S(t)\phi + \mathcal{J}^3[v](t) + \mathcal{K}^5[v](t). \tag{7}$$

It is expected that the following Picard iteration

$$P_0(\phi)=S(t)\phi \text{ and } P_j(\phi)=S(t)\phi+\mathcal{J}^3\left[P_{j-1}(\phi)\right]+\mathcal{K}^5\left[P_{j-1}(\phi)\right],\ j\in\mathbb{N},$$

converges to a solution v to (3) at least for short times. If this is the case, then v can be expressed as a series of multilinear terms in ϕ:

$$v(t) = \sum_{j\in 3(\mathbb{N}\cup\{0\})+5(\mathbb{N}\cup\{0\})}^{\infty} H_j[\phi](t), \tag{8}$$

where $H_j[\phi]$ consists of all homogeneous multilinear terms in ϕ of degree j. For instance,

$$\begin{aligned}
&H_0[\phi](t) := S(t)\phi;\\
&H_3[\phi](t) := \mathscr{J}^3[S(t)\phi];\\
&H_5[\phi](t) := \mathscr{K}^5[S(t)\phi];\\
&H_6[\phi](t) := \mathscr{K}^3\big[\mathscr{J}^3[S(t)\phi]\big];\\
&H_8[\phi](t) := \mathscr{K}^5\big[\mathscr{J}^3[S(t)\phi]\big] + \mathscr{J}^3\big[\mathscr{K}^5[S(t)\phi]\big];\\
&\qquad\qquad \cdots\cdots
\end{aligned} \tag{9}$$

Since the nonlinearity of (3) consists of both cubic and quintic nonlinearities, the series expansion (8) and (9) gets more involved as j gets larger. Inspired by [9, 27], we shall use trees to index the series (8). To this purpose, we introduce the following notion of ternary-quinary trees:

Definition 7.

(i) *Given a partially ordered set $\mathscr{T}$ with partial order $\le$, we say that $b \in \mathscr{T}$ with $b \le a$ and $b \ne a$ is a child of $a \in \mathscr{T}$, if $b \le c \le a$ implies either $c = a$ or $c = b$. If the latter condition holds, we also say that a is the parent of b.*

(ii) *A ternary-quinary tree $\mathscr{T}$ is a finite partially ordered set, satisfying the following properties:*
- *Let $a_1, a_2, a_3, a_4 \in \mathscr{T}$. If $a_4 \le a_2 \le a_1$ and $a_4 \le a_3 \le a_1$, then we have $a_2 \le a_3$ or $a_3 \le a_2$,*
- *A node $a \in \mathscr{T}$ is called terminal, if it has no child. A non-terminal node $a \in \mathscr{T}$ is a node with exactly* three *or* five *children,*
- *There exists a maximal element $r \in \mathscr{T}$ (called the root node) such that $a \le r$ for all $a \in \mathscr{T}$,*
- *$\mathscr{T}$ consists of the disjoint union of $\mathscr{T}^0$ and $\mathscr{T}^\infty$, where $\mathscr{T}^0$ and $\mathscr{T}^\infty$ denote the collections of non-terminal nodes and terminal nodes, respectively.*

Given a ternary-quinary tree $\mathscr{T}$, we denote by $\mathfrak{n}_3(\mathscr{T})$ (resp. $\mathfrak{n}_5(\mathscr{T})$) the number of non-terminal nodes which have three (resp. five) children.

We also denote the collection of ternary-quinary trees in the $(k,p)^{\text{th}}$ generation (i.e. with k parental nodes with three children and p parental nodes with five children) by $\mathbf{T}^{3,5}(k,p)$:

$$\mathbf{T}^{3,5}(k,p) := \big\{\mathscr{T} : \mathscr{T} \text{ is a tree with } (\mathfrak{n}_3(\mathscr{T}), \mathfrak{n}_5(\mathscr{T})) = (k,p)\big\}. \tag{10}$$

Note that the number $|\mathscr{T}|$ of nodes in a tree $\mathscr{T} \in \mathbf{T}^{3,5}(k,p)$ is $3k+5p+1$ for $k,p \in \mathbb{N}\cup\{0\}$. In particular, the number of non-terminal nodes is $|\mathscr{T}^0| = k+p$ and the number of terminal nodes is $|\mathscr{T}^\infty| = 2k+4p+1$.

Remark 8. The ternary-quinary trees defined in Definition 7 generalize the ternary trees introduced in [27] by allowing some parental nodes to have five children. It is easy to see that $\mathbf{T}^{3,5}(k,0)$ and $\mathbf{T}^{3,5}(0,p)$ consist of only ternary trees and only quinary trees, respectively.

Remark 9. Let $\mathscr{T} = \{a\}_{a\in\mathscr{T}}$ be a tree. Then, for every $a \in \mathscr{T}$, we denote by $\mathscr{T}_a$ the sub-tree whose root node is a. $\mathscr{T}_a^0$ and $\mathscr{T}_a^\infty$ are the associated sets of non-terminal and terminal nodes respectively. Let $r \in \mathscr{T}$ be the root node of $\mathscr{T}$, then

$$\mathscr{T} = \mathscr{T}_r.$$

And similarly, $\mathscr{T}^0 = \mathscr{T}_r^0$ and $\mathscr{T}^\infty = \mathscr{T}_r^\infty$.

We have the following bound on the number of trees in $\mathbf{T}^{3,5}(k,p)$:

Lemma 10. *Let* $\mathbf{T}^{3,5}(k,p)$ *be as in* (10). *Then, there exists* $C>0$ *such that*

$$\#\mathbf{T}^{3,5}(k,p) \le C^{k+p},$$

for all $(k,p)\in(\mathbb{N}\cup\{0\})^2$.

Proof. We observe that

$$\#\mathbf{T}^{3,5}(k,p) \le \#\mathbf{T}^{3,5}(0,k+p),$$

where $\mathbf{T}^{3,5}(0,k+p)$ consists of all quinary trees in the $(k+p)$-th generation. Then the bound follows from the same argument as [27, Lemma 2.3]. □

Next, we associate operators to trees in the following manner. Fix ϕ a function. Let $\mathcal{T}\in\mathbf{T}^{3,5}(k,p)$ for $(k,p)\in\mathbb{N}^2\cup\{(0,0)\}$ be a tree. Given functions $\phi_1,\cdots,\phi_{2k+4p+1}$, we formally associate $\Psi(\mathcal{T},\phi_1,\cdots,\phi_{2k+4p+1})$, a multilinear operator, by the following rules:

- Replace a non-terminal node by the Duhamel integral operator $\mathcal{J}$ (resp. $\mathcal{K}$) defined in (5) with its three (resp. five) children as arguments u_1,u_2 and u_3 (resp. u_1,u_2,u_3,u_4 and u_5).
- Replace a terminal node by the linear solution $S(t)\phi_j$, $j=1,\cdots,2k+4p+1$.

In the following, we set $\Psi_\phi(\mathcal{T})=\Psi(\mathcal{T},\phi,\cdots,\phi)$. Therefore, Ψ_ϕ denotes a mapping from $\bigcup_{k,p\ge0}\mathbf{T}^{3,5}(k,p)$ to $\mathcal{D}'(\mathbb{R}\times(-T,T))$. Note that, if $\mathcal{T}\in\mathbf{T}^{3,5}(k,p)$, then $\Psi_\phi(\mathcal{T})$ is $(2k+4p+1)$-linear in ϕ. At times, we might identify the multilinear expression $\Psi_\phi(\mathcal{T})$ with its associated tree $\mathcal{T}$ when the base function ϕ is fixed.

For $k,p\ge0$, we define $\Xi_{0,0}(t)=S(t)\phi$ and

$$\Xi^{3,5}_{k,p}(\phi) := \sum_{\mathcal{T}\in\mathbf{T}^{3,5}(k,p)} \Psi_\phi(\mathcal{T}), \tag{11}$$

for $k+p\ge1$. Finally, we can rewrite (8) the series expansion of the solution v to (7) as:

$$v=\sum_{k,p\ge0}\Xi^{3,5}_{k,p}(\phi), \tag{12}$$

where $\Xi^{3,5}_{k,p}(\phi)$ consists of homogeneous multilinear terms in ϕ of degree $2k+4p+1$. We group all the j-th generation of the Picard iterations in $\Xi_j(\phi)$, i.e.

$$\Xi_j(\phi) := \sum_{j=k+p}\Xi^{3,5}_{k,p}(\phi), \tag{13}$$

Then (12) can be further written as

$$v=\sum_{j=0}^{\infty}\Xi_j(\phi). \tag{14}$$

In the following, we shall show the convergence of the series (14) with some specific initial data ϕ.

2.2. *Multilinear estimates*

In this subsection, we establish some multilinear estimates with special initial data. Fix $N\gg1$ (to be chosen later). We define ϕ by setting

$$\widehat{\phi}(\xi)=R\left\{\mathbf{1}_{2N+Q_A}(\xi)+\mathbf{1}_{3N+Q_A}(\xi)\right\}, \tag{15}$$

where $Q_A=[-\frac{A}{2},\frac{A}{2})$, $R=R(N)$ is a real parameter, and $A=A(N)\gg1$, satisfying $A\ll N$, is to be chosen later. Note that we have

$$\|\phi\|_{H^s}\sim N^sRA^{\frac12} \quad\text{and}\quad \|\phi\|_{\mathcal{F}L^1}\sim RA \tag{16}$$

for any $s \in \mathbb{R}$. Here, and in what follows $\mathscr{F}L^p(\mathbb{R})$, denotes the Fourier–Lebesgue space of order p. It is defined by the norm

$$\|f\|_{\mathscr{F}L^p} = \left\|\widehat{f}(\xi)\right\|_{L^p(\mathbb{R})}.$$

Now we are ready to state our key multilinear estimates.

Lemma 11. *Let ϕ be as in* (15). *There exists a constant $C > 0$ such that for $t \geq 0$, we have*

$$\left\|\Xi^{3,5}_{k,p}(\phi)(t)\right\|_{\mathscr{F}L^1} \leq (Ct)^{k+p} N^k (RA)^{2k+4p+1}, \tag{17}$$

$$\left\|\Xi^{3,5}_{k,p}(\phi)(t)\right\|_{\mathscr{F}L^\infty} \leq (Ct)^{k+p} N^k (RA)^{2k+4p} R, \tag{18}$$

$$\left\|\partial_x\left(\Xi^{3,5}_{k,p}(\phi)\right)(t)\right\|_{\mathscr{F}L^\infty} \leq (Ct)^{k+p} N^{k+1} (RA)^{2k+4p} R, \tag{19}$$

for all $(k,p) \in (\mathbb{N}\cup\{0\})^2$ and $\mathscr{T} \in \mathbf{T}^{3,5}(k,p)$.

Proof. By Lemma 10 and (11), we only need to prove (17), (18), and (19) with $\Xi^{3,5}_{k,p}(\phi)$ replaced by $\Psi_\phi(\mathscr{T})$, where $\mathscr{T}$ is an element of $\mathbf{T}^{3,5}(k,p)$.

To prove (17), we proceed by induction on $n = |\mathscr{T}|$. If $|\mathscr{T}| = 1$, i.e. the tree $\mathscr{T}$ only has a single node (and thus $\Psi_\phi(\mathscr{T})(t) = \Xi_{0,0}(t) = S(t)\phi$), then (17) follows from (16). Fix $n \geq 1$ and assume that (17) holds for all trees $\mathscr{T} \in \mathbf{T}^{3,5}$ with $|\mathscr{T}| \leq n$. Let $\mathscr{T} \in \mathbf{T}^{3,5}(k,p)$ with $|\mathscr{T}| = n+1$, i.e. $n+1 = 3k+5p+1$ for some k and p. Since $\Psi_\phi(\mathscr{T})$ is $|\mathscr{T}^\infty|$-linear in ϕ, we observe from (15) that the function $\mathscr{F}[\Psi_\phi(\mathscr{T})]$ is supported on $\{\xi \in \mathbb{R} : |\xi| \leq 3|\mathscr{T}^\infty|(N+A)\}$. We divide our argument into two cases depending on the number of children of the root node.

Case 1: the root node a of $\mathscr{T}$ has three children. We denote these three children by a_s, $s \in \{1,2,3\}$. By the notations in Remark 9, let $j_s := |\mathscr{T}_{a_s}| =: 3k_s + 5p_s + 1$ for $s \in \{1,2,3\}$. Then it follows that

$$k_1 + k_2 + k_3 + 1 = k, \ p_1 + p_2 + p_3 = p \tag{20}$$

and

$$\Psi_\phi(\mathscr{T})(t) = -i\int_0^t S(t-t')\Big(\Psi_\phi(\mathscr{T}_{a_1})(t')\Psi_\phi(\mathscr{T}_{a_2})(t')\partial_x\overline{\Psi_\phi(\mathscr{T}_{a_3})}(t')\Big)dt'.$$

By unitarity of the linear propagator $S(t)$ in $\mathscr{F}L^1$, Young's inequality and the induction hypothesis, we have

$$\begin{aligned}
\|\Psi_\phi(\mathscr{T})(t)\|_{\mathscr{F}L^1} &\leq \int_0^t \left\|\Psi_\phi(\mathscr{T}_{a_1})(t')\Psi_\phi(\mathscr{T}_{a_2})(t')\overline{\partial_x\Psi_\phi(\mathscr{T}_{a_3})}(t')\right\|_{\mathscr{F}L^1} dt' \\
&\leq 3\cdot\left|\mathscr{T}^\infty_{a_3}\right|(N+A)\int_0^t \prod_{i=1}^3 \left\|\Psi_\phi(\mathscr{T}_{a_i})(t')\right\|_{\mathscr{F}L^1} dt' \\
&\leq 3\cdot\left|\mathscr{T}^\infty_{a_3}\right|(N+A)\int_0^t \prod_{i=1}^3 (Ct')^{k_i+p_i} N^{k_i}(RA)^{2k_i+4p_i+1} dt' \\
&= 3\cdot\left|\mathscr{T}^\infty_{a_3}\right|(N+A)\int_0^t (Ct')^{k+p-1} N^{k-1}(RA)^{2(k-1)+4p+3} dt' \\
&\leq 6\frac{|\mathscr{T}^\infty|}{k+p} C^{k+p-1} t^{k+p} N^k (RA)^{2k+4p+1},
\end{aligned}$$

where we used $N \geq A$ in the last inequality. Then the desired estimate follows by choosing C large enough and noting that $|\mathscr{T}^\infty| = 2k+4p+1 \leq 5(k+p)$.

Case 2: the root node a has five children. We denote them by a_s for $s \in \{1,2,3,4,5\}$. Let $j_s := |\mathscr{T}_{a_s}| =: 3k_s + 5p_s + 1$ for $s \in \{1,2,3,4,5\}$. We notice that

$$k_1 + k_2 + k_3 + k_4 + k_5 = k, \ p_1 + p_2 + p_3 + p_4 + p_5 + 1 = p, \tag{21}$$

$$\Psi_\phi(\mathcal{T})(t) = -i\int_0^t S(t-t')\Big(\Psi_\phi(\mathcal{T}_{a_1})\overline{\Psi_\phi(\mathcal{T}_{a_2})}\Psi_\phi(\mathcal{T}_{a_3})\overline{\Psi_\phi(\mathcal{T}_{a_4})}\Psi_\phi(\mathcal{T}_{a_5})\Big)dt'.$$

As in the previous case, we bound

$$\begin{aligned}\big\|\Psi_\phi(\mathcal{T})(t)\big\|_{\mathcal{F}L^1} &\le \int_0^t \Big\|\Psi_\phi(\mathcal{T}_{a_1})\overline{\Psi_\phi(\mathcal{T}_{a_2})}\Psi_\phi(\mathcal{T}_{a_3})\overline{\Psi_\phi(\mathcal{T}_{a_4})}\Psi_\phi(\mathcal{T}_{a_5})\Big\|_{\mathcal{F}L^1} dt' \\ &\le \int_0^t \prod_{i=1}^5 \big\|\Psi_\phi(\mathcal{T}_{a_i})(t')\big\|_{\mathcal{F}L^1} dt' \\ &\le \int_0^t \prod_{i=1}^5 (Ct')^{k_i+p_i} N^{k_i}(RA)^{2k_i+4p_i+1} dt' \\ &\le \int_0^t (Ct')^{k+p-1} N^k (RA)^{2k+4(p-1)+5} dt' \\ &\le C^{k+p-1} t^{k+p} N^k (RA)^{2k+4p+1},\end{aligned}$$

which again gives (17) provided $C \ge 1$. This finishes the proof of (17). The proofs of (18) and (19) follow from the similar arguments and we omit details. □

Lemma 12. *Let ϕ be as in* (15)*. Fix $s < 0$. Then, there exists a constant $C > 0$ such that for any $t \ge 0$, we have*

$$\Big\|\Xi^{3,5}_{k,p}(\phi)(t)\Big\|_{H^s} \le C^{k+p} f_s(A)\, t^{k+p} N^k (RA)^{2k+4p} R. \tag{22}$$

Here, $f_s(A)$ is given by

$$f_s(A) = \begin{cases} 1, & \text{if } s < -\frac{1}{2}, \\ (\log A)^{\frac{1}{2}}, & \text{if } s = -\frac{1}{2}, \\ A^{\frac{1}{2}+s} & \text{if } s > -\frac{1}{2}. \end{cases} \tag{23}$$

Proof. Recall that $|\mathcal{T}| = 3k+5p+1$ for $\mathcal{T} \in \mathbf{T}^{3,5}(k,p)$. In view of Lemma 10, (22) follows from the bound

$$\big\|\Psi_\phi(\mathcal{T})(t)\big\|_{H^s} \le C^{k+p} f_s(A)\, t^{k+p} N^k (RA)^{2k+4p} R, \tag{24}$$

for all $(k,p) \in (\mathbb{N}\cup\{0\})^2$ and $\mathcal{T} \in \mathbf{T}^{3,5}(k,p)$. We note that $\Psi_\phi(\mathcal{T})$ is $|\mathcal{T}^\infty|$-linear in ϕ, and thus the support of $\mathcal{F}(\Psi_\phi(\mathcal{T}))$ is contained in at most $2^{|\mathcal{T}^\infty|}$ intervals of length $|\mathcal{T}^\infty|\cdot A$. Furthermore, since $\langle\xi\rangle^s$ is a decreasing function in $|\xi|$ for $s<0$, we thus have

$$\big\|\langle\xi\rangle^s\big\|_{L^2(\operatorname{supp}\mathcal{F}[\Psi_\phi(\mathcal{T})(t)])} \le 2^{\frac{|\mathcal{T}^\infty|}{2}} f_s\big(|\mathcal{T}^\infty|A\big), \tag{25}$$

uniformly in $t \ge 0$.

If $|\mathcal{T}| = 1$, then the claimed result follows from (16). We now assume that $|\mathcal{T}| > 1$ so that $k + p \ge 1$. Let us first assume that the root node of $\mathcal{T}$ has three children a_s for $s \in \{1,2,3\}$. Then by (25), Lemma 11, and (20), we get

$$\begin{aligned}
\|\Psi_\phi(\mathcal{T})(t)\|_{H^s} &\le \|\langle\xi\rangle^s\|_{L^2(\operatorname{supp}\mathcal{F}[\Psi_\phi(\mathcal{T})(t)])} \int_0^t \Big\|\Psi_\phi(\mathcal{T}_{a_1})\Psi_\phi(\mathcal{T}_{a_2})\partial_x\overline{\Psi_\phi(\mathcal{T}_{a_3})}\Big\|_{\mathcal{F}L^\infty} dt' \\
&\le 2^{\frac{|\mathcal{T}^\infty|}{2}} f_s\big(|\mathcal{T}^\infty|A\big)\int_0^t \|\Psi_\phi(\mathcal{T}_{a_1})\|_{\mathcal{F}L^1}\|\Psi_\phi(\mathcal{T}_{a_2})\|_{\mathcal{F}L^1}\|\partial_x\Psi_\phi(\mathcal{T}_{a_3})\|_{\mathcal{F}L^\infty} dt' \\
&\le 2^{\frac{|\mathcal{T}^\infty|}{2}} f_s\big(|\mathcal{T}^\infty|A\big)\int_0^t \prod_{i=1}^{2}(Ct')^{k_i+p_i}N^{k_i}(RA)^{2k_i+4p_i+1} \\
&\qquad\qquad \times (Ct')^{k_3+p_3}N^{k_3+1}(RA)^{2k_3+4p_3}R\,dt' \\
&\le 2^{\frac{|\mathcal{T}^\infty|}{2}} \frac{|\mathcal{T}^\infty|}{k+p} f_s(A)(Ct)^{k+p}N^k(RA)^{2k+4p}R \\
&\le C^{k+p} f_s(A)t^{k+p}N^k(RA)^{2k+4p}R,
\end{aligned}$$

for $C > 0$ large enough. Note that in the last inequality we used $|\mathcal{T}^\infty| = 2k+4p+1 \le 5(k+p)$.

We then consider the case when the root node of $\mathcal{T}$ has five children a_s for $s \in \{1,2,3,4,5\}$. Then by (25), Lemma 11, and (21), we get

$$\begin{aligned}
\|\Psi_\phi(\mathcal{T})(t)\|_{H^s} &\le \|\langle\xi\rangle^s\|_{L^2(\operatorname{supp}\mathcal{F}[\Psi_\phi(\mathcal{T})(t)])} \int_0^t \Big\|\prod_{i=1}^{5}\Psi_\phi(\mathcal{T}_{a_i})\Big\|_{\mathcal{F}L^\infty} dt' \\
&\le 2^{\frac{|\mathcal{T}^\infty|}{2}} f_s\big(|\mathcal{T}^\infty|A\big)\int_0^t \prod_{i=1}^{4}\|\Psi_\phi(\mathcal{T}_{a_i})\|_{\mathcal{F}L^1}\|\Psi_\phi(\mathcal{T}_{a_5})\|_{\mathcal{F}L^\infty} dt' \\
&\le 2^{\frac{|\mathcal{T}^\infty|}{2}} f_s\big(|\mathcal{T}^\infty|A\big)\int_0^t \prod_{i=1}^{4}(Ct)^{k_i+p_i}N^{k_i}(RA)^{2k_i+4p_i+1} \\
&\qquad\qquad \times (Ct)^{k_5+p_5}N^{k_5}(RA)^{2k_5+4p_5}R\,dt' \\
&\le 2^{\frac{|\mathcal{T}^\infty|}{2}} \frac{|\mathcal{T}^\infty|}{k+p} f_s(A)(Ct)^{k+p}N^k(RA)^{2k+4p}R \\
&\le C^{k+p} f_s(A)t^{k+p}N^k(RA)^{2k+4p}R,
\end{aligned}$$

for $C > 0$ large enough. This shows (24) and finishes the proof of Lemma 23. □

As a consequence of the last lemma, we have the following estimate on the power expansion (14).

Lemma 13. *Let ϕ be as in* (15) *and fix $s < 0$. Let $R^2A^2 \gg N$. Then, there exists a constant $C > 0$ such that for any $j \ge 1$ and $t \ge 0$, we have*

$$\|\Xi_j(\phi)(t)\|_{H^s} \le C^j f_s(A)t^j(RA)^{4j}R, \tag{26}$$

where Ξ_j is given in (13).

Proof. Fix $j \ge 1$ and $t \ge 0$. By Lemma 10 and (13), it suffices to show

$$\Big\|\Xi^{3,5}_{k,p}(\phi)(t)\Big\|_{H^s} \le C^j f_s(A)t^j(RA)^{4j}R$$

with $k + p = j$. By Lemma 12, it suffices to show

$$N^{j-p}(RA)^{2(j+p)} \ll (RA)^{4j},$$

which is a consequence of the assumption $R^2A^2 \gg N$. □

Remark 14. Note that for $j \ge 0$, the bound (26) corresponds to the upper bound on $\Xi^{3,5}_{0,j}$ given in (22). Thus, under the conditions $R^2A^2 \gg N$, we can essentially bound the size of (14) by the contribution of quinary trees.

The following lemma shows that the multilinear expressions Ξ_j defined in (13) are stable under suitable perturbations.

Lemma 15. *Let ϕ be as in* (15) *and $\psi \in \mathscr{F}^{-1}C_0^\infty(\mathbb{R})$ with $\|\psi\|_{\mathscr{F}L^1} \lesssim RA$ and* $\operatorname{supp}(\widehat{\psi}) \subset [-M, M]$ *for some $M \geq 0$. We further assume $R^2A^2 \gg N$ and $N \gg M$. Then, there exists $C > 0$ such that for any $j \geq 0$ and $t \geq 0$, we have*

$$\left\|\Xi_j(\phi+\psi)(t) - \Xi_j(\phi)(t)\right\|_{L^2} \leq C^j \|\psi\|_{L^2} \left(tR^4A^4\right)^j$$

Proof. From (11) and (13), we have

$$\begin{aligned}\Xi_j(\phi+\psi) - \Xi_j(\phi) &= \sum_{\substack{\mathscr{T}\in\mathbf{T}^{3,5}(k,p)\\ j=k+p}} \left(\Psi_{\phi+\psi}(\mathscr{T}) - \Psi_\phi(\mathscr{T})\right)\\ &= \sum_{\substack{\mathscr{T}\in\mathbf{T}^{3,5}(k,p)\\ j=k+p}} \sum_{\phi_i\in\{\phi,\psi\}} \Psi\left(\mathscr{T};\phi_1,\dots,\phi_{2k+4p+1}\right),\end{aligned} \tag{27}$$

where the second summation in $\phi_1,\dots,\phi_{2k+4p+1}$ is over all possible combinations of $\phi_i \in \{\phi,\psi\}$ with at least one occurrence of ψ. Given $\mathscr{T} \in \mathbf{T}^{3,5}(k,p)$ with $(k,p) \in (\mathbb{N}\cup\{0\})^2$, we have that $\operatorname{supp}\mathscr{F}[\Psi(\mathscr{T};\phi_1,\dots,\phi_{2k+4p+1})] \subset \{\xi\in\mathbb{R} : |\xi| \leq 6(2k+4p+1)N\}$ provided $A, M \ll N$. Without loss of generality, we assume $\phi_1 = \psi$. This consideration on the Fourier support of $\Psi(\mathscr{T};\phi_1,\dots,\phi_{2k+4p+1})$ together with a similar induction argument as in Lemma 11, Young's inequality and the fact that $\|\psi\|_{\mathscr{F}L^1}, \|\phi\|_{\mathscr{F}L^1} \lesssim RA$ then yields

$$\begin{aligned}\left\|\Psi\left(\mathscr{T};\psi,\phi_2,\dots,\phi_{2k+4p+1}\right)(t)\right\|_{L^2} &\leq C^j t^j \|\psi\|_{L^2} N^k \prod_{j=2}^{2k+4p+1} \|\phi_j\|_{\mathscr{F}L^1}\\ &\leq C^j t^j \|\psi\|_{L^2} N^k \cdot (RA)^{2k+4p}\\ &\leq C^j t^j \|\psi\|_{L^2} \cdot (RA)^{4k+4p},\end{aligned}$$

provided $R^2A^2 \gg N$. □

To conclude this subsection, we remark that, by Lemma 13 and Lemma 15, and under the choice of parameters $T(RA)^4 \ll 1$ and $R^2A^2 \gg N \gg M$, then the series

$$v_1(t) = \sum_{j=0}^\infty \Xi_j(\phi+\psi)(t)$$

converges in $H^s(\mathbb{R})$, $s < 0$, for any $0 \leq t \leq T$, as long as $\|\psi\|_{\mathscr{F}L^1} \lesssim RA$. Let us note that the function v_1 solves the initial value problem (3) and (7) with initial data ϕ replaced by $\phi+\psi$.

2.3. *Second Picard iterate*

In this subsection, we obtain a lower bound on the second Picard iterate ($\Xi_1(\phi)$ in (13)). This will allow us to prove later that this term represents the main contribution to the series expansion (14) and to the norm inflation phenomena. See Subsection 3.1. To this purpose, we first recall the following elementary bounds on characteristic functions of intervals.

Lemma 16. *For any $a,b,c,d,e,\xi \in \mathbb{R}$ and $A \geq 1$, there exists $c > 0$ such that*

$$\mathbf{1}_{a+Q_A} * \mathbf{1}_{b+Q_A} * \mathbf{1}_{c+Q_A} * \mathbf{1}_{d+Q_A} * \mathbf{1}_{e+Q_A}(\xi) \geq cA^4 \mathbf{1}_{a+b+c+d+e+Q_A}(\xi).$$

We now state the lower bound on $\Xi_1(\phi)$.

Proposition 17. *Let ϕ be as in* (15). *Then, for $0 < t \ll N^{-2}$ and $R^2A^2 \gg N$, we have*

$$\left\|\Xi_1(\phi)(t)\right\|_{H^s} \gtrsim f_s(A)\cdot tR^5A^4, \tag{28}$$

where $f_s(A)$ is the function defined in (23).

Proof. From (11), we have,

$$\Xi_1(\phi)(t) := \Xi_{1,0}^{3,5}(\phi)(t) + \Xi_{0,1}^{3,5}(\phi)(t) = \Psi_\phi(\mathcal{T}_3)(t) + \Psi_\phi(\mathcal{T}_5)(t), \tag{29}$$

where $\mathcal{T}_3$ (resp. $\mathcal{T}_5$) is the tree with one parent node and three (resp. five) descendents. Namely, $\Psi_\phi(\mathcal{T}_3)(t) = \mathcal{J}^3[S(t)\phi]$ and $\Psi_\phi(\mathcal{T}_5)(t) = \mathcal{K}^5[S(t)\phi]$, see (6). From Lemma 12, we have

$$\big\|\Psi_\phi(\mathcal{T}_3)(t)\big\|_{H^s} \lesssim f_s(A)\cdot tN(RA)^2R. \tag{30}$$

We now turn to the quintic term $\Psi_\phi(\mathcal{T}_5)$ and have

$$\begin{aligned}\mathcal{F}\left[\Psi_\phi(\mathcal{T}_5)(t)\right](\xi) = -\frac{1}{2}e^{-i|\xi|^2 t} &\int_{\xi=\xi_1-\xi_2+\cdots+\xi_5} \\ &\times \int_0^t e^{it'(|\xi|^2-|\xi_1|^2+|\xi_2|^2-|\xi_3|^2+|\xi_4|^2-|\xi_5|^2)}\,dt' \\ &\times \widehat{\phi}(\xi_1)\overline{\widehat{\phi}}(\xi_2)\widehat{\phi}(\xi_3)\overline{\widehat{\phi}}(\xi_4)\widehat{\phi}(\xi_5)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4 d\xi_5.\end{aligned} \tag{31}$$

From (15), we have $|\xi_j| \lesssim N$ for $\xi_j \in \operatorname{supp}\widehat{\phi}$. Then, since $\xi = \xi_1 - \xi_2 + \xi_3 - \xi_4 + \xi_5$ we have

$$\big||\xi|^2 - |\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 + |\xi_4|^2 - |\xi_5|^2\big| \lesssim N^2,$$

which implies that,

$$\operatorname{Re}\left(e^{it'(|\xi|^2-|\xi_1|^2+|\xi_2|^2-|\xi_3|^2+|\xi_4|^2-|\xi_5|^2)}\right) \geq \frac{1}{2} \tag{32}$$

holds for all $0 < t' \ll N^{-2}$. Thus, by (31), (32), and Lemma 16, we arrive at

$$\big|\mathcal{F}\left[\Psi_\phi(\mathcal{T}_5)(t)\right](\xi)\big| \gtrsim tR^5A^4\cdot \mathbf{1}_{Q_A}(\xi). \tag{33}$$

Noting that $\|\langle\xi\rangle^s\|_{L^2_\xi(Q_A)} \sim f_s(A)$, we obtain from (33),

$$\big\|\Psi_\phi(\mathcal{T}_5)(t)\big\|_{H^s} \gtrsim f_s(A)\cdot t(RA)^4R. \tag{34}$$

Finally, by collecting (29), (30), (34) along with the the assumption $R^2A^2 \gg N$, we obtain (28). □

3. Proof of Theorem 2

In this section, we present the proof of Theorem 2. A density argument reduces the proof of Theorem 2 to the following statement.

Proposition 18. *Let $s < 0$. Fix $\psi \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\psi} \in C_0^\infty(\mathbb{R})$. Then, given any $n \in \mathbb{N}$, there exist a solution v_n to* (3) *and $t_n \in \left(0, \frac{1}{n}\right)$ such that*

$$\big\|v_n(0) - \psi\big\|_{H^s} < \tfrac{1}{n} \qquad \textit{and} \qquad \|v_n(t_n)\|_{H^s} > n. \tag{35}$$

In what follows, we prove Proposition 18. Given $n \in \mathbb{N}$, let $N_n, A_n, R_n, T_n \gg 1$ to be chosen later. We omit the dependence in n of these constants from now on for convenience. We set $v_0 = \psi + \phi$ with ϕ as in (15).

Denote by $v = v_n$ the global solution to (3) with initial data $v(0) = v_0$.

3.1. *Proof of Proposition 18*

We now prove Proposition 18 We claim that it suffices to show that the following properties hold:

$$N^s R A^{\frac{1}{2}} \ll \frac{1}{n}, \tag{i}$$

$$T R^4 A^4 \ll 1, \tag{ii}$$

$$n \ll f_s(A) \cdot T R^5 A^4, \tag{iii}$$

$$1 \ll N \ll R^2 A^2, \tag{iv}$$

$$1 \ll A \ll N, \tag{v}$$

$$T \ll N^{-2}, \tag{vi}$$

for some A, R, T, and $N \gg 1$, depending on n and ψ.

We first prove the above claim, i.e. we show how conditions (i)-(vi) imply Proposition 18. By Lemma 13 and Lemma 15[2], we have the following series expansion representation of solution v

$$v(t) := \sum_{j=0}^{\infty} \Xi_j(v_0)(t), \tag{36}$$

provided conditions (ii), (iv) and (v) hold. By (16), the first condition (i) ensures that the first inequality in (35) holds. By (36), (16), Proposition 17, Lemma 13, and Lemma 15 (which hold under (iv), (v) and (vi)), we have

$$\begin{aligned} \|v(T)\|_{H^s} &\geq \big\|\Xi_1(\phi)(T)\big\|_{H^s} - \big\|\Xi_0(\phi+\psi)(T)\big\|_{H^s} \\ &\quad - \sum_{j=1}^{\infty} \big\|\Xi_j(\phi)(T) - \Xi_j(\psi+\phi)(T)\big\|_{H^s} - \sum_{j=2}^{\infty} \big\|\Xi_j(\phi)(T)\big\|_{H^s} \\ &\gtrsim f_s(A)\cdot T R^5 A^4 - \Big(1 + N^s R A^{\frac{1}{2}}\Big) - T R^4 A^4 - f_s(A)\cdot T^2 R^9 A^8 \\ &\sim T R^5 A^4 \cdot f_s(A) \geq n, \end{aligned}$$

where we used (i), (ii) and (iii) in the last inequality. Finally, choosing N sufficiently large such that $R^4 A^4 \geq n$, together with (ii), imply $t_n = T \in (0, \frac{1}{n})$. This proves Proposition 18.

It thus remains to verify the conditions (i)-(vi). In what follows, we consider the following three cases:

Case 1: $s < -\frac{1}{2}$**.** We set

$$A = N^{\frac{\delta}{5}}, \quad R = N^{\frac{1}{2}}, \quad \text{and} \quad T = N^{-2-\delta},$$

with $\delta > 0$ sufficiently small such that $s + \frac{1}{2} + \frac{\delta}{10} < 0$. The conditions (ii), (iv), (v) and (vi) are trivially satisfied for $N \gg 1$. By choosing N large enough, we have

$$N^s R A^{\frac{1}{2}} = N^{s+\frac{1}{2}+\frac{\delta}{10}} \ll \frac{1}{n},$$

$$T R^5 A^4 = N^{\frac{1}{2}-\frac{\delta}{5}} \gg n,$$

which verify the conditions (i) and (iii) as $f_s(A) = 1$ in this case.

Case 2: $s = -\frac{1}{2}$**.** In this case we have $f_s(A) = (\log A)^{\frac{1}{2}}$. Set

$$A = (\log N)^{\frac{3}{2}}, \quad R = \frac{N^{\frac{1}{2}}}{\log N}, \quad \text{and} \quad T = N^{-2-\delta},$$

[2]In applying Lemma 15, we need to ensure that $\|\psi\|_{\mathscr{F}L^1} \lesssim RA$ and $N \gg M$ where $\mathrm{supp}(\widehat{\psi}) \subset [-M, M]$. This is guaranteed by taking N large enough (by (iv)). We omit the dependence of the constants in ψ in the remaining part of our argument.

with $0 < \delta \ll 1$. The conditions (ii), (iv), (v) and (vi) are trivially satisfied for $N \gg 1$. By choosing N large enough, we have

$$N^{-\frac{1}{2}} R A^{\frac{1}{2}} = (\log N)^{-\frac{1}{4}} \ll \frac{1}{n},$$
$$T R^5 A^4 f_s(A) \gtrsim N^{\frac{1}{4}} \gg n.$$

Thus, the conditions (i) and (iii) are satisfied.

Case 3: $-\frac{1}{2} < s < 0$. In this case, we have $f_s(A) = A^{s+\frac{1}{2}}$. Choose

$$A = N^{1+2s+\frac{9}{4}\delta}, \quad R = N^{-\frac{1}{2}-2s-\frac{17}{8}\delta}, \quad \text{and} \quad T = N^{-2-\delta},$$

with $0 < \delta \ll 1$ satisfying $1 + 2s + \frac{9}{4}\delta > 0$, $2s + \frac{9}{4}\delta < 0$ and $2s^2 - \frac{3\delta}{2} + \frac{9}{4}\delta s > 0$. We then have

$$N^s R A^{\frac{1}{2}} = N^{-\delta} \ll \frac{1}{n},$$
$$T R^4 A^4 = N^{-\frac{\delta}{2}} \ll 1,$$
$$T R^5 A^{s+\frac{9}{2}} = N^{2s^2-\frac{3\delta}{2}+\frac{9}{4}\delta s} \gg n,$$
$$R^2 A^2 = N^{1+\frac{\delta}{4}} \gg N,$$
$$1 \ll A = N^{1+2s+\frac{9}{4}\delta} \ll N,$$

provided N large enough. This verifies (i) - (vi).

Remark 19. This framework fails to provide the norm inflation at the endpoint regularity $s = 0$, which concurs with Remark 3. As a matter of fact, if $s = 0$, the condition (i) writes $R \ll A^{-\frac{1}{2}}$, which implies $R^2 A^2 \ll A$. This is incompatible with conditions (iv) and (v).

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