



ACADÉMIE  
DES SCIENCES  
INSTITUT DE FRANCE

# *Comptes Rendus*

---

# *Mathématique*


Fatiha Alabau-Boussouira, Piermarco Cannarsa and Cristina Urbani

**Bilinear control of evolution equations with unbounded lower order terms.  
Application to the Fokker–Planck equation**

Volume 362 (2024), p. 511-545

Online since: 31 May 2024

<https://doi.org/10.5802/crmath.567>

 This article is licensed under the  
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.  
<http://creativecommons.org/licenses/by/4.0/>



*The Comptes Rendus. Mathématique* are a member of the  
Mersenne Center for open scientific publishing  
[www.centre-mersenne.org](http://www.centre-mersenne.org) — e-ISSN : 1778-3569



Research article / *Article de recherche*  
Control theory / *Théorie du contrôle*

# Bilinear control of evolution equations with unbounded lower order terms. Application to the Fokker–Planck equation

*Contrôle bilinéaire d'équations d'évolution avec opérateur de contrôle non borné. Application à l'équation de Fokker–Planck*

Fatiha Alabau-Boussouira <sup>a</sup>, Piermarco Cannarsa <sup>b</sup> and Cristina Urbani <sup>©,c</sup>

<sup>a</sup> Laboratoire Jacques-Louis Lions, Sorbonne Université, 75005, Paris, Université de Lorraine, France

<sup>b</sup> Dipartimento di Matematica, Università di Roma Tor Vergata, 00133, Roma, Italy

<sup>c</sup> Dipartimento di Scienze Tecnologiche e dell'Innovazione, Universitas Mercatorum, Piazza Mattei 10, 00186, Roma, Italy

*E-mails:* fatiha.alabau@sorbonne-universite.fr (F. Alabau-Boussouira),  
cannarsa@mat.uniroma2.it (P. Cannarsa), cristina.urban@unimercurum.it  
(C. Urbani)

**Abstract.** We study the exact controllability of the evolution equation

$$u'(t) + Au(t) + p(t)Bu(t) = 0$$

where  $A$  is a nonnegative self-adjoint operator on a Hilbert space  $X$  and  $B$  is an unbounded linear operator on  $X$ , which is dominated by the square root of  $A$ . The control action is bilinear and only of scalar-input form, meaning that the control is the scalar function  $p$ , which is assumed to depend only on time. Furthermore, we only consider square-integrable controls. Our main result is the local exact controllability of the above equation to the ground state solution, that is, the evolution through time, of the first eigenfunction of  $A$ , as initial data.

The analogous problem (in a more general form) was addressed in our previous paper [Exact controllability to eigensolutions for evolution equations of parabolic type via bilinear control, Alabau-Boussouira F., Cannarsa P. and Urbani C., *Nonlinear Diff. Eq. Appl.* (2022)] for a bounded operator  $B$ . The current extension to unbounded operators allows for many more applications, including the Fokker–Planck equation in one space dimension, and a larger class of control actions.

**Résumé.** Nous étudions la contrôlabilité exacte de l'équation d'évolution

$$u'(t) + Au(t) + p(t)Bu(t) = 0$$

où  $A$  est un opérateur auto-adjoint non négatif dans l'espace de Hilbert  $X$  et  $B$  est un opérateur linéaire non borné de  $X$ , dominé par la racine carrée de l'opérateur  $A$ . L'action du contrôle est bilinéaire et sous forme d'entrée scalaire, ce qui signifie que le contrôle est la fonction scalaire  $p$ , supposée ne dépendre que du temps et être de carré intégrable. Notre résultat principal est la contrôlabilité locale exacte de l'équation d'évolution au voisinage de la solution de l'état fondamental, c'est-à-dire au voisinage de la solution de l'équation d'évolution de donnée initiale égale à la première fonction propre de  $A$ .

Nous avons traité un problème similaire sous une forme plus générale dans notre article précédent [Contrôlabilité exacte aux solutions propres pour les équations d'évolution de type parabolique via contrôle bilinéaire, Alabau-Boussouira F, Cannarsa P et Urbani C., *Nonlinear Diff. Eq. Appl.* (2022)] dans les cas où l'opérateur  $B$  est borné. L'extension actuelle aux opérateurs non bornés permet de nombreuses autres applications, comme celle de l'équation de Fokker-Planck en dimension un d'espace. Elle permet également de considérer une classe plus large d'opérateurs de contrôle.

**2020 Mathematics Subject Classification.** 35Q93, 93C25, 93C10, 93B05, 35K90.

**Funding.** This paper was partly supported by the INdAM National Group for Mathematical Analysis, Probability and their Applications (GNAMPA) and by the French-German-Italian Laboratoire International Associé (LIA) COPDESC.

The second and third authors acknowledge support from the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

The third author acknowledges support from Accademia Nazionale dei Lincei, Postdoc scholarship "Beniamino Segre" and from Università Italo Francese (Vinci Project 2018).

*Manuscript received 12 March 2023, revised 6 July 2023 and 3 September 2023, accepted 5 September 2023.*

## 1. Introduction

In a series of recent papers (see [3, 4, 16]), we have studied stabilization and exact controllability to eigensolutions for evolution equations of the form

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0, T) \\ u(0) = u_0 \end{cases} \quad (1)$$

where  $u_0$  belongs to a Hilbert space  $(X, \langle \cdot, \cdot \rangle, \|\cdot\|)$  and

- (i)  $A : D(A) \subset X \rightarrow X$  is a self-adjoint operator with compact resolvent and such that  $A \geq -\sigma I$ , with  $\sigma \geq 0$ ,
- (ii)  $B : X \rightarrow X$  is a bounded linear operator,
- (iii)  $p \in L^2(0, T)$  is a bilinear control.

We refer to equation (1) as being of parabolic type, since assumption i) is usually satisfied by parabolic operators.

The scalar-input bilinear controllability problem has been addressed by several authors, starting with the negative result by Ball, Marsden, Slemrod [7]. Controllability issues are interesting also in the hyperbolic or diffusive context, where several results are now available to describe the reachable set of specific partial differential equations in 1-D, such as the Schrödinger equation [5, 8, 10] and the classical [5, 9] and degenerate [13] wave equation. The above problem enters in the so-called class of nonlinear control problems.

We refer the readers to the book by Coron[17] where general control systems are studied as well as mathematical methods to treat them, with a focus on systems for which the nonlinearities are determinant for controllability issues, including the Schrödinger equation.

Most of the above results devoted to scalar-input bilinear controllability issues have in common the fact that they address controllability properties of (1) near the ground state solution  $\psi_1(t) = e^{-\lambda_1 t} \varphi_1$  (see [9, 10, 13]) or, more in general, to eigensolutions  $\psi_j(t) = e^{-\lambda_j t} \varphi_j$  (see [3, 4]), that are solutions of the free dynamics ( $p = 0$ ) associated to (1), namely

$$u'(t) + Au(t) = 0,$$

with initial condition  $u_0 = \varphi_j$ , where we denote by  $\{\lambda_k\}_{k \in \mathbb{N}^*}$  the eigenvalues of  $A$ , and by  $\{\varphi_k\}_{k \in \mathbb{N}^*}$  the associated eigenfunctions.

To prove local controllability to trajectories, we have introduced in [3] the notion of  $j$ -null controllability in time  $T > 0$  for the pair  $\{A, B\}$ : we require the existence of a constant  $N_T > 0$  such that for any initial condition  $y_0 \in X$ , there exists a control  $p \in L^2(0, T)$  satisfying

$$\|p\|_{L^2(0,T)} \leq N_T \|y_0\|, \tag{2}$$

and for which  $y(T) = 0$ , where  $y(\cdot)$  is the solution of the following linear problem

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0, T] \\ y(0) = y_0. \end{cases} \tag{3}$$

We call the best constant, defined as

$$N(T) := \sup_{\|y_0\|=1} \inf \left\{ \|p\|_{L^2(0,T)} : y(T; y_0, p) = 0 \right\}, \tag{4}$$

the *control cost*.

Furthermore, we have shown that sufficient conditions for  $\{A, B\}$  to be  $j$ -null controllable are a gap condition of the eigenvalues of  $A$  and a rank condition on  $B$ : we assume  $B$  to spread  $\varphi_j$  in all directions (see [4, Theorem 1.2]). When, for instance,  $X = L^2(0, 1)$ ,  $B$  could be taken as a multiplicative operator, i.e.,

$$(B\varphi)(x) = \mu(x)\varphi(x), \quad x \in (0, 1),$$

where  $\mu$  has to be chosen in order to guarantee the dispersive action mentioned above (see [5] for a general method to construct infinite classes of such  $\mu$ , including polynomial type classes, and for various PDE's, as well as various boundary conditions).

Another common feature of the aforementioned references is that  $B$  is assumed to be bounded. In many applications, however, one is forced to consider a bilinear control acting on a drift term rather than a potential, which leads to allowing  $B$  to be unbounded.

A typical example of such a situation occurs when dealing with the Fokker–Planck equation

$$u_t - u_{xx} - p(t) (\mu(x)u)_x = 0,$$

which is satisfied by the probability density of the diffusion process associated with

$$dX_t = p(t)\mu(X_t)dt + \sqrt{2}dW_t,$$

where  $W_t$  is the standard Wiener process on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ .

This paper aims to extend the theory of [4] to unbounded operators  $B : D(B) \subset X \rightarrow X$  satisfying

$$D(A_\sigma^{1/2}) \hookrightarrow D(B), \quad \text{and} \quad \|B\varphi\| \leq C \left( \|A_\sigma\varphi\|^2 + \|\varphi\|^2 \right)^{1/2},$$

for some constant  $C > 0$ , where  $A_\sigma := A + \sigma I$  (the aforementioned  $j$ -null controllability property of the pair  $\{A, B\}$  has to be assumed unchanged).

We consider (1) where  $A : D(A) \subset X \rightarrow X$  is a self-adjoint accretive operator with compact resolvent and  $B$  is an unbounded linear operator. By adapting the approach of [4], we first obtain in Section 3 a local controllability result (in the topology of  $D(A^{1/2})$ ), to the first eigensolution  $\psi_1$  (the ground state), see Theorem 7. Then, in Section 4 we derive two semi-global results, Theorems 9 and 10. Finally, in Section 5 we discuss applications to partial differential equations including

- (a) the Fokker–Planck equation,
- (b) the heat equation with control on the drift under Neumann boundary conditions,
- (c) a class of degenerate parabolic equations under Dirichlet or Neumann boundary conditions.

## 2. Preliminaries

Let  $(X, \langle \cdot, \cdot \rangle, \|\cdot\|)$  be a separable Hilbert space. Let  $A : D(A) \subset X \rightarrow X$  be a densely defined linear operator with the following properties:

- (a)  $A$  is self-adjoint,
- (b)  $\langle Ax, x \rangle \geq 0, \forall x \in D(A)$ ,
- (c)  $\exists \lambda > 0 : (\lambda I + A)^{-1} : X \rightarrow X$  is compact.

We recall that under the above assumptions  $A$  is a closed operator and  $D(A)$  is itself a Hilbert space with respect to the scalar product

$$(x, y)_{D(A)} = \langle x, y \rangle + \langle Ax, Ay \rangle, \quad \forall x, y \in D(A). \tag{6}$$

Moreover,  $-A$  is the infinitesimal generator of a strongly continuous semigroup of contractions on  $X$  which is also analytic and will be denoted by  $e^{-tA}$ .

In view of the above assumptions, there exists an orthonormal basis  $\{\varphi_k\}_{k \in \mathbb{N}^*}$  in  $X$  of eigenfunctions of  $A$ , that is,  $\varphi_k \in D(A)$  and  $A\varphi_k = \lambda_k \varphi_k \forall k \in \mathbb{N}^*$ , where  $\{\lambda_k\}_{k \in \mathbb{N}^*} \subset \mathbb{R}$  denote the corresponding eigenvalues. We recall that  $\lambda_k \geq 0, \forall k \in \mathbb{N}^*$  and we suppose — without loss of generality — that  $\{\lambda_k\}_{k \in \mathbb{N}^*}$  is ordered so that  $0 \leq \lambda_k \leq \lambda_{k+1} \rightarrow \infty$  as  $k \rightarrow \infty$ . The associated semigroup has the following representation

$$e^{-tA}\varphi = \sum_{k=1}^{\infty} \langle \varphi, \varphi_k \rangle e^{-\lambda_k t} \varphi_k, \quad \forall \varphi \in X. \tag{7}$$

For any  $s \geq 0$  we can define the operator  $A^s : D(A^s) \subset X \rightarrow X$  which is the  $s$ -fractional power of  $A$  (see [21]). Under our assumptions, such a linear operator is characterized as follows

$$D(A^s) = \left\{ x \in X \mid \sum_{k \in \mathbb{N}^*} \lambda_k^{2s} |\langle x, \varphi_k \rangle|^2 < \infty \right\} \tag{8}$$

$$A^s x = \sum_{k \in \mathbb{N}^*} \lambda_k^s \langle x, \varphi_k \rangle \varphi_k, \quad \forall x \in D(A^s).$$

The space  $D(A^s)$ , equipped with the norm

$$\|x\|_{D(A^s)} := \left( \|x\|^2 + \|A^s x\|^2 \right)^{1/2}, \quad \forall x \in D(A^s),$$

induced by the scalar product

$$\langle x, y \rangle_s = \langle x, y \rangle + \langle A^s x, A^s y \rangle$$

is a Hilbert space for any  $s \geq 0$ . Note that we have trivially the inequality

$$\|x\|_{D(A^s)} \leq (\|x\| + \|A^s x\|), \quad \forall x \in D(A^s). \tag{9}$$

Of course, the right hand side also defines an equivalent norm to  $\|\cdot\|_{D(A^s)}$  on  $D(A^s)$ , but is not associated to a scalar product. We will make use of the above inequality in some computations without further referring to it.

We indicate by  $B_{R,s}(x)$  the unit ball of radius  $R$  with respect to the norm  $\|\cdot\|_{D(A^s)}$  centered at  $x$ .

Let  $T > 0$  and consider the following problem

$$\begin{cases} u'(t) + Au(t) = f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{10}$$

where  $u_0 \in X$  and  $f \in L^2(0, T; X)$ . We now recall two definitions of solution of problem (10):

- the function  $u \in C([0, T], X)$  defined by

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(s) ds$$

is called the *mild solution* of (10),

- a function

$$u \in H^1(0, T; X) \cap L^2(0, T; D(A)) \tag{11}$$

is called a *strict solution* of (10) if  $u(0) = u_0$  and  $u$  satisfies the equation in (10) for a.e.  $t \in [0, T]$ .

The well-posedness of the Cauchy problem (10) is a classical result (see, for instance, [11, Theorem 3.1, p. 143]). We observe that the space

$$W(D(A); X) = H^1(0, T; X) \cap L^2(0, T; D(A))$$

is a Banach space with the norm

$$\|\varphi\|_W = \left( \|\varphi\|_{H^1(0, T; X)}^2 + \|\varphi\|_{L^2(0, T; D(A))}^2 \right)^{1/2}$$

for all  $\varphi \in W(D(A); X)$ .

**Theorem 1.** *Let  $u_0 \in D(A^{1/2})$  and  $f \in L^2(0, T; X)$ . Under hypothesis (5), the mild solution of system (10)*

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}f(s)ds \tag{12}$$

is a strict solution.

Moreover, there exists a constant  $C > 0$  such that

$$\|u\|_W \leq C \left( \|f\|_{L^2(0, T; X)} + \|u_0\|_{D(A^{1/2})} \right). \tag{13}$$

The regularity (11) of the function  $u$  given by (12) is called *maximal regularity*. Under our assumptions such a property is due to the analyticity of  $e^{-tA}$ . Observe that (13) ensures that the strict solution of (10) is unique.

We now recall a useful result which derives from spaces interpolation.

**Proposition 2.** *Let  $u \in W(D(A); X)$  then*

$$u \in C([0, T]; D(A^{1/2})).$$

We refer to [20, Proposition 2.1, p. 22 and Theorem 3.1, p. 23]) for the proof.

From Proposition 2 we deduce the following regularity property for the solution of problem (10).

**Corollary 3.** *Let  $u_0 \in D(A^{1/2})$  and  $f \in L^2(0, T; X)$ . Then, the strict solution  $u$  of (10) is such that  $u \in C([0, T]; D(A^{1/2}))$  and there exists  $C_0 > 0$  such that*

$$\sup_{t \in [0, T]} \|u(t)\|_{D(A^{1/2})} \leq C_0 \left( \|f\|_{L^2(0, T; X)} + \|u_0\|_{D(A^{1/2})} \right). \tag{14}$$

Given  $T > 0$ , let  $B : D(B) \subset X \rightarrow X$  be a linear unbounded operator such that

$$D(A^{1/2}) \hookrightarrow D(B), \tag{15}$$

namely  $D(A^{1/2}) \subset D(B)$  and there exists a constant  $C_B > 0$  (that we can suppose, without loss of generality, to be greater than one) such that

$$\|B\varphi\| \leq C_B \|\varphi\|_{D(A^{1/2})}, \quad \forall \varphi \in D(A^{1/2}). \tag{16}$$

In the proposition that follows we show the well-posedness of the bilinear control problem with a source term

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) + f(t) = 0, & t \in [0, T] \\ u(0) = u_0. \end{cases} \tag{17}$$

We introduce the following notation:  $\forall s \geq 0$  we set

$$\begin{aligned} \|\varphi\|_s &:= \|\varphi\|_{D(A^s)}, & \forall \varphi \in D(A^s) \\ \|f\|_{2,s} &:= \|f\|_{L^2(0,T;D(A^s))}, & \forall f \in L^2(0,T;D(A^s)) \\ \|f\|_{\infty,s} &:= \|f\|_{C([0,T];D(A^s))} = \sup_{t \in [0,T]} \|A^s f(t)\|, & \forall f \in C([0,T];D(A^s)). \end{aligned}$$

**Proposition 4.** *Let  $T > 0$  and fix  $p \in L^2(0, T)$ . Then, for all  $u_0 \in D(A^{1/2})$  and  $f \in L^2(0, T; X)$  there exists a unique mild solution  $u$  of (17).*

*Moreover,  $u \in C([0, T], D(A^{1/2}))$  and the following equality holds for every  $t \in [0, T]$*

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} [p(s)Bu(s) + f(s)] ds. \tag{18}$$

*Furthermore,  $u$  is a strict solution of (17) and there exists a constant  $C_1 = C_1(p) > 0$  such that*

$$\|u\|_{\infty,1/2} \leq C_1 (\|u_0\|_{1/2} + \|f\|_{2,0}). \tag{19}$$

Hereafter, we denote by  $C$  a generic positive constant which may differ from line to line. Constants which play a specific role will be distinguished by an index i.e.,  $C_0, C_B, \dots$

**Proof.** The existence and uniqueness of the solution of (17) comes from a fixed point argument which is based on Theorem 1.

For any  $\xi \in C([0, T]; D(A^{1/2}))$ , let us consider the map

$$\Phi(\xi)(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} [p(s)B\xi(s) + f(s)] ds.$$

We want to prove that  $\Phi$  is a contraction. First, we prove that  $\Phi$  maps  $C([0, T]; D(A^{1/2}))$  into itself. Since  $u_0 \in D(A^{1/2})$ ,  $\xi \in C([0, T]; D(A^{1/2}))$  and (15) holds true, then  $p(\cdot)B\xi(\cdot), f(\cdot) \in L^2(0, T; X)$ . Hence, by applying Theorem 1 and Corollary 3 it turns out that  $\Phi(\xi) \in C([0, T]; D(A^{1/2}))$ .

Now we have to prove that  $\Phi$  is a contraction. Thanks to (14), we have

$$\begin{aligned} \|\Phi(\xi) - \Phi(\zeta)\|_{\infty,1/2} &= \sup_{t \in [0,T]} \left\| \int_0^t e^{-(t-s)A} p(s)B(\xi - \zeta)(s) ds \right\|_{1/2} \\ &\leq C_0 \|pB(\xi - \zeta)\|_{2,0} \\ &= C_0 \left( \int_0^T |p(s)|^2 \|B(\xi - \zeta)(s)\|^2 ds \right)^{1/2} \\ &\leq C_0 C_B \left( \int_0^T |p(s)|^2 \|(\xi - \zeta)(s)\|_{1/2}^2 ds \right)^{1/2} \\ &\leq C_0 C_B \|p\|_{L^2(0,T)} \|\xi - \zeta\|_{\infty,1/2} \end{aligned} \tag{20}$$

where we have used the fact that  $D(A^{1/2}) \hookrightarrow D(B)$ . If  $C_0 C_B \|p\|_{L^2(0,T)} < 1$ , then (20) shows that  $\Phi$  is a contraction. If this quantity is larger than one, we subdivide the interval  $[0, T]$  into  $N$  subintervals  $[T_0, T_1], [T_1, T_2], \dots, [T_{N-1}, T_N]$ , with  $T_0 = 0, T_N = T$ , such that  $C_0 C_B \|p\|_{L^2(T_j, T_{j+1})} \leq 1/2$  in  $[T_j, T_{j+1}], \forall j = 0, \dots, N - 1$  and we repeat the contraction argument in each interval.

It remains to prove (19). By using once again (14), we get

$$\begin{aligned}
 \|u\|_{\infty,1/2} &\leq \|u_0\|_{1/2} + \sup_{t \in [0,T]} \left\| \int_0^t e^{-(t-s)A} [p(s)Bu(s) + f(s)] ds \right\|_{1/2} \\
 &\leq \|u_0\|_{1/2} + C_0 \left( \|pBu\|_{2,0} + \|f\|_{2,0} \right) \\
 &= \|u_0\|_{1/2} + C_0 \left( \int_0^T |p(s)|^2 \|Bu(s)\|^2 ds \right)^{1/2} + C_0 \|f\|_{2,0} \\
 &\leq \|u_0\|_{1/2} + C_0 C_B \left( \int_0^T |p(s)|^2 \|u(s)\|_{1/2}^2 ds \right)^{1/2} + C_0 \|f\|_{2,0} \\
 &= \|u_0\|_{1/2} + C_0 C_B \|p\|_{L^2(0,T)} \|u\|_{\infty,1/2} + C_0 \|f\|_{2,0},
 \end{aligned} \tag{21}$$

which implies

$$\left(1 - C_0 C_B \|p\|_{L^2(0,T)}\right) \|u\|_{\infty,1/2} \leq \|u_0\|_{1/2} + C_0 \|f\|_{2,0}.$$

If  $C_0 C_B \|p\|_{L^2(0,T)} \leq 1/2$ , then we have inequality (19). Otherwise, to get the conclusion, we proceed subdividing the interval  $[0, T]$  into smaller subintervals, as explained above.  $\square$

We now consider the following bilinear control problem with a specific source term

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, & t \in [0, T] \\ v(0) = v_0. \end{cases} \tag{22}$$

Let us denote by  $v(\cdot; v_0, p)$  the solution of (22) associated to the initial condition  $v_0$  and control  $p$ . The result that follows provides an estimate of  $\sup_{t \in [0,T]} \|v(t; v_0, p)\|_{1/2}$  in terms of the initial condition  $v_0$ .

**Proposition 5.** *Let  $T > 0$ . Let  $A$  and  $B$  satisfy hypotheses (5) and (15), respectively. Let  $v_0 \in D(A^{1/2})$  and  $p \in L^2(0, T)$  be such that*

$$\|p\|_{L^2(0,T)} \leq N_T \|v_0\| \tag{23}$$

with  $N_T$  a positive constant.

Then, the solution of (22) satisfies

$$\sup_{t \in [0,T]} \|v(t; v_0, p)\|_{1/2}^2 \leq C_{1,1}(T, \|v_0\|_{1/2}) \|v_0\|_{1/2}^2, \tag{24}$$

where

$$\begin{aligned}
 C_{1,1}(T, \|v_0\|_{1/2}) &:= e^{C_B N_T (\frac{3}{2} C_B N_T \|v_0\|_{1/2} + 2\sqrt{T}) \|v_0\|_{1/2} + T} \\
 &\cdot \left( 1 + \frac{5}{2} C_B^2 (1 + \lambda_1^{1/2})^2 N_T^2 + \frac{3}{2} C_B^2 N_T^2 \left( C_B^2 (1 + \lambda_1^{1/2})^2 N_T^2 + 1 \right) \|v_0\|_{1/2}^2 \right) \tag{25}
 \end{aligned}$$

**Proof.** For the sake of compactness, sometimes we will denote the solution of (22), by omitting the reference to the initial condition and the control, as  $v(\cdot)$ . We perform energy estimates of the equation satisfied by  $v$  by taking first the scalar product with  $v(t)$

$$\langle v'(t), v(t) \rangle + \langle Av(t), v(t) \rangle + p(t) \langle Bv(t) + B\varphi_1, v(t) \rangle = 0,$$

from which we have that

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \|A^{1/2} v(t)\|^2 &\leq |p(t)| \|Bv(t)\| \|v(t)\| + |p(t)| \|B\varphi_1\| \|v(t)\| \\
 &\leq C_B |p(t)| \left( \|v(t)\|^2 + \|A^{1/2} v(t)\|^2 \right)^{1/2} \|v(t)\| + C_B (1 + \lambda_1)^{1/2} |p(t)| \|v(t)\| \\
 &\leq C_B |p(t)| \left( \|v(t)\| + \|A^{1/2} v(t)\| \right) \|v(t)\| + C_B (1 + \lambda_1^{1/2}) |p(t)| \|v(t)\| \\
 &\leq C_B |p(t)| \|v(t)\|^2 + \frac{1}{2} \|A^{1/2} v(t)\|^2 + \frac{C_B^2}{2} |p(t)|^2 \|v(t)\|^2 + \frac{C_B^2}{2} (1 + \lambda_1^{1/2})^2 |p(t)|^2 + \frac{1}{2} \|v(t)\|^2
 \end{aligned} \tag{26}$$



where we used (9) twice to get the third line of (26). Therefore, from the above inequality it follows that

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 \leq \left( C_B |p(t)| + \frac{C_B^2}{2} |p(t)|^2 + \frac{1}{2} \right) \|v(t)\|^2 + \frac{C_B^2}{2} (1 + \lambda_1^{1/2})^2 |p(t)|^2.$$

We integrate from 0 to  $t$  and thanks to Gronwall's Lemma we deduce that

$$\sup_{t \in [0, T]} \|v(t; v_0, p)\|^2 \leq \left( \|v_0\|^2 + C_B^2 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^2 \right) e^{C_2(T)}, \quad (27)$$

where  $C_2(T) := 2C_B \sqrt{T} \|p\|_{L^2(0, T)} + C_B^2 \|p\|_{L^2(0, T)}^2 + T$ .

Now, we multiply the equation in (22) by  $v'(t)$  and we obtain

$$\langle v'(t), v'(t) \rangle + \langle Av(t), v'(t) \rangle + p(t) \langle Bv(t) + B\varphi_1, v'(t) \rangle = 0.$$

We recall that under our assumptions on  $A$  the function

$$t \mapsto \langle Av(t), v(t) \rangle$$

is absolutely continuous on  $[0, T]$ . Thus, it is differentiable almost everywhere on  $[0, T]$ , its derivative is Lebesgue integrable on  $[0, T]$  and we have

$$\frac{d}{dt} \langle Av(t), v(t) \rangle = 2 \langle Av(t), v'(t) \rangle, \quad \text{a.e. on } [0, T]$$

Therefore, we have for almost every  $t \in [0, T]$

$$\begin{aligned} \|v'(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|A^{1/2} v(t)\|^2 &\leq |p(t)| \|Bv(t)\| \|v'(t)\| + |p(t)| \|B\varphi_1\| \|v'(t)\| \\ &\leq C_B |p(t)| (\|v(t)\| + \|A^{1/2} v(t)\|) \|v'(t)\| + C_B |p(t)| (1 + \lambda_1^{1/2}) \|v'(t)\| \\ &\leq \frac{3}{4} C_B^2 |p(t)|^2 \|v(t)\|^2 + \frac{1}{3} \|v'(t)\|^2 + \frac{3}{4} C_B^2 |p(t)|^2 \|A^{1/2} v(t)\|^2 \\ &\quad + \frac{1}{3} \|v'(t)\|^2 + \frac{3}{4} C_B^2 (1 + \lambda_1^{1/2})^2 |p(t)|^2 + \frac{1}{3} \|v'(t)\|^2, \end{aligned}$$

that gives for almost every  $t \in [0, T]$

$$\frac{d}{dt} \|A^{1/2} v(t)\|^2 \leq \left( \frac{3}{2} C_B^2 |p(t)|^2 \|v(t)\|^2 + \frac{3}{2} C_B^2 (1 + \lambda_1^{1/2})^2 |p(t)|^2 \right) + \frac{3}{2} C_B^2 |p(t)|^2 \|A^{1/2} v(t)\|^2.$$

By Gronwall's Lemma and using the previous energy estimate (27), we deduce that for all  $t \in [0, T]$

$$\begin{aligned} \|A^{1/2} v(t)\|^2 &\leq \left( \|A^{1/2} v_0\|^2 + \frac{3}{2} C_B^2 \int_0^t |p(s)|^2 \|v(s)\|^2 ds + \frac{3}{2} C_B^2 (1 + \lambda_1^{1/2})^2 \int_0^t |p(s)|^2 ds \right) e^{\frac{3}{2} C_B^2 \int_0^t |p(s)|^2 ds} \\ &\leq \left( \|A^{1/2} v_0\|^2 + \frac{3}{2} C_B^2 \|p\|_{L^2(0, T)}^2 \sup_{t \in [0, T]} \|v(t)\|^2 + \frac{3}{2} C_B^2 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^2 \right) e^{\frac{3}{2} C_B^2 \|p\|_{L^2(0, T)}^2} \\ &\leq \left( \|A^{1/2} v_0\|^2 + \frac{3}{2} C_B^2 \|p\|_{L^2(0, T)}^2 \left( \|v_0\|^2 + C_B^2 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^2 \right) e^{C_2(T)} \right. \\ &\quad \left. + \frac{3}{2} C_B^2 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^2 \right) e^{\frac{3}{2} C_B^2 \|p\|_{L^2(0, T)}^2} \\ &\leq \left( \|A^{1/2} v_0\|^2 + \frac{3}{2} C_B^2 \|p\|_{L^2(0, T)}^2 \|v_0\|^2 + \frac{3}{2} C_B^4 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^4 \right. \\ &\quad \left. + \frac{3}{2} C_B^2 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^2 \right) e^{C_3(T)} \end{aligned}$$

with  $C_3(T) := \frac{3}{2} C_B^2 \|p\|_{L^2(0, T)}^2 + C_2(T)$ .

Taking the supremum over the interval  $[0, T]$ , we have that

$$\sup_{t \in [0, T]} \|A^{1/2} v(t)\|^2 \leq \left( \|A^{1/2} v_0\|^2 + \frac{3}{2} C_B^2 \|p\|_{L^2(0, T)}^2 \|v_0\|^2 + \frac{3}{2} C_B^4 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^4 + \frac{3}{2} C_B^2 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^2 \right) e^{C_3(T)}. \quad (28)$$

Finally, combining (27) and (28), we find that

$$\begin{aligned} \sup_{t \in [0, T]} \|v(t)\|_{1/2}^2 &\leq \sup_{t \in [0, T]} \|v(t)\|^2 + \sup_{t \in [0, T]} \|A^{1/2} v(t)\|^2 \\ &\leq e^{C_3(T)} \left( \|v_0\|_{1/2}^2 + \frac{5}{2} C_B^2 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^2 + \frac{3}{2} C_B^2 \|p\|_{L^2(0, T)}^2 \|v_0\|^2 + \frac{3}{2} C_B^4 (1 + \lambda_1^{1/2})^2 \|p\|_{L^2(0, T)}^4 \right) \end{aligned}$$

and using estimate (23) we conclude that

$$\sup_{t \in [0, T]} \|v(t)\|_{1/2}^2 \leq e^{C_3(T)} \left( 1 + \frac{5}{2} C_B^2 (1 + \lambda_1^{1/2})^2 N_T^2 + \frac{3}{2} C_B^2 N_T^2 (C_B^2 (1 + \lambda_1^{1/2})^2 N_T^2 + 1) \|v_0\|_{1/2}^2 \right) \|v_0\|_{1/2}^2.$$

Thus, we get (24) with  $C_{1,1}(T, \|v_0\|)$  defined in (25). □

We now turn our attention to the following control problem

$$\begin{cases} w'(t) + Aw(t) + p(t)Bv(t) = 0, & t \in [0, T] \\ w(0) = 0 \end{cases} \quad (29)$$

where  $v$  solves (22). Observe that  $p(\cdot)Bv(\cdot) \in L^2(0, T; X)$  so, thanks to Corollary 3, there exists a unique strict solution of (29) and it also belongs to  $C([0, T]; D(A^{1/2}))$ . We will denote by  $w(\cdot; 0, p)$  the solution of (29). In the result that follows we give a quadratic estimate of  $w(\cdot; 0, p)$  in terms of the initial condition of the problem solved by  $v$ .

**Proposition 6.** *Let  $T > 0$  and let  $A$  and  $B$  satisfy hypotheses (5) and (15), respectively. Let  $p \in L^2(0, T)$  satisfy (23) with  $N_T =: N(T)$  and  $v_0 \in D(A^{1/2})$  be such that*

$$N(T) \|v_0\|_{1/2} \leq 1. \quad (30)$$

Then, it holds that

$$\|w(T; 0, p)\|_{1/2} \leq K(T) \|v_0\|_{1/2}^2, \quad (31)$$

where

$$K(T)^2 := 2e^{C_4(T)} C_B^2 N(T)^2 C_5(T) \quad (32)$$

with

$$C_4(T) := C_B \left( \frac{5}{2} C_B + 2\sqrt{T} \right) + 2T,$$

$$C_5(T) = 1 + \frac{5}{2} C_B^2 (1 + \lambda_1^{1/2})^2 N(T)^2 + \frac{3}{2} C_B^2 (C_B^2 (1 + \lambda_1^{1/2})^2 N(T)^2 + 1).$$

**Proof.** For the sake of compactness, sometimes we will denote the solution of (29) by  $w(\cdot)$ . First, we multiply the equation in (29) by  $w(t)$ , and we get

$$\langle w'(t), w(t) \rangle + \langle Aw(t), w(t) \rangle + p(t) \langle Bv(t), w(t) \rangle = 0,$$

that implies

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w(t)\|^2 &\leq |p(t)| \|Bv(t)\| \|w(t)\| \\ &\leq C_B |p(t)| \|v(t)\|_{1/2} \|w(t)\| \\ &\leq \frac{C_B^2}{2} |p(t)|^2 \|v(t)\|_{1/2}^2 + \frac{1}{2} \|w(t)\|^2. \end{aligned}$$

Applying Gronwall's Lemma we obtain

$$\|w(t)\|^2 \leq C_B^2 e^t \int_0^t |p(s)|^2 \|v(s)\|_{1/2}^2 ds,$$

and therefore

$$\sup_{t \in [0, T]} \|w(t)\|^2 \leq C_B^2 e^T \|p\|_{L^2(0, T)}^2 \sup_{t \in [0, T]} \|v(t)\|_{1/2}^2. \tag{33}$$

Now, we multiply the equation in (29) by  $w'(t)$ ,

$$\langle w'(t), w'(t) \rangle + \langle Aw(t), w'(t) \rangle + p(t) \langle Bv(t), w'(t) \rangle = 0,$$

from which we deduce

$$\begin{aligned} \|w'(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|A^{1/2} w(t)\|^2 &\leq |p(t)| \|Bv(t)\| \|w'(t)\| \\ &\leq C_B |p(t)| \|v(t)\|_{1/2} \|w'(t)\| \\ &\leq \frac{C_B^2}{2} |p(t)|^2 \|v(t)\|_{1/2}^2 + \frac{1}{2} \|w'(t)\|^2. \end{aligned}$$

Therefore, it holds that

$$\frac{d}{dt} \|A^{1/2} w(t)\|^2 \leq C_B^2 |p(t)|^2 \|v(t)\|_{1/2}^2,$$

that yields the following estimate

$$\sup_{t \in [0, T]} \|A^{1/2} w(t)\|^2 \leq C_B^2 \|p\|_{L^2(0, T)}^2 \sup_{t \in [0, T]} \|v(t)\|_{1/2}^2. \tag{34}$$

Combining (33) and (34) we have

$$\begin{aligned} \sup_{t \in [0, T]} \|w(t)\|_{1/2}^2 &\leq \sup_{t \in [0, T]} \|w(t)\|^2 + \sup_{t \in [0, T]} \|A^{1/2} w(t)\|^2 \\ &\leq 2e^T C_B^2 \|p\|_{L^2(0, T)}^2 \sup_{t \in [0, T]} \|v(t)\|_{1/2}^2 \end{aligned}$$

and thanks to the estimates of  $\sup_{t \in [0, T]} \|v(t)\|_{1/2}^2$  given by (24), (25) and of  $\|p\|_{L^2(0, T)}$  given by (23), we deduce that

$$\sup_{t \in [0, T]} \|w(t)\|_{1/2}^2 \leq 2e^T C_B^2 C_{1,1}(T, \|v_0\|_{1/2}) N(T)^2 \|v_0\|_{1/2}^4. \tag{35}$$

Finally, thanks to hypothesis (30), we obtain the claim. □

### 3. Main result

Let  $T > 0$ . In a separable Hilbert space  $(X, \langle \cdot, \cdot \rangle, \|\cdot\|)$ , we consider the control problem

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0 \in D(A^{1/2}) \end{cases} \tag{36}$$

where  $A : D(A) \subset X \rightarrow X$  is a densely defined linear operator satisfying (5),  $B : D(B) \subset X \rightarrow X$  is an unbounded linear operator which satisfies (15) and  $p \in L^2(0, T)$  is a bilinear control. We denote by  $u(\cdot; u_0, p)$  the solution of (36) associated to the initial condition  $u_0$  and control  $p$ . We call the

“ground state solution” of problem (36) the function  $\psi_1(t) = e^{-\lambda_1 t} \varphi_1$ , where  $\lambda_1 \geq 0$  is the first eigenvalue of  $A$  and  $\varphi_1$  is the associated eigenfunction.

For any  $0 \leq s_0 \leq s_1$  we also introduce the linear control problem

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_1 = 0 & t \in [s_0, s_1] \\ y(s_0) = y_0 \end{cases} \tag{37}$$

and we denote by  $y(\cdot; y_0, s_0, p)$  its solution associated to the initial condition  $y_0$ , attained at time  $s_0$ , and control  $p$ .

We recall that the pair  $\{A, B\}$  is called 1-null controllable in time  $T$  if there exists a constant  $N_T > 0$  such that for any  $y_0 \in X$  there exists a control  $p \in L^2(0, T)$  with

$$\|p\|_{L^2(0,T)} \leq N_T \|y_0\|$$

such that  $y(T; y_0, 0, p) = 0$ . If  $\{A, B\}$  is 1-null controllable in time  $T$ , we call the constant  $N(T)$  defined in (4) the control cost.

We now state our main controllability result.

**Theorem 7.** *Let  $A : D(A) \subset X \rightarrow X$  be a densely defined linear operator that satisfies (5). Let  $B : D(B) \subset X \rightarrow X$  be a linear unbounded operator such that (15) holds. Let  $\{A, B\}$  be 1-null controllable in any  $T > 0$  with control cost  $N(\cdot)$  such that there exist  $\nu, T_0 > 0$  for which*

$$N(\tau) \leq e^{\nu/\tau}, \quad \forall 0 < \tau \leq T_0. \tag{38}$$

*Then, for any  $T > 0$ , there exists a constant  $R_T > 0$  such that, for any  $u_0 \in B_{R_T, 1/2}(\varphi_1)$ , there exists a control  $p \in L^2(0, T)$  for which system (36) is locally controllable to the ground state solution in time  $T$ , that is,  $u(T; u_0, p) = \psi_1(T)$ .*

*Moreover, the following estimate holds*

$$\|p\|_{L^2(0,T)} \leq \frac{e^{-\pi^2 \Gamma_0 / T}}{e^{2\pi^2 \Gamma_0 / (3T)} - 1}, \tag{39}$$

where

$$\begin{aligned} \Gamma_0 &:= 2\nu + \max\{\log D, 0\} \\ R_T &:= e^{-6\Gamma_0 / T_1} \end{aligned} \tag{40}$$

with

$$\begin{aligned} D &:= 2\sqrt{2}C_B e^{C_B(\frac{5}{4}C_B+1)+1} \left( \max \left\{ 1 + \frac{3}{2}C_B^2, \frac{C_B^2}{2} (1 + \lambda_1^{1/2})^2 (5 + 3C_B^2) \right\} \right)^{1/2} \\ T_1 &= \min \left\{ \frac{6}{\pi^2} T, 1, T_0 \right\}. \end{aligned} \tag{41}$$

**Remark 8.** Theorem 7 requires assumptions (5) and, in particular, it requires that all the eigenvalues of  $A$  are nonnegative. We shall show how assumptions (5) can be weakened in order to include the cases when  $A$  admits negative eigenvalues. Let us thus assume that  $A : D(A) \subset X \rightarrow X$  satisfies the following weakened assumptions:

- (a)  $A$  is self-adjoint,
- (b')  $\langle Ax, x \rangle \geq -\sigma \|x\|^2, \forall x \in D(A)$ ,
- (c')  $\exists \lambda > \sigma : (\lambda I + A)^{-1} : X \rightarrow X$  is compact.

Note that this situation has to be considered only if the smallest eigenvalue  $\lambda_1$  of  $A$  is strictly negative. For such a situation the best constant  $\sigma$  in (42), is  $\sigma = -\lambda_1$ . We claim that if the smallest eigenvalue of  $A$  is strictly negative, we can still prove a local exact controllability result with the appropriate changes in the assumptions. In Theorem 7 we replace (15) by the assumption  $D((A + \sigma I)^{\frac{1}{2}}) \hookrightarrow D(B)$ , and (5) by assumption (42).

Let us show how to proceed to derive a local exact controllability result to the ground state. We assume that the smallest eigenvalue of  $A$  is strictly negative and that (42) holds. We denote by  $\{\lambda_k\}_{k \in \mathbb{N}^*}$  and  $\{\varphi_k\}_{k \in \mathbb{N}^*}$  the eigenvalues and the corresponding eigenfunctions of  $A$ . Without loss of generality, we can assume the eigenvalues ordered as  $\lambda_k \leq \lambda_{k+1}$  for every  $k \in \mathbb{N}^*$ .

It is easy to check that  $\sigma$  can be supposed to satisfy in addition

$$\sigma = \sup_{x \in D(A), x \neq 0} \left( \frac{\langle (-A)x, x \rangle}{\|x\|^2} \right),$$

so that  $\sigma = -\lambda_1$ . Hence, we assume from now on, without loss of generality, that  $\sigma = -\lambda_1$ . The ground state solution associated to  $A$  is  $\psi_1(t) = e^{-\lambda_1 t} \varphi_1 = e^{\sigma t} \varphi_1$ .

We now introduce the operator  $A_\sigma = A + \sigma I$ . Then it is easy to show that  $A_\sigma$  satisfies hypothesis (5). One can write  $((\lambda - \sigma)I + A_\sigma)^{-1} = (\lambda + A)^{-1}$ , so that  $A_\sigma$  satisfies (c) of (5) with  $\mu = \lambda - \sigma > 0$ . The eigenvalues of  $A_\sigma$  are given by  $\mu_k = \lambda_k + \sigma$  and the associated eigenfunctions are the same of those of  $A$ . The ground state solution associated to  $A_\sigma$  is therefore  $\psi_1^\sigma(t) = e^{-\mu_1 t} \varphi_1 = \varphi_1$ .

By the change of variables  $z(t; u_0, p) = e^{-\sigma t} u(t; u_0, p)$ , one has that  $z$  solves the following problem

$$\begin{cases} z'(t) + A_\sigma z(t) + p(t)Bz(t) = 0 \\ z(0) = u_0. \end{cases} \tag{43}$$

Since by hypothesis  $D((A_\sigma)^{\frac{1}{2}}) \hookrightarrow D(B)$ , we can then apply Theorem 7 and deduce that the solution of (43) is controllable in time  $T$  to the first eigensolution  $\psi_1^\sigma$ . Finally, from this latter result we get

$$u(T) - \psi_1(T) = e^{\sigma T} z(T) - e^{\sigma T} \varphi_1 = e^{\sigma T} (z(T) - \varphi_1) = e^{\sigma T} (z(T) - \psi_1^\sigma(T)) = 0$$

which implies the controllability of  $u$  to the ground state solution  $\psi_1$ .

### 3.1. Proof of Theorem 7

Our aim is to show the local controllability of system (36) to the ground state solution  $\psi_1(t) = e^{-\lambda_1 t} \varphi_1$ , that is the solution of (36) when  $p = 0$  and  $u_0 = \varphi_1$ . We recall that  $\{\varphi_k\}_{k \in \mathbb{N}^*}$  is a basis of  $X$  of orthonormal eigenfunctions of the operator  $A$ , associated to the eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}^*}$ :  $A\varphi_k = \lambda_k \varphi_k$  for all  $k \in \mathbb{N}^*$ . The proof of Theorem 7 is divided into two parts: we first consider the case  $\lambda_1 = 0$  and prove the controllability result to the corresponding stationary eigensolution  $\psi_1(t) \equiv \varphi_1$ . Then, we recover the result for the case  $\lambda_1 > 0$  in the second part.

#### 3.1.1. Case $\lambda_1 = 0$

We define the constant

$$T_f := \min \left\{ T, \frac{\pi^2}{6}, \frac{\pi^2}{6} T_0 \right\}, \tag{44}$$

where  $T > 0$  and  $T_0$  is defined in (38). In what follows we construct a control  $p \in L^2(0, T_f)$  which drives the solution of (36) to  $\psi_1 \equiv \varphi_1$  in time  $T_f$ .

Set

$$T_1 := \frac{6}{\pi^2} T_f. \tag{45}$$

It is easy to see that  $0 < T_1 \leq 1$ . We now define the sequence  $\{T_j\}_{j \in \mathbb{N}^*}$  by

$$T_j := \frac{T_1}{j^2}, \tag{46}$$

and the time steps

$$\tau_n = \sum_{j=1}^n T_j, \quad \forall n \in \mathbb{N}, \tag{47}$$

with the convention that  $\sum_{j=1}^0 T_j = 0$ . Notice that  $\sum_{j=1}^\infty T_j = \frac{\pi^2}{6} T_1 = T_f$ .

Let  $v := u - \varphi_1$ . Then, for any  $0 \leq s_0 \leq s_1$ ,  $v$  solves the following bilinear control problem

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0 & t \in [s_0, s_1] \\ v(s_0) = v_0 \end{cases} \tag{48}$$

with  $v_0 = u(s_0) - \varphi_1$ . We denote by  $v(\cdot; v_0, s_0, p)$  the solution of (48) associated to the initial condition  $v_0$  attained at  $s_0$  and control  $p$ . Observe that showing local controllability of  $u$  to  $\varphi_1$  in time  $T_f$  is equivalent to prove local null controllability for the solution  $v$  of (48):  $v(T_f; u_0 - \varphi_1, 0, p) = 0$ .

We follow the strategy of the proof of [4, Theorem 1.1] which consists first of obtaining an estimate of the solution of (48) at  $T_1$  by the square of the norm of the initial condition thanks to the construction of a suitable control  $p_1 \in L^2(0, T_1)$ . Then, the same procedure is iterated in consecutive time steps  $[\tau_{n-1}, \tau_n]$  in which we build a control  $p_n \in L^2(\tau_{n-1}, \tau_n)$  such that, setting

$$\begin{aligned} q_n &:= \sum_{j=1}^n p_j(t) \chi_{[\tau_{j-1}, \tau_j]}(t), \\ v_n &:= v(\tau_n; v_0, 0, q_n), \end{aligned} \tag{49}$$

where  $\chi_A(\cdot)$  denotes the indicator function of the set  $A$ , it holds that

$$\begin{aligned} 1. & \|p_n\|_{L^2(\tau_{n-1}, \tau_n)} \leq N(T_n) \|v_{n-1}\|, \\ 2. & y(\tau_n; v_{n-1}, \tau_{n-1}, p_n) = 0, \\ 3. & \|v(\tau_n; v_{n-1}, \tau_{n-1}, p_n)\|_{1/2} \leq e^{(\sum_{j=1}^n 2^{n-j} j^2 - 2^n 6)\Gamma_0/T_1}, \\ 4. & \|v(\tau_n; v_{n-1}, \tau_{n-1}, p_n)\|_{1/2} \leq \prod_{j=1}^n K(T_j)^{2^{n-j}} \|v_0\|_{1/2}^2, \end{aligned} \tag{50}$$

where  $y(\cdot; v_{n-1}, \tau_{n-1}, p_n)$  is the solution of (37) in  $[\tau_{n-1}, \tau_n]$  with initial condition  $v_{n-1}$  and control  $p_n$ . We recall that  $K(\cdot)$  is defined in (32).

**First step:** we consider problem (48) with  $[s_0, s_1] = [0, T_1]$ . Since by hypothesis the pair  $\{A, B\}$  is 1-null controllable in any time  $T > 0$ , then for any  $v_0 \in D(A^{1/2})$  there exists a control  $p_1 \in L^2(0, T_1)$  such that

$$\|p_1\|_{L^2(0, T_1)} \leq N(T_1) \|v_0\| \quad \text{and} \quad y(T_1; v_0, 0, p_1) = 0$$

with  $N(\cdot)$  the control cost defined in (38) and  $y(\cdot; v_0, 0, p_1)$  the solution of (37) in  $[s_0, s_1] = [0, T_1]$ . Therefore, 1. and 2. of (50) are satisfied. We apply now Proposition 5 with  $N_T := N(T_1)$  and we obtain

$$\sup_{t \in [0, T_1]} \|v(t)\|_{1/2}^2 \leq C_{1,1}(T_1, \|v_0\|_{1/2}) \|v_0\|_{1/2}^2$$

with  $C_{1,1}$  defined in (25).

In order to prove 3. and 4. of (50) we introduce the function  $w(t) := v(t; v_0, 0, p_1) - y(t; v_0, 0, p_1)$ . It is easy to see that  $w$  solves (29) with  $T = T_1$  and  $p = p_1$ . We then apply Proposition 6 and we deduce that if

$$N(T_1) \|v_0\|_{1/2} \leq 1 \tag{51}$$

then

$$\|w(T_1; 0, p_1)\|_{1/2} = \|v(T_1; v_0, 0, p_1)\|_{1/2} \leq K(T_1) \|v_0\|_{1/2}^2 \tag{52}$$

where  $K(\cdot)$  is defined in (32). Observe that, thanks to the definition of  $T_1$  and to (38), we infer that there exists a constant  $\Gamma_0 > \nu$  such that

$$K(\tau) \leq e^{\Gamma_0/\tau}, \quad 0 < \tau \leq T_1. \tag{53}$$

Observe that, a possible choice of  $\Gamma_0$  is given by (40).

Let  $R_T = e^{-6\Gamma_0/T_1}$  with  $T_1$  defined as in (45). Let us prove that if  $v_0 \in B_{R_T,1/2}(0)$  then (51) is satisfied:

$$N(T_1) \|v_0\| \leq N(T_1) \|v_0\|_{1/2} \leq e^{\nu/T_1} e^{-6\Gamma_0/T_1} \leq e^{-5\Gamma_0/T_1} \leq 1$$

where we have used that  $\Gamma_0 > \nu$ , so that (52) holds.

Thus, from (52) we deduce that

$$\|v(T_1; v_0; 0, p_1)\|_{1/2} \leq K(T_1) \|v_0\|_{1/2}^2 \leq e^{\Gamma_0/T_1} e^{-12\Gamma_0/T_1} = e^{-11\Gamma_0/T_1}$$

which proves 3. and 4. of (50).

**Iterative step:** for the sake of completeness we report the proof of [4, Section 3.1.2] adapted to the current choice of functional setting.

Suppose that for every  $j = 1, \dots, n-1$  we have built controls  $p_j \in L^2(\tau_{j-1}, \tau_j)$  such that (50) is satisfied. At the step  $n-1$  we have construct  $p_{n-1} \in L^2(\tau_{n-2}, \tau_{n-1})$  such that

1.  $\|p_{n-1}\|_{L^2(\tau_{n-2}, \tau_{n-1})} \leq N(T_{n-1}) \|v_{n-2}\|,$
2.  $y(\tau_{n-1}; v_{n-2}, \tau_{n-2}, p_{n-1}) = 0,$
3.  $\|v(\tau_{n-1}; v_{n-2}, \tau_{n-2}, p_{n-1})\|_{1/2} \leq e^{\left(\sum_{j=1}^{n-1} 2^{n-1-j} j^2 - 2^{n-1} 6\right)\Gamma_0/T_1},$  (54)
4.  $\|v(\tau_{n-1}; v_{n-2}, \tau_{n-2}, p_{n-1})\|_{1/2} \leq \prod_{j=1}^{n-1} K(T_j) 2^{n-1-j} \|v_0\|_{1/2}^{2^{n-1}}.$

Let us prove that there exists  $p_n \in L^2(\tau_{n-1}, \tau_n)$  which verifies (50). First, we define  $q_{n-1}$  and  $v_{n-1}$  as in (49). Consider then the problem

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0 & t \in [\tau_{n-1}, \tau_n] \\ v(\tau_{n-1}) = v_{n-1} \end{cases} \tag{55}$$

where the control  $p$  has to be properly chosen. We apply the change of variables  $s = t - \tau_{n-1}$  so that we shift the problem into the interval  $[0, T_n]$ . By introducing the new variables  $\tilde{v}(s) = v(s + \tau_{n-1})$  and  $\tilde{p}(t) = p(s + \tau_{n-1})$ , problem (55) becomes

$$\begin{cases} \tilde{v}'(t) + A\tilde{v}(t) + \tilde{p}(t)B\tilde{v}(t) + \tilde{p}(t)B\varphi_1 = 0 & t \in [0, T_n] \\ \tilde{v}(0) = v_{n-1}. \end{cases} \tag{56}$$

Since  $\{A, B\}$  is 1-null controllable in any positive time, there exists  $\tilde{p}_n \in L^2(0, T_n)$  such that

$$\|\tilde{p}_n\|_{L^2(0, T_n)} \leq N(T_n) \|v_{n-1}\| \quad \text{and} \quad \tilde{y}(T_n; v_{n-1}, 0, \tilde{p}_n) = 0$$

with  $\tilde{y}(\cdot; v_{n-1}, 0, \tilde{p}_n)$  solves (37) on  $[0, T_n]$ . Moreover, since  $v_{n-1} = v(\tau_{n-1}; v_0, 0, q_{n-1}) = v(\tau_{n-1}; v_{n-2}, \tau_{n-2}, p_{n-1})$ , we deduce that

$$\begin{aligned} N(T_n) \|v_{n-1}\|_{1/2} &\leq e^{\nu n^2/T_1} e^{\left(\sum_{j=1}^{n-1} 2^{n-1-j} j^2 - 2^{n-1} 6\right)\Gamma_0/T_1} \\ &\leq e^{(n^2 + (-(n-1)^2 - 4(n-1) + 2^{n-1} 6 - 6 - 2^{n-1} 6))\Gamma_0/T_1} \\ &= e^{-(2n+3)\Gamma_0/T_1} \leq 1 \end{aligned} \tag{57}$$

from 3. of (54). Observe that we have used that  $\nu \leq \Gamma_0$  and the identity

$$\sum_{j=0}^n \frac{j^2}{2^j} = 2^{-n} (-n^2 - 4n + 6(2^n - 1)), \quad n \geq 0$$

that can be proved by induction. We now choose  $\tilde{p} = \tilde{p}_n$  in (56) and we keep denoting by  $\tilde{v}$  the corresponding solution. Define  $w = \tilde{v} - \tilde{y}$  and observe that  $w$  solves (29) with  $v = \tilde{v}$ ,  $T = T_n$  and control  $p = \tilde{p}_n$ . Thanks to (57), we can apply Proposition 6 with  $T = T_n$  and we deduce that

$$\|w(T_n; 0, \tilde{p}_n)\|_{1/2} = \|\tilde{v}(T_n; v_{n-1}, 0, \tilde{p}_n)\|_{1/2} \leq K(T_n) \|v_{n-1}\|_{1/2}^2.$$

We define  $p_n(t) := \tilde{p}_n(t - \tau_{n-1})$ . By shifting back the problem into the interval  $[\tau_{n-1}, \tau_n]$  we obtain

$$\|p_n\|_{L^2(\tau_{n-1}, \tau_n)} \leq N(T_n) \|v_{n-1}\| \quad \text{and} \quad y(\tau_n; v_{n-1}, \tau_{n-1}, p_n) = 0$$

and

$$\|v(\tau_n; v_{n-1}, \tau_{n-1}, p_n)\|_{1/2} \leq K(T_n) \|v_{n-1}\|_{1/2}^2. \tag{58}$$

Thus, the first two items of (50) are proved. Using 3. of (54) and (53), we deduce that

$$\begin{aligned} \|v(\tau_n; v_{n-1}, \tau_{n-1}, p_n)\|_{1/2} &\leq e^{\Gamma_0 n^2 / T_1} \left[ e^{(\sum_{j=1}^{n-1} 2^{n-1-j} j^2 - 2^{n-1} 6) \Gamma_0 / T_1} \right]^2 \\ &= e^{(\sum_{j=1}^n 2^{n-j} j^2 - 2^n 6) \Gamma_0 / T_1}. \end{aligned}$$

Therefore, item 3. of (50) is verified. Finally, using again (58) and 4. of (54) we get

$$= \prod_{j=1}^n K(T_j)^{2^{n-j}} \|v_0\|_{1/2}^{2^n}.$$

The induction argument is then concluded.

We can now complete the proof of our theorem for the case  $\lambda_1 = 0$ . Notice that for all  $n \in \mathbb{N}^*$  we have that

$$\begin{aligned} \|v(\tau_n; v_{n-1}, \tau_{n-1}, p_n)\|_{1/2} &\leq \prod_{j=1}^n K(T_j)^{2^{n-j}} \|v_0\|_{1/2}^{2^n} \leq \prod_{j=1}^n \left( e^{\Gamma_0 j^2 / T_1} \right)^{2^{n-j}} \|v_0\|_{1/2}^{2^n} \\ &\leq e^{\Gamma_0 2^n / T_1 \sum_{j=1}^n j^2 / 2^j} \|v_0\|_{1/2}^{2^n} \leq \left( e^{6\Gamma_0 / T_1} \|v_0\|_{1/2} \right)^{2^n}. \end{aligned}$$

The above inequality is equivalent to

$$\|v(\tau_n; v_0, 0, q_n)\|_{1/2} \leq \left( e^{6\Gamma_0 / T_1} \|v_0\|_{1/2} \right)^{2^n}$$

where  $q_n$  is defined in (49).

Taking the limit as  $n \rightarrow +\infty$  we deduce that

$$\|u(T_f; u_0, q_\infty) - \varphi_1\|_{1/2} = \|v(T_f; v_0, 0, q_\infty)\|_{1/2} \leq 0$$

where  $T_f$  is defined in (44) and  $q_\infty(t) = \sum_{n=1}^\infty p_n(t) \chi_{[\tau_{n-1}, \tau_n]}(t)$ . Indeed, by hypothesis  $u_0 \in B_{R_T, 1/2}(\varphi_1)$ , with  $R_T$  defined in (40), and so  $\|v_0\|_{1/2} < e^{-6\Gamma_0 / T_1}$ . This means that we have construct a control  $p \in L^2_{loc}([0, +\infty))$ , defined as follows

$$p(t) = \begin{cases} \sum_{n=1}^\infty p_n(t) \chi_{[\tau_{n-1}, \tau_n]}(t) & t \in (0, T_f] \\ 0 & t > T_f, \end{cases}$$

that steers the solution  $u$  of (36) to the ground state solution in time  $T_f$ , less or equal to  $T$ .

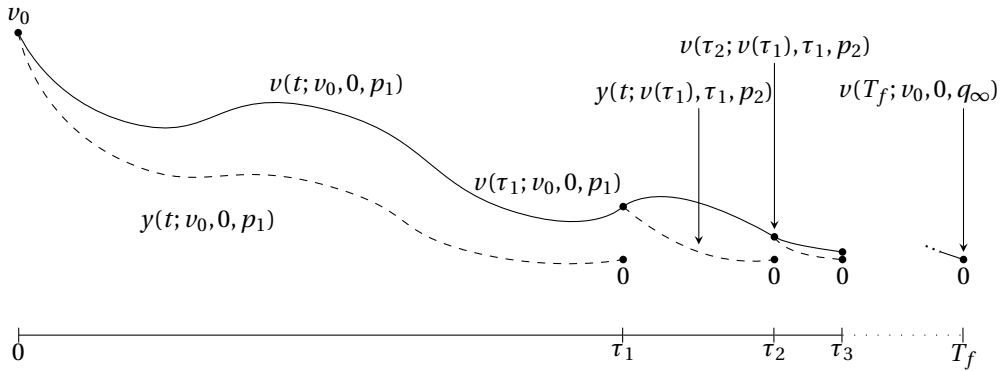
Moreover, we can give an upper bound of the  $L^2$ -norm of the control:

$$\begin{aligned} \|p\|_{L^2(0, T)}^2 &= \sum_{n=1}^\infty \|p_n\|_{L^2(\tau_{n-1}, \tau_n)}^2 \leq \sum_{n=1}^\infty (N(T_n) \|v_{n-1}\|_{1/2})^2 \\ &\leq \sum_{n=1}^\infty e^{-2(2n+3)\Gamma_0 / T_1} \leq \frac{e^{-6\Gamma_0 / T_1}}{e^{4\Gamma_0 / T_1} - 1} = \frac{e^{-\pi^2 \Gamma_0 / T_f}}{e^{2\pi^2 \Gamma_0 / 3T_f} - 1} \end{aligned}$$

where we have used 1. of (50) and (57). Recalling that  $T_f \leq T$ , we obtain (39).

In the figure below we have sketched a drawing representing the iterative process we have just described in the proof





3.1.2. Case  $\lambda_1 > 0$

If  $\lambda_1 > 0$  the result is easily deducible from the previous case. Indeed, we introduce the operator

$$A_1 := A - \lambda_1 I.$$

Observe that

- $A_1$  satisfies (5),
- $A_1$  has the same eigenfunctions of  $A$ ,  $\{\varphi_j\}_{j \in \mathbb{N}^*}$ , and the first eigenvalue of  $A_1$  is equal to  $\mu_1 = \lambda_1 - \lambda_1 = 0$ ,
- $\{A_1, B\}$  is 1-null controllable with associated control cost  $N_1(\cdot)$  that satisfies (38).

Hence, once proved the exact controllability in time  $T$  of the following problem

$$\begin{cases} z'(t) + A_1 z(t) + p(t)Bz(t) = 0, & t \in [0, T] \\ z(0) = u_0 \end{cases}$$

to the associated ground state solution  $\widetilde{\psi}_1 = e^{-\mu_1 t} \varphi_1 = \varphi_1$ , we introduce the function  $u(t) := z(t)e^{-\lambda_1 t}$  that is the solution of

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{59}$$

and satisfies

$$\|u(T) - \psi_1(T)\| = \|e^{-\lambda_1 T} z(T) - e^{-\lambda_1 T} \varphi_1\| = e^{-\lambda_1 T} \|z(T) - \varphi_1\| = 0.$$

Therefore, we have shown that (59) is exactly controllable to the ground state solution  $\psi_1(t) = e^{-\lambda_1 t} \varphi_1$  in time  $T$ .

4. Semi-global results

In this section we present two semi-global results for the exact controllability to the ground state solution of problem (36). In the first one, Theorem 9, we show that the solution of (36) with initial condition  $u_0$  lying in a suitable strip (see condition (60)), reaches the ground state in finite time  $T_R$ .

**Theorem 9.** *Let  $A : D(A) \subset X \rightarrow X$  be a densely defined linear operator such that (5) holds, and let  $B : D(B) \subset X \rightarrow X$  be an unbounded linear operator that verifies (15). Let  $\{A, B\}$  be a 1-null*

controllable pair with control cost that satisfies (38). Then there exists a constant  $r_1 > 0$  such that for any  $R > 0$  there exists  $T_R > 0$  such that for all  $u_0 \in D(A^{1/2})$  that satisfy

$$|\langle u_0, \varphi_1 \rangle_{1/2} - 1| < r_1, \tag{60}$$

$$\|u_0 - \langle u_0, \varphi_1 \rangle_{1/2} \varphi_1\|_{1/2} \leq R,$$

problem (36) is exactly controllable to the ground state solution  $\psi_1(t) = e^{-\lambda_1 t} \varphi_1$  in time  $T_R$ .

Our second semi-global result, Theorem 10 below, ensures the exact controllability of all initial states  $u_0 \in D(A^{1/2}) \setminus \varphi_1^\perp$  to the evolution of their orthogonal projection along the ground state defined by

$$\phi_1(t) = \langle u_0, \varphi_1 \rangle_{1/2} \psi_1(t), \quad \forall t \geq 0, \tag{61}$$

where  $\psi_1$  is the ground state solution.

**Theorem 10.** Let  $A : D(A) \subset X \rightarrow X$  be a densely defined linear operator such that (5) holds, and let  $B : D(B) \subset X \rightarrow X$  be an unbounded linear operator that verifies (15). Let  $\{A, B\}$  be a 1-null controllable pair with control cost that satisfies (38).

Then, for any  $R > 0$  there exists  $T_R > 0$  such that for all  $u_0 \in D(A^{1/2})$  satisfying

$$\|u_0 - \langle u_0, \varphi_1 \rangle_{1/2} \varphi_1\|_{1/2} \leq R |\langle u_0, \varphi_1 \rangle_{1/2}| \tag{62}$$

system (36) is exactly controllable to  $\phi_1$ , defined in (61), in time  $T_R$ .

To prove Theorems 9 and 10, one may follow the strategies described in [4, Section 5]. The only difference with respect to [4, Section 5] is that, in the current setting,  $u_0 \in D(A^{1/2})$  and this yields to the following definition of the controllability time

$$T_R := 1 + \frac{1}{\lambda_2} \log \left( \frac{R^2}{r_1^2} \right).$$

### 5. Applications

In this section we discuss applications of Theorem 7 to parabolic equations. In [4] we have provided sufficient conditions for a pair of linear operators  $A$  and  $B$ , to be  $j$ -null controllable with control cost that fulfills (38). Let us recall that a pair  $\{A, B\}$  is called  $j$ -null controllable in time  $T > 0$  if there exists a constant  $N(T) > 0$  such that, for any initial condition  $y_0 \in X$ , it is possible to find a control  $p \in L^2(0, T)$  satisfying

$$\|p\|_{L^2(0, T)} \leq N(T) \|y_0\|,$$

and for which  $y(T) = 0$ , where  $y(\cdot)$  is the solution of the following linear problem

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0, T] \\ y(0) = y_0. \end{cases}$$

We state [4, Theorem 1.2] in the case of an accretive operator  $A$  and of controllability to the first eigensolution.

**Theorem 11.** Let  $A : D(A) \subset X \rightarrow X$  be such that

- (a)  $A$  is self-adjoint,
  - (b)  $\langle Ax, x \rangle \geq 0, \forall x \in D(A),$
  - (c)  $\exists \lambda > 0 : (\lambda I + A)^{-1} : X \rightarrow X$  is compact
- (63)

and suppose that there exists a constant  $\alpha > 0$  for which the eigenvalues of  $A$  verify the gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \alpha, \quad \forall k \in \mathbb{N}^*. \tag{64}$$

Let  $B : D(B) \subset X \rightarrow X$  be a linear operator such that there exist  $b, q > 0$  for which

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \text{and} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b, \quad \forall k \geq 2. \tag{65}$$

Then, the pair  $\{A, B\}$  is 1-null controllable in any time  $T > 0$  with control cost  $N(T)$  that satisfies (38).

**Remark 12.** Notice that, for accretive operators, hypothesis

$$\sqrt{\lambda_{k+1} - \lambda_1} - \sqrt{\lambda_k - \lambda_1} \geq \alpha, \quad \forall k \in \mathbb{N}^*,$$

of [4, Theorem 1.2] can be replaced by (64), see [4, Remark 6.1].

Moreover, property

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \quad \text{and} \quad |\lambda_k - \lambda_1|^q |\langle B\varphi_1, \varphi_k \rangle| \geq b, \quad \forall k \geq 2,$$

of [4, Theorem 1.2] can be replaced by (65). Indeed, if (65) holds, then by observing that

$$1 - \frac{\lambda_1}{\lambda_k} \geq 1 - \frac{\lambda_1}{\lambda_2}$$

we conclude that

$$|\lambda_k - \lambda_1|^q |\langle B\varphi_1, \varphi_k \rangle| \geq \bar{b}, \quad \forall k \geq 2$$

with

$$\bar{b} = \left| 1 - \frac{\lambda_1}{\lambda_2} \right|^q b.$$

### 5.1. Fokker–Planck equation

The Fokker–Planck equation describes the evolution of the probability density  $u(t, x)$  of a real-valued random variable  $X_t$ , which is associated with an Ito stochastic differential equation driven by a standard Wiener process  $W_t$

$$\begin{cases} dX_t = v(t, X_t)dt + \sigma(t, X_t)dW_t \\ X(t=0) = X_0 \end{cases} \tag{66}$$

with drift  $v(t, x)$ , diffusion coefficient  $D(t, x) = \sigma^2(t, x)/2$  and initial condition  $X_0$ . We recall that given any  $a \leq b$ , denoting by  $u(t, x)$  the probability density associated to  $X_t$ , the following identity holds

$$\mathbb{P}(a \leq X_t \leq b) = \int_a^b u(t, x)dx.$$

Under suitable assumptions on the coefficients  $v$  and  $\sigma$ , the equation satisfied by  $u(t, x)$ , named after A. D. Fokker and M. Planck, is the following one

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} (D(t, x)u(t, x)) - \frac{\partial}{\partial x} (v(t, x)u(t, x)), & (t, x) \in [0, T] \times \mathbb{R} \\ u(0, x) = u_0(x) \end{cases}$$

where  $u_0$  is the density associated to  $X_0$ .

Physically, the probability density  $u(t, x)$  can be interpreted as a quantity proportional to the number of particles in a flow of an abstract substance. For instance, it can reflect the concentration of this substance at the point  $x$  at time  $t$ .

Let us first recall that (66) admits a strong solution (see [6, Definition 9.1]) which is pathwise unique (see [6, Definition 9.4]) under the following assumptions

$$\left\{ \begin{array}{l} 1. (t, x) \mapsto v(t, x) \text{ and } (t, x) \mapsto \sigma(t, x) \text{ are measurable functions on } [0, T] \times \mathbb{R} \\ 2. \exists M > 0 : |v(t, x)| \leq M(1 + |x|) \text{ and } |\sigma(t, x)| \leq M(1 + |x|), \forall (t, x) \in [0, T] \times \mathbb{R} \\ 3. \exists L > 0 : |v(t, x) - v(t, y)| \leq L|x - y| \\ \qquad \qquad \qquad \text{and } |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|, \forall t \in [0, T], \forall x, y \in \mathbb{R} \end{array} \right. \quad (H1)$$

(see [6, Theorem 9.2]).

We are interested in studying the possibility to find a drift  $v$  such that the probability density  $u$  reaches the associated ground state solution in finite time. Moreover, as suggested by the results of the previous sections, we would like to take a drift of the form

$$v(t, x) = p(t)\mu(x), \quad (t, x) \in [0, T] \times \mathbb{R}.$$

However, the sublinear growth assumption 2. of (H1) on  $v$  requires an essential bound on the scalar control  $p$ . Since our controls  $p$  are only locally square integrable, we need to weaken (H1). Thus, in Appendix A, we adapt the proof of the existence and uniqueness of a strong solution of (66) under the following weaker assumptions:

$$\left\{ \begin{array}{l} 1. (t, x) \mapsto \sigma(t, x) \text{ is a measurable function on } [0, T] \times \mathbb{R} \\ 2. p \in L^2(0, T) \\ 3. \exists M > 0 : |\sigma(t, x)| \leq M(1 + |x|), \forall (t, x) \in [0, T] \times \mathbb{R} \\ 4. \exists L > 0 : |\mu(x) - \mu(y)| \leq L|x - y|, |\sigma(t, x) - \sigma(t, y)| \\ \qquad \qquad \qquad \leq L|x - y|, \forall t \in [0, T], \forall x, y \in \mathbb{R} \end{array} \right. \quad (H2)$$

From now on we will consider a constant diffusion  $\sigma(t, x) \equiv \sqrt{2}$  and a drift of the form  $v(t, x) = p(t)\mu(x)$ , where  $p \in L^2(0, T)$  and  $\mu : [0, 1] \rightarrow \mathbb{R}$  is at least Lipschitz continuous. Then, by extending  $\mu(\cdot)$  outside the interval  $[0, 1]$  as follows

$$\tilde{\mu}(x) = \begin{cases} \mu(0) & x \leq 0 \\ \mu(x) & 0 < x < 1 \\ \mu(1) & x \geq 1 \end{cases} \quad (67)$$

it is clear that  $\tilde{\mu}$  satisfies assumption (H2).

The Fokker–Planck equation can be studied also on bounded domains under suitable boundary conditions such as perfectly reflecting, partially reflective, and non reflecting boundary conditions. We now describe such conditions for the equation

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) - p(t) \frac{\partial}{\partial x} (\mu(x)u(t, x)), \quad (t, x) \in [0, T] \times [0, 1] \\ u(0, x) = u_0(x) \end{array} \right.$$

with their physical aspects and the mathematical difficulties they generate.

- The perfectly reflecting boundary conditions are given by:

$$\frac{\partial}{\partial x} u(t, 1) - p(t)\mu(1)u(t, 1) = \frac{\partial}{\partial x} u(t, 0) - p(t)\mu(0)u(t, 0) = 0. \quad (68)$$

Let us note that, for such boundary conditions, the total mass is conserved in the interval  $[0, 1]$ . Thus, if the initial data  $u_0$  has a total mass 1, then through time, the probability density  $u$  satisfies

$$\int_0^1 u(t, x) dx = 1 \quad \forall t \in (0, T), \quad (69)$$

that is, the probability to find particles in the interval  $[0, 1]$  is equal to 1. Indeed, by using the equation solved by  $u$  we get

$$\int_0^1 \frac{\partial}{\partial t} u(t, x) dx = \int_0^1 \left( \frac{\partial^2}{\partial x^2} u(t, x) - p(t) \frac{\partial}{\partial x} (\mu(x) u(t, x)) \right) dx = 0,$$

hence

$$\frac{\partial}{\partial t} \int_0^1 u(t, x) dx = 0 \quad \forall t \in (0, T)$$

which implies condition (69). However, these perfect boundary conditions have a great impact on the functional framework in which one can set up the abstract formulation of the Fokker-Planck equation: the domain of the abstract operator  $A$  will have to include the dependence on time and control  $p$ , that is, one should have to handle  $D(A(t, p)) = \{u(t, \cdot) \in H^2(0, 1), u_x(t, 1) = p(t)\mu(1)u(t, 1), u_x(t, 0) = p(t)\mu(0)u(t, 0)\}$  for all  $t \in [0, T]$ . As far as we know such mathematical difficulties have not yet been studied in bilinear control, the strongest difficulty being that the domain depends itself on the scalar control  $p$ .

- The partially reflecting boundary conditions (i.e. reflecting only the diffusive part of the process, the Brownian motion), are given by

$$\frac{\partial}{\partial x} u(t, 1) = \frac{\partial}{\partial x} u(t, 0) = 0$$

which leads to the following problem

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) + p(t) \frac{\partial}{\partial x} (\mu(x) u(t, x)) = 0 & (t, x) \in [0, T] \times [0, 1] \\ \frac{\partial}{\partial x} u(t, 1) = \frac{\partial}{\partial x} u(t, 0) = 0 & t \in [0, T] \\ u(0, x) = u_0(x) & x \in [0, 1] \end{cases} \quad (70)$$

However, when considering such boundary conditions, the total mass is, in general, no more conserved through time. Indeed, considering again the equation solved by  $u$ , this time we find that

$$\int_0^1 \frac{\partial}{\partial t} u(t, x) dx = \int_0^1 \left( \frac{\partial^2}{\partial x^2} u(t, x) - p(t) \frac{\partial}{\partial x} (\mu(x) u(t, x)) \right) dx = -p(t) [\mu(1)u(t, 1) - \mu(0)u(t, 0)].$$

- The absorbing (Dirichlet) boundary conditions

$$u(t, 1) = u(t, 0) = 0$$

leads to the following problem

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) + p(t) \frac{\partial}{\partial x} (\mu(x) u(t, x)) = 0 & (t, x) \in [0, T] \times [0, 1] \\ u(t, 1) = u(t, 0) = 0 & t \in [0, T] \\ u(0, x) = u_0(x) & x \in [0, 1]. \end{cases} \quad (71)$$

The total density is again in general, not preserved

$$\int_0^1 \frac{\partial}{\partial t} u(t, x) dx = \int_0^1 \left( \frac{\partial^2}{\partial x^2} u(t, x) - p(t) \frac{\partial}{\partial x} (\mu(x) u(t, x)) \right) dx = \frac{\partial}{\partial x} u(t, 1) - \frac{\partial}{\partial x} u(t, 0).$$

In view of the mathematical difficulties generated by the perfectly reflecting boundary conditions, we shall consider in the present paper, only the partially reflecting boundary conditions, and the absorbing ones.

Let us start by studying local controllability to the ground state for problem (70), where our bilinear control will be the time dependent part of the drift  $p(\cdot)$ . We set  $I = (0, 1)$ . We recast the problem in the general setting of (36) by introducing the operators  $A$  and  $B$  defined by

$$\begin{aligned} D(A) &= \left\{ \varphi \in H^2(I) : \frac{\partial}{\partial x} \varphi(0) = \frac{\partial}{\partial x} \varphi(1) = 0 \right\}, & A\varphi &= -\frac{d^2 \varphi}{dx^2} \\ D(B) &= \left\{ \varphi \in L^2(I) : \frac{d}{dx}(\mu\varphi) \in L^2(I) \right\}, & B\varphi &= \frac{d}{dx}(\mu\varphi) \end{aligned}$$

where  $\mu$  is a real-valued function in  $H^3(I)$  to be chosen later on in order to fulfill the rank condition (65).

$A$  satisfies all the properties in (5) and its eigenvalues and eigenfunctions have the following explicit expressions

$$\begin{aligned} \lambda_0 &= 0, & \varphi_0 &= 1 \\ \lambda_k &= (k\pi)^2, & \varphi_k(x) &= \sqrt{2} \cos(k\pi x), \forall k \geq 1. \end{aligned}$$

It is straightforward to prove that the eigenvalues fulfill the required gap property. Indeed,

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi, \quad \forall k \in \mathbb{N},$$

so that (64) is satisfied.

Observe that, in this context, we have an explicit description of the spaces  $D(A^{s/2})$ , see [22, Section 4.3.3] for a general result. For example, for  $s = 1$

$$D(A^{1/2}) = H^1(I).$$

In order to apply Theorem 7, we have to check that  $D(A^{1/2}) \hookrightarrow D(B)$ . This is easily proved as follows:

$$\|\varphi\|_{D(B)} = \|(\mu\varphi)_x\|_{L^2(I)} \leq C_\mu (\|\varphi_x\|_{L^2(I)} + \|\varphi\|_{L^2(I)}) \leq C\|\varphi\|_{1/2}$$

for all  $\varphi$  in  $D(A^{1/2})$ .

To prove local controllability of (70) to the ground state solution  $\psi_0(t, x) \equiv 1$ , we want to use Theorem 11 for  $j = 0$  (note also that, due to the Neumann boundary conditions, one has to consider  $j = 0$  instead of  $j = 1$  and for  $k$  varying in  $\mathbb{N}$  instead of  $\mathbb{N}^*$ ) in order to apply Theorem 7. Thus, we have to check that (65) is satisfied for all  $k \in \mathbb{N}$ , that is, we have to choose some suitable function  $\mu$ . By definition of  $B$ , we have  $B\varphi_0 = \mu'$ , hence one can observe that the choice  $\mu(x) = x$  for all  $x \in I$  leads to  $\langle B\varphi_0, \varphi_k \rangle = 0$ , which is not suitable for our purposes.

Let us first examine more precisely  $\langle B\varphi_0, \varphi_k \rangle$  for all  $k \in \mathbb{N}^*$ :

$$\begin{aligned} \langle B\varphi_0, \varphi_k \rangle &= \sqrt{2} \int_0^1 \mu'(x) \cos(k\pi x) dx = \sqrt{2} \mu'(x) \frac{\sin(k\pi x)}{k\pi} \Big|_0^1 - \sqrt{2} \int_0^1 \mu''(x) \frac{\sin(k\pi x)}{k\pi} dx \\ &= \sqrt{2} \mu''(x) \frac{\cos(k\pi x)}{(k\pi)^2} \Big|_0^1 - \sqrt{2} \int_0^1 \mu'''(x) \frac{\cos(k\pi x)}{(k\pi)^2} dx \\ &= \frac{\sqrt{2}}{(k\pi)^2} [\mu''(1)(-1)^k - \mu''(0)] - \sqrt{2} \int_0^1 \mu'''(x) \frac{\cos(k\pi x)}{(k\pi)^2} dx. \end{aligned}$$

By the Riemann–Lebesgue Lemma, the last integral term on the right-hand side of the above identity converges to 0 as  $k$  goes to  $+\infty$ . Thus, if we choose  $\mu$  such that  $\mu''(1) \neq \pm\mu''(0)$  and  $\langle B\varphi_0, \varphi_k \rangle \neq 0$  for all  $k \geq 1$ , then

$$\exists b > 0 : \lambda_k |\langle B\varphi_0, \varphi_k \rangle| \geq b, \quad \text{for all } k \geq 1.$$

Furthermore, if  $\mu(1) \neq \mu(0)$ , we deduce that

$$\langle B\varphi_0, \varphi_0 \rangle = \int_0^1 \mu'(x) dx \neq 0.$$

**Remark 13.** Observe that if  $\mu''(1) \neq \pm\mu''(0)$ , by the Riemann–Lebesgue Lemma, there exists  $k_1 \geq 0$  such that

$$\exists b > 0 : \lambda_k |\langle B\varphi_0, \varphi_k \rangle| \geq b, \quad \text{for all } k > k_1.$$

Thus, if  $\mu(1) \neq \mu(0)$ , it would be enough to check that  $\langle B\varphi_0, \varphi_k \rangle \neq 0$  for a finite number of indices  $k = 1, \dots, k_1$ . However,  $k_1$  depends on  $\mu$  and therefore, in general, is not explicit. Nevertheless, it is possible to build constructive strategies, as in [5], to determine classes of functions  $\mu$  that satisfy such infinite set of non vanishing conditions.

To sum up, every function  $\mu$  such that

$$\begin{cases} \mu(1) \neq \mu(0) \\ \mu''(1) \neq \pm\mu''(0) \\ \langle B\varphi_0, \varphi_k \rangle \neq 0, & k \in \mathbb{N}^* \end{cases}$$

is suitable for our controllability purposes. For instance, any power  $\mu(x) = x^n$ ,  $x \in [0, 1]$ , with  $n > 2$  satisfies the above conditions at the boundary. Moreover, once extending  $\mu$  as in (67), it clearly satisfies (H2)<sup>1</sup>. So, one has just to check that  $\langle B\varphi_0, \varphi_k \rangle \neq 0$  for  $k = 1, \dots, k_1$ . For example, for  $n = 3$  one easily proves that

$$\langle B\varphi_0, \varphi_k \rangle = \begin{cases} 6\sqrt{2} \frac{(-1)^k}{(k\pi)^2} & k \geq 1 \\ 1 & k = 0. \end{cases}$$

Another suitable choice is  $\mu(x) = \sin(\alpha x)$  for all  $x \in I$ , where  $\alpha$  is as any positive real number chosen in  $[0, \infty) \setminus \pi\mathbb{N}$ . In this case one can check that  $\mu''(1) = -\alpha^2 \sin(\alpha) \neq 0 = \mu''(0)$ ,  $\mu(1) = \sin(\alpha) \neq 0 = \mu(0)$ , and

$$\langle B\varphi_0, \varphi_k \rangle = \frac{\sqrt{2}}{\alpha^2 - (k\pi)^2} (-1)^k \alpha^2 \sin(\alpha).$$

Thus, any choice for  $\mu$  of the above forms meets all the required properties, and also the assumptions in (H2) for the well-posedness of the stochastic differential equation.

Hence, thanks to Theorems 11 and 7, it is possible to build a control  $p \in L^2(0, T)$  such that the solution of the Fokker–Planck equation (70), with initial condition in a neighbourhood of  $\varphi_0 = 1$ , partially reflecting boundary conditions and drift  $v(t, x) = p(t)\mu(x)$ , is controllable to  $\psi_0 = 1$  in time  $T$ . This means that the probability to find the particle in the interval  $[0, 1]$  at time  $T$  is equal to 1 (then the event happens almost surely) even if we are in presence of non-perfectly reflecting walls. This is due to the appropriate choice of the drift.

We move now to the Fokker–Planck equation with absorbing (Dirichlet) boundary conditions. In this case the eigenvalues and eigenfunctions of the Laplacian are the following

$$\lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2} \sin(k\pi x), \quad \forall k \in \mathbb{N}^*.$$

Since we have that

$$\int_0^1 \varphi_1(x) dx = \sqrt{2} \int_0^1 \sin(\pi x) dx = \frac{2\sqrt{2}}{\pi}$$

controlling the solution to the ground state means that we are forcing some mass to remain in the interval  $[0, 1]$  after time  $T$  (in the sense that with probability equal to  $\frac{2\sqrt{2}}{\pi} \cong 0.9$  we find a particle in the interval  $[0, 1]$ ), even though we are in presence of absorbing boundary conditions.

<sup>1</sup>We observe that by multiplying  $\mu$  by a cut-off function, it is possible to construct a smooth extension of  $\mu$  which satisfies (H2).

In order to apply Theorem 11 for  $j = 1$ , and deduce the controllability of (71) to the ground state by applying Theorem 7, we have to verify the rank condition (65) of the unbounded operator  $B$ . Let us compute the following scalar product

$$\begin{aligned} \langle (\mu\varphi_1)', \varphi_k \rangle &= \sqrt{2} \int_0^1 (\mu\varphi_1)'(x) \sin(k\pi x) dx \\ &= \sqrt{2} \left( -(\mu\varphi_1)'(x) \frac{\cos(k\pi x)}{k\pi} \Big|_0^1 + \int_0^1 (\mu\varphi_1)''(x) \frac{\cos(k\pi x)}{k\pi} dx \right) \\ &= \frac{2}{k} \left( \mu(1)(-1)^k + \mu(0) \right) + \frac{\sqrt{2}}{k\pi} \int_0^1 (\mu\varphi_1)''(x) \cos(k\pi x) dx. \end{aligned}$$

Thus, if  $\mu(1) \neq \pm\mu(0)$  and  $\langle (\mu\varphi_1)', \varphi_k \rangle \neq 0, \forall k \in \mathbb{N}^*$ , there exists a constant  $b$  such that

$$\lambda_k^{1/2} |\langle B\varphi_1, \varphi_k \rangle| \geq b, \quad \forall k \in \mathbb{N}^*.$$

Hence, any potential  $\mu(x) = x^n$ , with  $n \geq 1$ , extended to the real line as in the previous example, is admissible for having exact controllability of (71) to the ground state.

**Remark 14.** If we consider  $\mu(x) = x$ , we can directly check that the Fourier coefficients of  $B\varphi_1$  do not vanish for every  $k \in \mathbb{N}^*$

$$\langle (\mu\varphi_1)', \varphi_k \rangle = \begin{cases} \frac{(-1)^k 2k}{k^2 - 1}, & k \geq 2 \\ \frac{1}{2} & k = 1 \end{cases}$$

and the lower bound (65) is satisfied with  $q = \frac{1}{2}$  and  $b = 2\pi$ .

### 5.2. Diffusion equation with Neumann boundary conditions

Now, we consider a diffusion equation with a controlled potential subject to Neumann boundary conditions. Let  $I = (0, 1)$  and consider the following problem

$$\begin{cases} u_t(t, x) - \partial_x^2 u(t, x) + p(t)\mu(x) (u_x(t, x) + u(t, x)) = 0 & (t, x) \in [0, T] \times I \\ u_x(t, 0) = 0, \quad u_x(t, 1) = 0 & t \in [0, T] \\ u(0, x) = u_0(x). & x \in I \end{cases} \tag{72}$$

Let  $X = L^2(I)$ , we rewrite (72) in abstract form by defining the operators  $A$  and  $B$  as

$$\begin{aligned} D(A) &= \{ \varphi \in H^2(I) : \varphi'(0) = 0, \varphi'(1) = 0 \}, \quad A\varphi = -\frac{d^2\varphi}{dx^2} \\ D(B) &= D(A^{1/2}) = H^1(I), \quad B\varphi = \mu \left( \frac{d}{dx} \varphi + \varphi \right) \end{aligned}$$

where  $\mu$  is a real-valued function in  $H^2(I)$ .

Operator  $A$  satisfies the assumptions in (5) and it is possible to compute explicitly its eigenvalues and eigenfunctions:

$$\begin{aligned} \lambda_0 &= 0, \varphi_0 = 1 \\ \lambda_k &= (k\pi)^2, \varphi_k(x) = \sqrt{2} \cos(k\pi x), \quad \forall k \geq 1. \end{aligned}$$

Since the eigenvalues are the same of those in Example 5.1 for  $k \geq 1$ , the gap condition is satisfied for all  $k \geq 0$ .

Furthermore, we have that

$$\|B\varphi\| = \|\mu(\varphi_x + \varphi)\| \leq C_\mu (\|A^{1/2}\varphi\| + \|\varphi\|) \leq C \|\varphi\|_{1/2},$$

thus, also hypothesis (16) is verified.



Let us compute the scalar product  $\langle B\varphi_0, \varphi_k \rangle$  to find a lower bound of the Fourier coefficients of  $B\varphi_0$ :

$$\begin{aligned} \langle \mu(\varphi'_0 + \varphi_0), \varphi_k \rangle &= \sqrt{2} \int_0^1 \mu(x) \cos(k\pi x) dx \\ &= \sqrt{2} \left( \mu(x) \frac{\sin(k\pi x)}{k\pi} \Big|_0^1 - \int_0^1 \mu'(x) \frac{\sin(k\pi x)}{k\pi} dx \right) \\ &= \sqrt{2} \left( \mu'(x) \frac{\cos(k\pi x)}{(k\pi)^2} \Big|_0^1 - \int_0^1 \mu''(x) \frac{\cos(k\pi x)}{(k\pi)^2} dx \right) \\ &= \frac{\sqrt{2}}{(k\pi)^2} \left( \mu'(1)(-1)^k - \mu'(0) \right) - \frac{\sqrt{2}}{(k\pi)^2} \int_0^1 \mu''(x) \cos(k\pi x) dx. \end{aligned}$$

Thus, reasoning as Example 5.1, if  $\langle B\varphi_0, \varphi_k \rangle \neq 0 \forall k \in \mathbb{N}$  and  $\mu'(1) \pm \mu'(0) \neq 0$ , then we have that

$$\exists C > 0 \text{ such that } |\langle B\varphi_0, \varphi_k \rangle| \geq Ck^{-2} = C\lambda_k^{-1}, \quad \forall k \in \mathbb{N}^* \tag{73}$$

and therefore hypothesis (65) is fulfilled.

**Remark 15.** An example of a suitable function  $\mu$  for problem (72) that satisfies the above hypothesis, is  $\mu(x) = x^2$ , for which

$$\langle B\varphi_0, \varphi_k \rangle = \frac{2\sqrt{2}(-1)^k}{(k\pi)^2}, \quad \forall k \geq 1, \quad \text{and} \quad \langle B\varphi_0, \varphi_0 \rangle = 1/3.$$

Applying Theorem 7, it follows that system (72) is exactly controllable to  $\psi_0$ .

### 5.3. Degenerate parabolic equation with Dirichlet boundary conditions

Let  $T > 0$ ,  $I = (0, 1)$ ,  $X = L^2(I)$  and consider the following degenerate bilinear control system

$$\begin{cases} u_t - (x^\alpha u_x)_x + p(t)\mu(x)u_x = 0, & (t, x) \in [0, T] \times I \\ u(t, 1) = 0, \quad \begin{cases} u(t, 0) = 0, & \text{if } \alpha \in [0, 1), \\ (x^\alpha u_x)(t, 0) = 0, & \text{if } \alpha \in [1, 2), \end{cases} \\ u(0, x) = u_0(x), & x \in I \end{cases} \tag{74}$$

with  $\mu(x) = x$  and  $\alpha \in [0, 2)$  the degeneracy parameter. We recall that when  $\alpha \in [0, 1)$  problem (74) is called *weakly degenerate*, while for  $\alpha \in [1, 2)$  *strongly degenerate*.

Let  $\alpha \in [0, 1)$  and we define the weighted Sobolev spaces

$$\begin{aligned} H_\alpha^1(I) &= \{u \in X : u \text{ is absolutely continuous on } [0, 1], x^{\alpha/2}u_x \in X\} \\ H_{\alpha,0}^1(I) &= \{u \in H_\alpha^1(I) : u(0) = 0, u(1) = 0\} \\ H_\alpha^2(I) &= \{u \in H_\alpha^1(I) : x^\alpha u_x \in H^1(I)\}, \end{aligned} \tag{75}$$

and the linear operator  $A : D(A) \subset X \rightarrow X$  by

$$\begin{cases} \forall u \in D(A), \quad Au := -(x^\alpha u_x)_x, \\ D(A) := \{u \in H_{\alpha,0}^1(I), x^\alpha u_x \in H^1(I)\}. \end{cases}$$

For  $\alpha \in [1, 2)$ , we define the spaces

$$\begin{aligned} H_\alpha^1(I) &= \{u \in X : u \text{ is absolutely continuous on } (0, 1], x^{\alpha/2}u_x \in X\} \\ H_{\alpha,0}^1(I) &:= \{u \in H_\alpha^1(I) : u(1) = 0\}, \\ H_\alpha^2(I) &= \{u \in H_\alpha^1(I) : x^\alpha u_x \in H^1(I)\} \end{aligned} \tag{76}$$

and the linear degenerate operator  $A : D(A) \subset X \rightarrow X$  by

$$\left\{ \begin{array}{l} \forall u \in D(A), \quad Au := -(x^\alpha u_x)_x, \\ D(A) := \left\{ u \in H^1_{\alpha,0}(I) : x^\alpha u_x \in H^1(I) \right\} \\ \quad = \left\{ u \in X : u \text{ is absolutely continuous in } (0,1], x^\alpha u \in H^1_0(I), \right. \\ \quad \quad \left. x^\alpha u_x \in H^1(I) \text{ and } (x^\alpha u_x)(0) = 0 \right\}. \end{array} \right.$$

In both cases of weak and strong degeneracy it can be proved that  $D(A)$  is dense in  $X$  and  $A : D(A) \subset X \rightarrow X$  is a self-adjoint accretive operator (see [12] and [14], respectively). Therefore,  $-A$  is the infinitesimal generator of an analytic semigroup of contractions on  $X$ . Furthermore, it has been showed (see, for instance, [1, Appendix]) that  $A$  has a compact resolvent.

Let  $\alpha \in [0, 2)$  and define

$$\nu_\alpha := \frac{|1 - \alpha|}{2 - \alpha}, \quad k_\alpha := \frac{2 - \alpha}{2}.$$

Given  $\nu \geq 0$ , we denote by  $J_\nu$  the Bessel function of the first kind and order  $\nu$  and by  $j_{\nu,1} < j_{\nu,2} < \dots < j_{\nu,k} < \dots$  the sequence of all positive zeros of  $J_\nu$ . It is possible to prove (see [18]) that the eigenvalues and eigenfunctions of  $A$  are given by

$$\lambda_{\alpha,k} = k_\alpha^2 j_{\nu_\alpha,k}^2, \tag{77}$$

$$\varphi_{\alpha,k}(x) = \frac{\sqrt{2k_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha}(j_{\nu_\alpha,k} x^{k_\alpha}) \tag{78}$$

for every  $k \in \mathbb{N}^*$ . Moreover, the family  $(\varphi_{\alpha,k})_{k \in \mathbb{N}^*}$  is an orthonormal basis of  $X$ , see [18]. Hereafter, we shall denote the eigenfunctions  $(\varphi_{\alpha,k})_{k \in \mathbb{N}^*}$  by  $(\varphi_k)_{k \in \mathbb{N}^*}$ , and the eigenvalues  $(\lambda_{\alpha,k})_{k \in \mathbb{N}^*}$  by  $(\lambda_k)_{k \in \mathbb{N}^*}$ .

It has been proved (see, for instance, [19, p. 135, Proposition 7.8] and [15, Corollary 1]) that the gap condition between the zeros of the Bessel functions is satisfied for all  $\alpha \in [0, 2)$ , which implies the required gap between the eigenvalues of the degenerate operator. More precisely, it is proved that:

- if  $\alpha \in [0, 1)$ ,  $\nu_\alpha = \frac{1-\alpha}{2-\alpha} \in (0, \frac{1}{2}]$ , the sequence  $(j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k})_{k \in \mathbb{N}^*}$  is nondecreasing and converges to  $\pi$ . Therefore,

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = k_\alpha (j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}) \geq k_\alpha (j_{\nu_\alpha,2} - j_{\nu_\alpha,1}) \geq \frac{7}{16} \pi > 0,$$

- if  $\nu_\alpha \geq \frac{1}{2}$ , the sequence  $(j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k})_{k \in \mathbb{N}^*}$  is nonincreasing and converges to  $\pi$ . Thus,

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = k_\alpha (j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}) \geq k_\alpha \pi \geq \frac{(2-\alpha)\pi}{2} > 0.$$

Therefore, the hypothesis (64) is fulfilled in both weak and strong degenerate problems with different constants. We now define the unbounded linear operator  $B$  by

$$\begin{aligned} B : D(B) = H^1(I) \subset X &\rightarrow X \\ \varphi &\mapsto \mu\varphi' \end{aligned}$$

where we recall that  $\mu(x) = x$ . We observe that  $D(A^{1/2}) = H^1_{\alpha,0}(I)$ , thus  $D(A^{1/2}) \subset D(B)$  and

$$\|\varphi\|_{D(B)} = \|\mu\varphi'\| \leq C \|x^{\alpha/2}\varphi'\| \leq C\|\varphi\|_{1/2}, \quad \forall \varphi \in D(A^{1/2}).$$

Hence (16) holds true. Finally, we need to prove the validity of (65). To this purpose, we study the Fourier coefficients of  $B\varphi_1$ :

$$\begin{aligned} \langle B\varphi_1, \varphi_k \rangle &= \int_0^1 x\varphi_1'(x)\varphi_k(x)dx = -\frac{1}{\lambda_k} \int_0^1 x\varphi_1'(x)(x^\alpha \varphi_k'(x))' dx \\ &= -\frac{1}{\lambda_k} \left[ x\varphi_1'(x)x^\alpha \varphi_k'(x) \Big|_0^1 - \int_0^1 (x\varphi_1')'(x)x^\alpha \varphi_k'(x)dx \right] \\ &= -\frac{1}{\lambda_k} \left[ \varphi_1'(1)\varphi_k'(1) - \int_0^1 (x\varphi_1')'(x)x^\alpha \varphi_k'(x)dx \right] \end{aligned}$$

where we adapted [2, Proposition 2.5, Property (2.14)] to the current setting to prove that

$$\lim_{x \rightarrow 0^+} (x\varphi_1'(x)x^\alpha \varphi_k'(x)) = 0,$$

so that, we have

$$\langle B\varphi_1, \varphi_k \rangle = -\frac{1}{\lambda_k} \left[ \varphi_1'(1)\varphi_k'(1) - (x\varphi_1')'(x)x^\alpha \varphi_k(x) \Big|_0^1 + \int_0^1 ((x\varphi_1')'x^\alpha)'(x)\varphi_k(x)dx \right].$$

We claim that for all  $\alpha \in [0, 2)$ , the following property holds:

$$(x\varphi_1'(x))'x^\alpha \varphi_k(x) \Big|_{x=0} \text{ is defined and is equal to 0.}$$

Let us consider  $x \in (0, 1)$ . We have

$$(x\varphi_1')'(x) = x^{1-\alpha}(x^\alpha \varphi_1')'(x) + (1-\alpha)\varphi_1'(x) = -\lambda_1 x^{1-\alpha}\varphi_1 + (1-\alpha)\varphi_1'(x). \tag{79}$$

Thus, we obtain

$$x^\alpha (x\varphi_1')'(x)\varphi_k(x) = -\lambda_1 x\varphi_1(x)\varphi_k(x) + (1-\alpha)(x^\alpha \varphi_1'(x))\varphi_k(x).$$

Observe that

$$|x\varphi_1(x)\varphi_k(x)| \leq \frac{1}{2}(x\varphi_1^2(x) + x\varphi_k^2(x)), \quad \forall x \in (0, 1].$$

Since  $\varphi_1$  and  $\varphi_k$  are in  $H_\alpha^1(I)$ , we can use [2, Proposition 2.5, Property (2.12)] in the particular case  $a(x) = x^\alpha$  for  $x \in [0, 1]$ , and successively with  $u = \varphi_1$ , and  $u = \varphi_k$ . Hence, we have

$$\lim_{x \rightarrow 0^+} (x\varphi_1^2(x) + x\varphi_k^2(x)) = 0,$$

so that

$$\lim_{x \rightarrow 0^+} (x\varphi_1(x)\varphi_k(x)) = 0.$$

Since  $\varphi_1 \in D(A)$  and  $\varphi_k$  is in  $H_\alpha^1(I)$ , we can use [2, Proposition 2.5, Property (2.15)], in the particular case  $a(x) = x^\alpha$  for  $x \in [0, 1]$ ,  $\phi = \varphi_k$ , and  $u = \varphi_1$ , noticing in addition that we have  $\phi(0) = 0$  if  $\alpha \in [0, 1)$ . Hence, we deduce

$$\lim_{x \rightarrow 0^+} (x^\alpha \varphi_1'(x)\varphi_k(x)) = 0.$$

Therefore, we have proved that for all  $\alpha \in [0, 2)$

$$(x\varphi_1'(x))'x^\alpha \varphi_k(x) \Big|_{x=0} \text{ is defined and is equal to 0.}$$

We easily prove that since  $\varphi_1(1) = \varphi_k(1) = 0$ , we have

$$(x\varphi_1'(x))'x^\alpha \varphi_k(x) \Big|_{x=1} = 0.$$

Thus, we obtain that

$$(x\varphi_1')'(x)x^\alpha \varphi_k(x) \Big|_0^1 = 0,$$

so that using this property in our previous computations for  $\langle B\varphi_1, \varphi_k \rangle$ , we get

$$\langle B\varphi_1, \varphi_k \rangle = -\frac{1}{\lambda_k} \left[ \varphi_1'(1)\varphi_k'(1) + \int_0^1 ((x\varphi_1')'x^\alpha)'(x)\varphi_k(x)dx \right].$$

Now we use the identity (79) which yields

$$\begin{aligned} (x^\alpha (x\varphi_1)')'(x)\varphi_k(x) &= \left(-\lambda_1 x\varphi_1(x) + (1-\alpha)x^\alpha \varphi_1'(x)\right)'(x)\varphi_k(x) \\ &= \left[-(2-\alpha)\lambda_1 \varphi_1(x)\varphi_k(x) - \lambda_1 x\varphi_1'(x)\varphi_k(x)\right]. \end{aligned}$$

Using this identity in our previous computations and the property that the family  $(\varphi_k)_{k \in \mathbb{N}^*}$  is an orthonormal basis of  $X$ , we obtain

$$\langle B\varphi_1, \varphi_k \rangle = -\frac{\varphi_1'(1)\varphi_k'(1)}{\lambda_k - \lambda_1}.$$

We have proved in [16, p. 12, formula (2.43)] that

$$\varphi_1'(1)\varphi_k'(1) = \frac{2k_\alpha^3 j_{\nu_\alpha,1} j_{\nu_\alpha,k}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,1})| |J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} J'_{\nu_\alpha}(j_{\nu_\alpha,1}) J'_{\nu_\alpha}(j_{\nu_\alpha,k}).$$

The above identity, together with (77), imply that there exists a constant  $C > 0$  such that

$$|\langle B\varphi_1, \varphi_k \rangle| \geq C\lambda_k^{-1/2}, \quad \forall k > 1.$$

For  $k = 1$ ,  $\langle B\varphi_1, \varphi_1 \rangle$  is given by

$$\langle B\varphi_1, \varphi_1 \rangle = \int_0^1 x\varphi_1'(x)\varphi_1(x)dx = \frac{1}{2} \int_0^1 x \frac{d}{dx} (\varphi_1^2(x)) dx = \frac{1}{2} \left[ x\varphi_1^2(x) \Big|_0^1 - \int_0^1 \varphi_1^2(x) dx \right] = -\frac{1}{2},$$

where we used once again [2, Proposition 2.5, Property (2.12)] with  $u = \varphi_1$ . Thus, we have  $\langle B\varphi_1, \varphi_1 \rangle \neq 0$ .

Therefore, we proved that all the hypotheses of Theorem 7 are satisfied. Applying such result, we deduce that for any  $T > 0$  and initial condition  $u_0 \in D(A^{1/2})$  sufficiently close to the ground state  $\varphi_1$ , there exists a control  $p \in L^2(0, T)$  that steers the solution of (74) to  $\psi_1(T, x) = e^{-\lambda_1 T} \varphi_1(x)$  in time  $T$ , by the iterative constructive process we set up in our abstract results.

#### 5.4. Degenerate parabolic equation with Neumann boundary conditions

In this section we study the controllability of the following degenerate problem

$$\begin{cases} u_t - (x^\alpha u_x)_x + p(t)\mu(x)(u_x + u) = 0, & (t, x) \in [0, T] \times I \\ (x^\alpha u_x)(t, 0) = 0, \quad u_x(t, 1) = 0, & t \in [0, T] \\ u(0, x) = u_0(x), & x \in I, \end{cases} \tag{80}$$

where  $T > 0$ ,  $I = (0, 1)$  and  $\mu(x) = x^{2-\alpha}$ . The control  $p \in L^2(0, T)$  is a real valued function and  $\mu$  represents an admissible potential.

Recalling the definitions of the weighted Sobolev spaces  $H_\alpha^1(I)$  and  $H_\alpha^2(I)$  in (75) and (76) for weak and strong degeneracy respectively, we define the second order linear operator

$$\begin{cases} \forall u \in D(A), \quad Au := -(x^\alpha u_x)_x, \\ D(A) := \{u \in H_\alpha^2(I), (x^\alpha u_x)(0) = 0, u_x(1) = 0\}. \end{cases}$$

In [13, Proposition 2.1, Proposition 2.2] it is proved that for any  $\alpha \in [0, 2)$  the operator  $A$  is self-adjoint accretive and has a dense domain. Therefore,  $-A$  is the infinitesimal generator of an analytic semigroup of contraction. Moreover, in [13, Proposition 3.1] the authors showed that the eigenvalues and eigenfunctions of the operator  $A$ , for any  $\alpha \in [0, 2)$ , are given by

$$\lambda_{\alpha,0} = 0, \quad \varphi_{\alpha,0}(x) = 1 \tag{81}$$

and for all  $m \geq 1$

$$\lambda_{\alpha,m} = \kappa_\alpha^2 j_{\nu_\alpha+1,m}^2, \tag{82}$$

$$\varphi_{\alpha,m}(x) = K_{\alpha,m} x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}(j_{\nu_\alpha+1,m} x^{\frac{2-\alpha}{2}}), \tag{83}$$

where

$$\kappa_\alpha := \frac{2-\alpha}{2}, \quad \nu_\alpha := \frac{\alpha-1}{2-\alpha},$$

$J_{\nu_\alpha}$  is the Bessel's function of order  $\nu_\alpha$ ,  $(j_{\nu_\alpha+1,m})_{m \geq 1}$  are the positive zeros of the Bessel's function  $J_{\nu_\alpha+1}$  and  $K_{\alpha,m}$  are positive constants.

Observe that, in the same paper [13, Propositions 3.1] it is proved that the eigenvalues  $\{\lambda_{\alpha,m}\}_{m \in \mathbb{N}}$  satisfy the following gap condition

$$\forall \alpha \in [0, 2), \sqrt{\lambda_{\alpha,m+1}} - \sqrt{\lambda_{\alpha,m}} \geq \frac{2-\alpha}{2}\pi.$$

We introduce the operator  $B : D(B) =: H^1(I) \subset X \rightarrow X$  defined by

$$B\varphi = \mu(\varphi' + \varphi), \quad \forall \varphi \in D(B),$$

where  $\mu(x) = x^{2-\alpha}$ . In order to have that  $D(A^{1/2}) \hookrightarrow D(B)$  (that is, hypothesis (16)), we need to restrict the analysis to the case  $\alpha \in [0, 4/3]$ . Indeed,

$$\|B\varphi\| = \|\mu(\varphi' + \varphi)\| \leq C(\|x^{\alpha/2}\varphi'\| + \|\varphi\|) = C(\|A^{1/2}\varphi\| + \|\varphi\|) \leq C\|\varphi\|_{1/2}, \quad \forall \varphi \in D(A^{1/2})$$

is satisfied if and only if  $\alpha \in [0, 4/3]$ .

Furthermore, it is possible to prove that also hypothesis (65) holds true. We compute the scalar product  $\langle B\varphi_{\alpha,0}, \varphi_{\alpha,m} \rangle$ , for all  $m \in \mathbb{N}$ :

$$\langle B\varphi_{\alpha,0}, \varphi_{\alpha,0} \rangle = \int_0^1 x^{2-\alpha} dx = \frac{1}{3-\alpha},$$

and, for all  $m \geq 1$  we have that

$$\begin{aligned} \langle B\varphi_{\alpha,0}, \varphi_{\alpha,m} \rangle &= \int_0^1 \mu(x)(\varphi'_{\alpha,0} + \varphi_{\alpha,0})\varphi_{\alpha,m} dx = \int_0^1 \mu(x)\varphi_{\alpha,m}(x) dx \\ &= \frac{1}{\lambda_{\alpha,m}} \int_0^1 \mu(x)(-x^\alpha \varphi'_{\alpha,m})'(x) dx \\ &= \frac{1}{\lambda_{\alpha,m}} \left( [-x^\alpha \mu(x)\varphi'_{\alpha,m}(x)]_0^1 + \int_0^1 x^\alpha \mu'(x)\varphi'_{\alpha,m}(x) dx \right). \end{aligned}$$

Recalling that  $\mu(x) = x^{2-\alpha}$ , we obtain

$$\begin{aligned} \int_0^1 x^\alpha \mu'(x)\varphi'_{\alpha,m}(x) dx &= (2-\alpha) \int_0^1 x\varphi'_{\alpha,m}(x) dx \\ &= (2-\alpha) [x\varphi_{\alpha,m}(x)]_0^1 - (2-\alpha) \int_0^1 \varphi_{\alpha,m}(x) dx \\ &= (2-\alpha) [x\varphi_{\alpha,m}(x)]_0^1 - (2-\alpha) \langle \varphi_{\alpha,0}, \varphi_{\alpha,m} \rangle. \end{aligned}$$

Since the eigenfunctions are orthogonal, we have that

$$\langle \varphi_{\alpha,0}, \varphi_{\alpha,m} \rangle = 0,$$

hence

$$\langle B\varphi_{\alpha,0}, \varphi_{\alpha,m} \rangle = \frac{1}{\lambda_{\alpha,m}} \left( [-x^2 \varphi'_{\alpha,m}(x)]_0^1 + (2-\alpha) [x\varphi_{\alpha,m}(x)]_0^1 \right).$$

From the Neumann boundary conditions satisfied by  $\varphi_{\alpha,m}$ , we know that  $x^\alpha \varphi'_{\alpha,m}(x) \rightarrow 0$  as  $x \rightarrow 0$  and  $x \rightarrow 1$ , thus

$$[-x^2 \varphi'_{\alpha,m}(x)]_0^1 = 0.$$

Furthermore, in [13, Lemma 5.1 and Lemma 5.2] it is shown that  $\varphi_{\alpha,m}$  has a finite limit as  $x \rightarrow 0$ , therefore

$$x\varphi_{\alpha,m}(x) \rightarrow 0, \quad \text{as } x \rightarrow 0,$$

and moreover that  $|\varphi_{\alpha,m}(1)| = \sqrt{2-\alpha}$ , which implies

$$\left| (2-\alpha) [x\varphi_{\alpha,m}(x)]_0^1 \right| = (2-\alpha)^{3/2}.$$

Therefore,

$$|\langle B\varphi_{\alpha,0}, \varphi_{\alpha,m} \rangle| = \frac{(2-\alpha)^{3/2}}{\lambda_{\alpha,m}},$$

that is condition (65) with  $q = 1$ .

By applying Theorem 7, we conclude that problem (80), for  $\alpha \in [0, 4/3]$ , is exactly controllable to the ground state solution  $\psi_0 \equiv 1$  in any time  $T > 0$ .

Let us now show an application of Theorem 10 to Example 5.4. We recall that  $\lambda_0 = 0$  is the first eigenvalue of  $A$ , and is associated to the eigenfunction  $\varphi_0 \equiv 1$ . We set

$$\phi_0(t) = \langle u_0, \varphi_0 \rangle_{1/2} \psi_0(t) = \int_0^1 u_0(x) dx, \quad \forall t \geq 0. \tag{84}$$

Theorem 10 applied to Example 5.4 gives:

for any  $R > 0$  there exists  $T_R > 0$  such that for all  $u_0 \in D(A^{1/2})$  satisfying

$$\int_0^1 u_0(x) dx \neq 0, \tag{85}$$

and

$$\left( \frac{\int_0^1 u_0^2(x) dx - \left(\int_0^1 u_0(x) dx\right)^2 + \int_0^1 x^\alpha |u_0'(x)|^2 dx}{\left|\int_0^1 u_0(x) dx\right|} \right)^{1/2} \leq R \tag{86}$$

system (80) is exactly controllable to  $\phi_0$ , defined in (84), in time  $T_R$ .

**Appendix A.**

$$\begin{cases} dX_t = v(t, X_t) dt + \sigma(t, X_t) dB_t \\ X(t=0) = X_0 \end{cases} \tag{1}$$

under assumptions (H2) which are weaker than assumptions (H1) in [6, Theorem 9.2]. We develop the computations for a drift  $v$  of the form

$$v(t, x) = p(t)\mu(x), \quad t \in [0, T], x \in \mathbb{R},$$

with  $p \in L^2(0, T)$ , which is the case of interest to this paper. A general drift  $v$  satisfying (H2) can be treated in a similar way.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. We recall that the process  $(X_t)_{t \in [0, T]}$  is a solution of (1) if

- (1)  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, (B_t)_{t \in [0, T]}, \mathbb{P})$  is a standard Brownian motion
- (2)  $(X_t)_{t \in [0, T]}$  is adapted to the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$
- (3) for every  $t \in [0, T]$ :

$$X_t = X_0 + \int_0^t p(s)\mu(X_s) ds + \int_0^t \sigma(s, X_s) dB_s.$$

We now prove the following preliminary result which adapts [6, Theorem 9.1] to the current setting.

**Theorem 16.** *Let  $X$  be a solution of*

$$X_t = X_0 + \int_0^t p(s)\mu(X_s) ds + \int_0^t \sigma(s, X_s) dB_s$$

where  $\mu, p$  and  $\sigma$  satisfy hypotheses 1., 2. and 3. of (H2) (see page 18), and  $X_0$  be a  $\mathcal{F}_0$ -measurable r.v. of  $L^2$ . Then,

$$\mathbb{E} \left( \sup_{0 \leq s \leq T} |X_s|^2 \right) \leq c(T, M) (1 + \mathbb{E}(|X_0|^2)) \tag{2}$$

$$\mathbb{E} \left( \sup_{0 \leq s \leq t} |X_s - X_0|^2 \right) \leq c(T, M) t (1 + \mathbb{E}(|X_0|^2)). \tag{3}$$

**Proof.** We follow the proof of [6, Theorem 9.1]. Fixed any  $R > 0$ , we define  $X_R(t) := X_{t \wedge \tau_R}$  where  $\tau_R := \inf\{t : 0 \leq t \leq T, |X_t| \geq R\}$  is the exit time of the process  $X$  from the open ball of radius  $R$ . If  $|X_t| < R$  for every  $t \in [0, T]$  then we set  $\tau_R = T$ . We have that

$$\begin{aligned} X_R(t) &= X_0 + \int_0^{t \wedge \tau_R} p(r)\mu(X_r)dr + \int_0^{t \wedge \tau_R} \sigma(r, X_r)dB_r \\ &= X_0 + \int_0^t p(r)\mu(X_r)\mathbb{1}_{\{r < \tau_R\}}dr + \int_0^t \sigma(r, X_r)\mathbb{1}_{\{r < \tau_R\}}dB_r \\ &= X_0 + \int_0^t p(r)\mu(X_R(r))\mathbb{1}_{\{r < \tau_R\}}dr + \int_0^t \sigma(r, X_R(r))\mathbb{1}_{\{r < \tau_R\}}dB_r. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_R(s)|^2 \right] &\leq 3\mathbb{E}[|X_0|^2] + 3\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s p(r)\mu(X_R(r))\mathbb{1}_{\{r < \tau_R\}}dr \right|^2 \right] \\ &\quad + 3\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(r, X_R(r))\mathbb{1}_{\{r < \tau_R\}}dB_r \right|^2 \right], \end{aligned} \tag{4}$$

where we have used the inequality

$$|x_1 + \dots + x_m|^p \leq m^{p-1} (|x_1|^p + \dots + |x_m|^p)$$

which holds, in general, for any  $x_1, \dots, x_m \in \mathbb{R}^d$ .

By Hölder’s inequality and hypothesis 3. of (H2) we obtain

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s p(r)\mu(X_R(r))\mathbb{1}_{\{r < \tau_R\}}dr \right|^2 \right] &\leq \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left( \left( \int_0^s |p(r)|^2 dr \right) \left( \int_0^s |\mu(X_R(r))|^2 \mathbb{1}_{\{r < \tau_R\}} dr \right) \right) \right] \\ &\leq \|p\|_{L^2(0,T)}^2 M^2 \mathbb{E} \left[ \sup_{0 \leq s \leq t} \int_0^s (1 + |X_R(r)|)^2 \mathbb{1}_{\{r < \tau_R\}} dr \right]. \end{aligned} \tag{5}$$

Whereas, by the  $L^2$  inequalities for stochastic integrals (see [6, Proposition 8.4]) we deduce

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(r, X_R(r))\mathbb{1}_{\{r < \tau_R\}}dB_r \right|^2 \right] &\leq c_2 \mathbb{E} \left[ \int_0^t |\sigma(r, X_R(r))|^2 dr \right] \\ &\leq c_2 M^2 \mathbb{E} \left[ \int_0^t (1 + |X_R(r)|)^2 dr \right], \end{aligned} \tag{6}$$

where in the last inequality we have used again hypothesis 3. Now, plugging (5) and (6) into (4), we get

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_R(s)|^2 \right] &\leq 3\mathbb{E}[|X_0|^2] + 3M^2 \left( \|p\|_{L^2(0,T)}^2 + c_2 \right) 2\mathbb{E} \left[ T + \int_0^t |X_R(r)|^2 dr \right] \\ &\leq c_1(T, M) (1 + \mathbb{E}[|X_0|^2]) + c_2(T, M) \int_0^t \mathbb{E}[|X_R(r)|^2] dr. \end{aligned}$$

If we set  $v(t) := \mathbb{E}[\sup_{0 \leq s \leq t} |X_R(s)|^2]$ , from the above estimate we obtain

$$v(t) \leq c_1(T, M) (1 + \mathbb{E}[|X_0|^2]) + c_2(T, M) \int_0^t v(r)dr.$$

Observe that, since  $|X_R(t)| = |X_0|$  if  $|X_0| > R$  and  $|X_R(t)| \leq R$  otherwise, it holds that  $|X_R(t)| \leq \max\{|X_0|, R\}$  and then  $v(t) \leq \mathbb{E}[\max\{|X_0|^2, R^2\}] < +\infty$ . Thus,  $v$  is bounded and we can apply Gronwall’s inequality:

$$v(T) = \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_R(s)|^2 \right] \leq c_1(T, M) (1 + \mathbb{E}[|X_0|^2]) e^{Tc_2(T, M)} = c(T, M) (1 + \mathbb{E}[|X_0|^2]).$$

Since the right-hand side does not depend on  $R$ , we can take the limit  $R \rightarrow +\infty$ .

Let us prove that  $\tau_R \rightarrow T$  as  $R \rightarrow +\infty$ . Since  $X$  is continuous,  $\sup_{0 \leq t \leq \tau_R} |X_t|^2 = R^2$  on  $\{\tau_R < T\}$ . Therefore, we have that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq \tau_R} |X_t|^2 \right] \geq R^2 \mathbb{P}(\tau_R < T)$$

so that

$$\mathbb{P}(\tau_R < T) \leq \frac{1}{R^2} \mathbb{E} \left[ \sup_{0 \leq t \leq \tau_R} |X_t|^2 \right] \leq \frac{c(T, M) (1 + \mathbb{E}(|X_0|^2))}{R^2}.$$

Hence,  $\mathbb{P}(\tau_R < T) \rightarrow 0$  as  $R \rightarrow +\infty$ . Since  $R \rightarrow \tau_R$  is increasing,  $\lim_{R \rightarrow +\infty} \tau_R = T$  a.s. and

$$\sup_{0 \leq s \leq T} |X_R(s)|^2 \rightarrow \sup_{0 \leq s \leq T} |X_s|^2 \quad \text{a.s., for } R \rightarrow +\infty.$$

By applying Fatou's lemma we conclude that (2) holds true. We refer to [6, Theorem 9.1] for the proof of (3).  $\square$

Let  $a, b \in \mathbb{R}$  such that  $a \leq b$ . We recall the definition of the spaces  $M^p([a, b])$ , with  $p \geq 1$ . First, we define the space  $M^p_{loc}([a, b])$  as the space of the equivalence classes of real-valued progressively measurable processes  $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{a \leq t \leq b}, (X_t)_{a \leq t \leq b}, \mathbb{P})$  such that

$$\int_a^b |X_s|^p ds < +\infty \quad \text{a.s.}$$

Then, by  $M^p([a, b])$  we denote the subspace of  $M^p_{loc}([a, b])$  of the processes such that

$$\mathbb{E} \left[ \int_a^b |X_s|^p ds \right] < +\infty.$$

**Theorem 17.** *Let  $X_0$  be a real-valued r.v.,  $\mathcal{F}_0$  measurable and square integrable. Then, under assumption (H2) (see page 18) there exists  $X \in M^2([0, T])$  that satisfies*

$$X_t = X_0 + \int_0^t p(s)\mu(X_s)ds + \int_0^t \sigma(s, X_s)dB_s. \tag{7}$$

Moreover, if  $X'$  is another solution of (7), then

$$\mathbb{P}(X_t = X'_t \text{ for every } t \in [0, T]) = 1 \tag{8}$$

(pathwise uniqueness).

**Proof.** We follow the proof of [6, Theorem 9.2]. We define recursively a sequence of processes by  $X_0(t) = X_0$  and

$$X_{m+1}(t) = X_0 + \int_0^t p(s)\mu(X_m(s)) ds + \int_0^t \sigma(s, X_m(s)) dB_s.$$

Our aim is to show that the sequence  $(X_m)_m$  converges uniformly on  $[0, T]$  to a process  $X$  which is solution of (7).

We first prove by induction that

$$\mathbb{E} \left[ \sup_{0 \leq r \leq t} |X_{m+1}(r) - X_m(r)|^2 \right] \leq \frac{(Rt)^{m+1}}{(m+1)!} \tag{9}$$

with  $R := 2(\|p\|_{L^2(0, T)}^2 + 4) \max\{2M^2(1 + \mathbb{E}[|X_0|^2]), L^2\}$ , where  $L, M > 0$  are the constants in hypotheses 3. and 4..

For  $m = 0$  we have that

$$\begin{aligned} \sup_{0 \leq r \leq t} |X_1(r) - X_0|^2 &\leq 2 \sup_{0 \leq r \leq t} \left| \int_0^r p(s)\mu(X_0)ds \right|^2 + 2 \sup_{0 \leq r \leq t} \left| \int_0^r \sigma(s, X_0)dB_s \right|^2 \\ &\leq 2 \|p\|_{L^2(0, T)}^2 \sup_{0 \leq r \leq t} \int_0^r |\mu(X_0)|^2 ds + 2 \sup_{0 \leq r \leq t} \left| \int_0^r \sigma(s, X_0)dB_s \right|^2 \\ &\leq 4M^2 \|p\|_{L^2(0, T)}^2 (1 + |X_0|^2) t + 2 \sup_{0 \leq r \leq t} \left| \int_0^r \sigma(s, X_0)dB_s \right|^2 \end{aligned}$$



where we have applied Hölder's inequality and hypothesis 3.

By using Doob's maximal inequality (see [6, formula (7.23), p. 195]) and hypothesis 3. of (H2) it is possible to prove that

$$\mathbb{E} \left[ \sup_{0 \leq r \leq t} \left| \int_0^r \sigma(s, X_s) dB_s \right|^2 \right] \leq 4\mathbb{E} \left[ \int_0^t |\sigma(s, X_s)|^2 ds \right] \leq 8M^2 \left( t + \mathbb{E} \left[ \int_0^t |X_s|^2 ds \right] \right).$$

Hence, we get that

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq r \leq t} |X_1(r) - X_0|^2 \right] &\leq 4M^2 \|p\|_{L^2(0,T)}^2 (1 + \mathbb{E}[|X_0|^2]) t + 16M^2 (t + t\mathbb{E}[|X_0|^2]) \\ &= 4M^2 (\|p\|_{L^2(0,T)}^2 + 4) (1 + \mathbb{E}[|X_0|^2]) t \leq Rt. \end{aligned}$$

Now, suppose that (9) holds till index  $m-1$  and let us prove it for  $m$ . Observe that, thanks to Hölder's inequality and hypothesis 4., it holds

$$\begin{aligned} \sup_{0 \leq r \leq t} |X_{m+1}(r) - X_m(r)|^2 &\leq 2 \sup_{0 \leq r \leq t} \left| \int_0^r p(s) [\mu(X_m(s)) - \mu(X_{m-1}(s))] ds \right|^2 \\ &\quad + 2 \sup_{0 \leq r \leq t} \left| \int_0^r [\sigma(s, X_m(s)) - \sigma(s, X_{m-1}(s))] dB_s \right|^2 \\ &\leq 2 \|p\|_{L^2(0,T)}^2 \sup_{0 \leq r \leq t} \int_0^r |\mu(X_m(s)) - \mu(X_{m-1}(s))|^2 ds \\ &\quad + 2 \sup_{0 \leq r \leq t} \left| \int_0^r [\sigma(s, X_m(s)) - \sigma(s, X_{m-1}(s))] dB_s \right|^2 \\ &\leq 2L^2 \|p\|_{L^2(0,T)}^2 \sup_{0 \leq r \leq t} \int_0^r |X_m(s) - X_{m-1}(s)|^2 ds \\ &\quad + 2 \sup_{0 \leq r \leq t} \left| \int_0^r [\sigma(s, X_m(s)) - \sigma(s, X_{m-1}(s))] dB_s \right|^2. \end{aligned}$$

We now compute the expected value

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq r \leq t} |X_{m+1}(r) - X_m(r)|^2 \right] &\leq 2L^2 \|p\|_{L^2(0,T)}^2 \int_0^t \mathbb{E}[|X_m(s) - X_{m-1}(s)|^2] ds \\ &\quad + 2\mathbb{E} \left[ \sup_{0 \leq r \leq t} \left| \int_0^r [\sigma(s, X_m(s)) - \sigma(s, X_{m-1}(s))] dB_s \right|^2 \right]. \end{aligned}$$

By using again Dobb's inequality, hypothesis 4. and the inductive step, we have

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq r \leq t} |X_{m+1}(r) - X_m(r)|^2 \right] &\leq 2L^2 \|p\|_{L^2(0,T)}^2 \int_0^t \mathbb{E}[|X_m(s) - X_{m-1}(s)|^2] ds \\ &\quad + 8\mathbb{E} \left[ \int_0^t |\sigma(s, X_m(s)) - \sigma(s, X_{m-1}(s))|^2 ds \right] \\ &\leq 2L^2 \|p\|_{L^2(0,T)}^2 \int_0^t \mathbb{E}[|X_m(s) - X_{m-1}(s)|^2] ds \\ &\quad + 8L^2 \int_0^t \mathbb{E}[|X_m(s) - X_{m-1}(s)|^2] ds \\ &\leq 2L^2 (\|p\|_{L^2(0,T)}^2 + 4) \int_0^t \frac{(Rs)^m}{m!} ds \leq \frac{(Rt)^{m+1}}{(m+1)!}. \end{aligned}$$

Thus, the proof of (9) is completed.

Now we apply Markov's inequality which gives

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} |X_{m+1}(t) - X_m(t)| > \frac{1}{2^m} \right) \leq 2^{2m} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{m+1}(t) - X_m(t)|^2 \right] \leq 2^{2m} \frac{(RT)^{m+1}}{(m+1)!}.$$

Since the right-hand side of the above inequality is summable, by the Borel-Cantelli lemma we get

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} |X_{m+1}(t) - X_m(t)| > \frac{1}{2^m} \text{ for infinitely many indices } m \right) = 0,$$

that is, for almost every  $\omega$  we eventually have

$$\sup_{0 \leq t \leq T} (X_{m+1}(t) - X_m(t)) \leq \frac{1}{2^m}$$

and therefore, for fixed  $\omega$  the series

$$X_0 + \sum_{k=0}^{m-1} |X_{k+1}(t) - X_k(t)| = X_m(t)$$

converges uniformly on  $[0, T]$  a.s. Let  $X_t = \lim_{m \rightarrow +\infty} X_m(t)$ . Then,  $X$  is continuous, being the uniform limit of continuous processes, and therefore  $X \in M^2_{loc}([0, T])$ .

We now prove that  $X$  is solution of (7). Recall that

$$X_{m+1}(t) = X_0 + \int_0^t p(s)\mu(X_m(s))ds + \int_0^t \sigma(s, X_m(s))dB_s. \tag{10}$$

Then, the left-hand side converges uniformly to  $X$ . From hypothesis 3. of (H2) we get that

$$|p(t)| |\mu(X_m(t)) - \mu(X_t)| \leq L |p(t)| |X_m(t) - X_t|.$$

By Lebesgue's dominated convergence theorem, we deduce that

$$\int_0^t p(s)\mu(X_m(s))ds \rightarrow \int_0^t p(s)\mu(X_s)ds \quad \text{a.s. as } m \rightarrow +\infty.$$

Moreover, since by hypothesis 3. of (H2)

$$|\sigma(s, X_m(s)) - \sigma(s, X_s)| \leq L |X_m(s) - X_s|$$

we deduce that, uniformly on  $[0, T]$  a.s.,

$$\lim_{m \rightarrow +\infty} \sigma(t, X_m(t)) = \lim_{m \rightarrow +\infty} \sigma(t, X_t)$$

and therefore

$$\int_0^t \sigma(s, X_m(s))dB_s \rightarrow \int_0^t \sigma(s, X_s)dB_s \quad \text{a.s. as } m \rightarrow +\infty.$$

Since a.s. convergence implies convergence in probability, we can take the limit in probability in (10) and obtain that

$$X_t = X_0 + \int_0^t p(s)\mu(X_s)ds + \int_0^t \sigma(s, X_s)dB_s,$$

that is,  $X$  is solution of the stochastic differential equation. Moreover,  $X \in M^2([0, T])$  thanks to Theorem 16.

We now prove uniqueness. Let  $X_1, X_2$  be two solutions of (7). Then, we have

$$|X_1(t) - X_2(t)| \leq \left| \int_0^t p(s) (\mu(X_1(s)) - \mu(X_2(s))) ds \right| + \left| \int_0^t (\sigma(s, X_1(s)) - \sigma(s, X_2(s))) dB_s \right|.$$

We compute the mean value of the supremum over  $[0, t]$  of the difference of the solutions

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq r \leq t} |X_1(r) - X_2(r)|^2 \right] &\leq 2\mathbb{E} \left[ \sup_{0 \leq r \leq t} \left| \int_0^r p(s) (\mu(X_1(s)) - \mu(X_2(s))) ds \right|^2 \right] \\ &\quad + 2\mathbb{E} \left[ \sup_{0 \leq r \leq t} \left| \int_0^r (\sigma(s, X_1(s)) - \sigma(s, X_2(s))) dB_s \right|^2 \right] \\ &\leq 2 \|p\|_{L^2(0,T)}^2 \mathbb{E} \left[ \int_0^t |\mu(X_1(s)) - \mu(X_2(s))|^2 ds \right] \\ &\quad + 8\mathbb{E} \left[ \int_0^t |\sigma(s, X_1(s)) - \sigma(s, X_2(s))|^2 ds \right] \\ &\leq (2L^2 \|p\|_{L^2(0,T)}^2 + 8L^2) \int_0^t \mathbb{E} [|X_1(s) - X_2(s)|^2] ds. \end{aligned}$$

If we set  $v(t) = \mathbb{E}[\sup_{0 \leq r \leq t} |X_1(r) - X_2(r)|^2]$ , then  $v$  is bounded thanks to Theorem 16 and from the above inequality we deduce

$$v(t) \leq C \int_0^t v(s) ds, \quad 0 \leq t \leq T$$

with  $C := 2L^2(\|p\|_{L^2(0,T)}^2 + 4)$ . From Gronwall's inequality we conclude that  $v \equiv 0$  on  $[0, T]$ . Thus, the two solutions  $X_1$  and  $X_2$  coincide.  $\square$

## Acknowledgements

The authors would like to thank the anonymous reviewer for reading carefully the paper and giving suggestions to improve it.

## Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

## References

- [1] F. Alabau-Boussouira, P. Cannarsa and G. Fragnelli, "Carleman estimates for degenerate parabolic operators with applications to null controllability", *J. Evol. Equ.* **6** (2006), no. 2, pp. 161–204.
- [2] F. Alabau-Boussouira, P. Cannarsa and G. Leugering, "Control and stabilization of degenerate wave equations", *SIAM J. Control Optim.* **55** (2017), no. 3, pp. 2052–2087.
- [3] F. Alabau-Boussouira, P. Cannarsa and C. Urbani, "Superexponential stabilizability of evolution equations of parabolic type via bilinear control", *J. Evol. Equ.* **21** (2021), no. 1, pp. 941–967.
- [4] F. Alabau-Boussouira, P. Cannarsa and C. Urbani, "Exact controllability to eigensolutions for evolution equations of parabolic type via bilinear control", *NoDEA, Nonlinear Differ. Equ. Appl.* **29** (2022), no. 4, article no. 38.
- [5] F. Alabau-Boussouira and C. Urbani, *A constructive mathematical algorithm for building mixing coupling real valued potentials with time control. Applications to exact controllability and stabilization in bilinear control*, 2020. work in progress, Chapter 5 of Urbani C. PhD Thesis, <http://hdl.handle.net/20.500.12571/10061>.

- [6] P. Baldi, “Stochastic calculus”, in *Stochastic Calculus. An introduction through theory and exercises*, Springer, 2017, pp. 215–254.
- [7] J. M. Ball, J. E. Marsden and M. Slemrod, “Controllability for distributed bilinear systems”, *SIAM J. Control Optim.* **20** (1982), no. 4, pp. 575–597.
- [8] K. Beauchard, “Local controllability of a 1-D Schrödinger equation”, *J. Math. Pures Appl.* **84** (2005), no. 7, pp. 851–956.
- [9] K. Beauchard, “Local controllability and non-controllability for a 1D wave equation with bilinear control”, *J. Differ. Equations* **250** (2011), no. 4, pp. 2064–2098.
- [10] K. Beauchard and C. Laurent, “Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control”, *J. Math. Pures Appl.* **94** (2010), no. 5, pp. 520–554.
- [11] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, *Representation and control of infinite dimensional systems. Volume 1*, Birkhäuser, 1992.
- [12] M. Campiti, G. Metafuno and D. Pallara, “Degenerate self-adjoint evolution equations on the unit interval”, *Semigroup Forum* **57** (1998), no. 1, pp. 1–36.
- [13] P. Cannarsa, P. Martinez and C. Urbani, “Bilinear control of a degenerate hyperbolic equation”, *SIAM J. Math. Anal.* **55** (2023), no. 6, pp. 6517–6553.
- [14] P. Cannarsa, P. Martinez and J. Vancostenoble, “Carleman estimate for a class of degenerate parabolic operators”, *SIAM J. Control Optim.* **47** (2008), no. 1, pp. 1–19.
- [15] P. Cannarsa, P. Martinez and J. Vancostenoble, “The cost of controlling weakly degenerate parabolic equations by boundary controls”, *Math. Control Relat. Fields* **7** (2017), no. 2, pp. 171–211.
- [16] P. Cannarsa and C. Urbani, “Superexponential stabilizability of degenerate parabolic equations via bilinear control”, in *Inverse Problems and Related Topics*, Springer, 2020, pp. 31–45.
- [17] J.-M. Coron, *Control and Nonlinearity*, American Mathematical Society, 2007.
- [18] M. Gueye, “Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations”, *SIAM J. Control Optim.* **52** (2014), no. 4, pp. 2037–2054.
- [19] V. Komornik and P. Loreti, *Fourier series in control theory*, Springer, 2005.
- [20] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications, vol. 1*, Dunod, 1968.
- [21] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer, 2012.
- [22] H. Triebel, *Interpolation Theory, Function Spaces and Differential Operators*, North-Holland, 1978.