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Hankel determinants and Jacobi continued fractions for q-Euler numbers

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Abstract. The q-analogs of Bernoulli and Euler numbers were introduced by Carlitz in 1948. Similar to recent results on the Hankel determinants for the q-Bernoulli numbers established by Chapoton and Zeng, we perform a parallel analysis for the q-Euler numbers. It is shown that the associated orthogonal polynomials for q-Euler numbers are given by a specialization of the big q-Jacobi polynomials, thereby leading to their corresponding Jacobi continued fraction expressions, which eventually serve as a key to our determinant evaluations.

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1. Introduction

A *Hankel matrix* $(M_{i,j})$ is a square matrix with constant skew diagonals, i.e., $M_{i,j} = M_{i',j'}$ whenever $i+j = i'+j'$. This terminology was named after Hermann Hankel, and in recent years Hankel matrices have exhibited substantial utility in data analysis, ranging from geophysics [12] to signal processing [21]. Letting $\{s_n\}_{n\geq 0}$ be a sequence in a field $\mathbb{K}$, one may define its associated Hankel matrices by $(s_{i+j})_{0\leq i,j\leq n}$. It is often of significance to evaluate the determinant of these matrices. Such determinants

$$\det_{0\leq i,j\leq n}(s_{i+j}) = \det\begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_n \\ s_1 & s_2 & s_3 & \cdots & s_{n+1} \\ s_2 & s_3 & s_4 & \cdots & s_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & s_{n+2} & \cdots & s_{2n} \end{pmatrix}$$

are usually called the *Hankel determinants* for $\{s_n\}_{n\geq 0}$.

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From a number-theoretic perspective, it is a natural game to take classical sequences of arithmetic meaning for s_n. As an example, considering the *Bernoulli numbers* B_n defined by the exponential generating function

$$\sum_{n\geq 0} B_n \frac{t^n}{n!} = \frac{t}{e^t-1},$$

it was shown by Al-Salam and Carlitz [1, p. 93, eq. (3.1)] that

$$\det_{0\leq i,j\leq n}(B_{i+j}) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \frac{(k!)^6}{(2k)!(2k+1)!}.$$

Another example treats the *Euler numbers* E_n given by

$$\sum_{n\geq 0} E_n \frac{t^n}{n!} = \frac{2}{e^t+e^{-t}}.$$

Al-Salam and Carlitz [1, p. 93, eq. (4.2)] also proved that

$$\det_{0\leq i,j\leq n}(E_{i+j}) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} (k!)^2.$$

More such Hankel determinant evaluations were nicely collected by Krattenthaler in his seminal surveys [16, Section 2.7] and [17, Section 5.4].

When it comes to generalizing number sequences, there are numerous ways available, with polynomialization standing out as one of the most evident choices. Simply speaking, one may construct a sequence of polynomials so that it reduces to the original number sequence when the argument is specifically chosen. Along this line, we can define *Bernoulli polynomials* $B_n(x)$ and *Euler polynomials* $E_n(x)$ by

$$\sum_{n\geq 0} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t-1},$$
$$\sum_{n\geq 0} E_n(x) \frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1}.$$

Then $B_n = B_n(0)$ and $E_n = 2^n E_n(\frac{1}{2})$. However, the Hankel determinant evaluations for $B_n(x)$ and $E_n(x)$ are not very exciting since

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$
$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x-\frac{1}{2}\right)^{n-k}$$

In this connection, there is a standard result [19, p. 419, Item 445] stating that

$$\det_{0\leq i,j\leq n}(s_{i+j}) = \det_{0\leq i,j\leq n}\big(s_{i+j}(x)\big),$$

where

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_k x^{n-k}.$$

Thus,

$$\det_{0\leq i,j\leq n}\big(B_{i+j}(x)\big) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \frac{(k!)^6}{(2k)!(2k+1)!}, \tag{1}$$

$$\det_{0\leq i,j\leq n}\big(E_{i+j}(x)\big) = (-\tfrac{1}{4})^{\binom{n+1}{2}} \prod_{k=1}^{n} (k!)^2, \tag{2}$$

which can be found in, for instance, [1, p. 94, eqs. (5.1) and (5.2)]. Although the above evaluations are independent of the variable x, the story becomes different when subsequences of Bernoulli or Euler polynomials are considered. As shown in a recent paper of Dilcher and Jiu [9, Theorem 1.1 and Corollary 5.2],

$$\det_{0\le i,j\le n}\left(B_{2(i+j)+1}\left(\tfrac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{k=1}^{n}\left(\frac{k^4(x^2-k^2)}{4(2k-1)(2k+1)}\right)^{n-k+1},$$

$$\det_{0\le i,j\le n}\left(E_{2(i+j)+1}\left(\tfrac{x+1}{2}\right)\right) = (-1)^{\binom{n+1}{2}}\left(\frac{x}{2}\right)^{n+1}\prod_{k=1}^{n}\left(\frac{k^2(x^2-4k^2)}{4}\right)^{n-k+1}.$$

In addition, results of a similar nature can be found in [10].

For other generalizations of number sequences, the q-version is sometimes of more "q-riosity." Usually, we have a sequence of rational functions in q, which yields our original sequence at the limiting case $q\to 1$. Throughout, we introduce the *q-integers* for $m\in\mathbb{Z}$,

$$[m]_q := \frac{1-q^m}{1-q},$$

and the *q-factorials* for $M\in\mathbb{N}$,

$$[M]_q! := \prod_{m=1}^{M}[m]_q.$$

We also require the *q-Pochhammer symbols* for $N\in\mathbb{N}\cup\{\infty\}$,

$$(A;q)_N := \prod_{k=0}^{N-1}(1-Aq^k),$$

and

$$(A,B,\dots,C;q)_N := (A;q)_N(B;q)_N\cdots(C;q)_N.$$

The q-analogs of Bernoulli and Euler numbers were introduced by Carlitz [4]:

$$\beta_n := \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k+1}{[k+1]_q}, \tag{3}$$

$$\epsilon_n := \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+q}{1+q^{k+1}}. \tag{4}$$

Alternatively, Carlitz's *q-Bernoulli numbers* β_n can be recursively defined by $\beta_0=1$ and for $n\ge 1$,

$$\sum_{k=0}^{n}\binom{n}{k}q^{k+1}\beta_k - \beta_n = \begin{cases}1, & \text{if } n=1,\\ 0, & \text{if } n\ge 2,\end{cases} \tag{5}$$

while the *q-Euler numbers* are given by $\epsilon_0=1$ and recursively for $n\ge 1$,

$$\sum_{k=0}^{n}\binom{n}{k}q^{k+1}\epsilon_k + \epsilon_n = 0. \tag{6}$$

It should be remarked that at the $q\to 1$ limit, ϵ_n reduces to

$$1, -\tfrac{1}{2}, 0, \tfrac{1}{4}, 0, -\tfrac{1}{2}, 0, \tfrac{17}{8}, 0, -\tfrac{31}{2}, \dots,$$

which is identical to $E_n(0)$ rather than $2^nE_n(\frac{1}{2})$, the Euler numbers E_n.

In regard to the Hankel determinants, interestingly, we still have neat evaluations for the q-Bernoulli numbers. It was obtained by Chapoton and Zeng [7, p. 359, eq. (4.7)] that

$$\det_{0\le i,j\le n}(\beta_{i+j}) = (-1)^{\binom{n+1}{2}}q^{\binom{n+1}{3}}\prod_{k=1}^{n}\frac{([k]_q!)^6}{[2k]_q![2k+1]_q!}.$$

Meanwhile, Chapoton and Zeng also evaluated $\det(\beta_{i+j+\ell})_{0\le i,j\le n}$ with $\ell\in\{1,2,3\}$. Now one may naturally ask if similar results exist for the q-Euler numbers. Our objective in this paper is to answer this question in the affirmative.

Theorem 1.

$$\det_{0\le i,j\le n}(\epsilon_{i+j})=\frac{(-1)^{\binom{n+1}{2}}q^{\frac14\binom{2n+2}{3}}}{(1-q)^{n(n+1)}}\prod_{k=1}^{n}\frac{(q^2,q^2;q^2)_k}{(-q,-q^2,-q^2,-q^3;q^2)_k},\tag{7}$$

$$\det_{0\le i,j\le n}(\epsilon_{i+j+1})=\frac{(-1)^{\binom{n+2}{2}}q^{\frac14\binom{2n+4}{3}}}{(1-q)^{n(n+1)}(1+q^2)^{n+1}}\prod_{k=1}^{n}\frac{(q^2,q^4;q^2)_k}{(-q^2,-q^3,-q^3,-q^4;q^2)_k},\tag{8}$$

$$\det_{0\le i,j\le n}(\epsilon_{i+j+2})=\frac{(-1)^{\binom{n+2}{2}}q^{\frac14\binom{2n+4}{3}}(1+q)^n\big(1-(-1)^nq^{(n+2)^2}\big)}{(1-q)^{n(n+1)}(1+q^2)^{2(n+1)}(1+q^3)^{n+1}}\prod_{k=1}^{n}\frac{(q^4,q^4;q^2)_k}{(-q^3,-q^4,-q^4,-q^5;q^2)_k}.\tag{9}$$

Remark 2. Taking $q\to1$, (7) becomes

$$\lim_{q\to1}\det_{0\le i,j\le n}(\epsilon_{i+j})=(-\tfrac{1}{16})^{\binom{n+1}{2}}\prod_{k=1}^{n}\big((2k)!!\big)^2=(-\tfrac14)^{\binom{n+1}{2}}\prod_{k=1}^{n}(k!)^2.$$

In light of the fact that $\lim_{q\to1}\epsilon_n=E_n(0)$, the above relation matches (2) with $x=0$.

Remark 3. It is notable that the evaluation of Hankel determinants for q-Euler numbers also has a close connection with q-analogs of values at negative integers for certain Dirichlet L-series, as advocated in a recent paper of Chapoton, Krattenthaler and Zeng [6]. In particular, our (7) can be recovered by a specialization of [6, Theorem 1.5], which suggests a route towards possible bivariate extensions.

It is well-known that the evaluation of Hankel determinants is closely related to orthogonal polynomials and continued fractions, so several preliminary lemmas are provided in Section 2. Then in Section 3, we introduce the big q-Jacobi polynomials, whose orthogonality will be used for our determinant calculations. In addition, to clearly characterize the orthogonality of the big q-Jacobi polynomials, we require a linear functional on $\mathbb{Q}(q)[z]$, as discussed in Section 4. With the above preparations, Section 5 is devoted to the continued fraction expressions for the generating functions of $\{\epsilon_k\}_{k\ge0}$ and $\{\epsilon_{k+1}\}_{k\ge0}$, thereby leading to the proof of Theorem 1. Finally, in Section 6, we close this paper with some additional discussions.

2. Preliminaries

A family of polynomials $\{p_n(z)\}_{n\ge0}$ with $p_n(z)$ of degree n is called *orthogonal* if there is a linear functional L on the space of polynomials in z such that $L\big(p_m(z)p_n(z)\big)=\delta_{m,n}\sigma_n$ where $\delta_{m,n}$ is the Kronecker delta and $\{\sigma_n\}_{n\ge0}$ is a fixed nonzero sequence. Notably, orthogonal polynomials can be characterized as follows:

Lemma 4 (cf. [22, p. 195, Theorem 50.1] or [16, p. 21, Theorem 12]). *Let $\{p_n(z)\}_{n\ge0}$ be a family of monic polynomials with $p_n(z)$ of degree n. Then they are orthogonal if and only if there exist sequences $\{a_n\}_{n\ge0}$ and $\{b_n\}_{n\ge1}$ with $b_n\ne0$ such that $p_0(z)=1$, $p_1(z)=a_0+z$, and for $n\ge1$,*

$$p_{n+1}(z)=(a_n+z)p_n(z)-b_np_{n-1}(z).\tag{10}$$

Remark 5. The above result is often referred to as the *Favard theorem*, named after Jean Favard [11], who discovered it in 1935. However, Favard's discovery is not the earliest, and indeed a similar result was already presented by Stieltjes [20] in 1894. For more historical discussions, see the survey by Marcellán and Álvarez-Nodarse [18].

Note that $\{z^k\}_{k\geq 0}$ forms a basis of the space of polynomials in z. Therefore, for a family of orthogonal polynomials, to explicitly express its associated linear functional L, it suffices to evaluate $L(z^k)$, which is called the *k-th moment associated with L*, for each $k \geq 0$. Such evaluations have a surprising connection with *Jacobi continued fractions*, or *J-fractions* for short.

Lemma 6 (cf. [22, p. 197, Theorem 51.1] or [16, p. 21, Theorem 13]). *Let L be an associated linear functional for a family of orthogonal monic polynomials $\{p_n(z)\}_{n\geq 0}$ with $p_n(z)$ of degree n. Then*

$$\sum_{k\geq 0} L(z^k)x^k = \cfrac{L(z^0)}{1+a_0x-\cfrac{b_1x^2}{1+a_1x-\cfrac{b_2x^2}{1+a_2x-\ddots}}}, \tag{11}$$

where $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 1}$ are as in (10).

Finally, given a J-fraction, Heilermann [14] established evaluations of the Hankel determinants for the sequence of coefficients in the series expansion of this continued fraction.

Lemma 7 (cf. [17, pp. 115–116, Theorem 29]). *Let $\{\mu_k\}_{k\geq 0}$ be a sequence such that its generating function $\sum_{k\geq 0}\mu_k x^k$ has the J-fraction expression*

$$\sum_{k\geq 0} \mu_k x^k = \cfrac{\mu_0}{1+a_0x-\cfrac{b_1x^2}{1+a_1x-\cfrac{b_2x^2}{1+a_2x-\ddots}}}.$$

Then for $n \geq 0$,

$$\det_{0\leq i,j\leq n}(\mu_{i+j}) = \mu_0^{n+1} b_1^n b_2^{n-1}\cdots b_{n-1}^2 b_n. \tag{12}$$

Further, with a_n and b_n from the J-fraction above, define $\{p_n(z)\}_{n\geq 0}$ a family of polynomials given by a three-term recursive relation for $n \geq 1$,

$$p_{n+1}(z) = (a_n+z)p_n(z) - b_n p_{n-1}(z),$$

with initial conditions $p_0(z) = 1$ and $p_1(z) = a_0 + z$. Then

$$\det_{0\leq i,j\leq n}(\mu_{i+j+1}) = \det_{0\leq i,j\leq n}(\mu_{i+j}) \cdot (-1)^{n+1} p_{n+1}(0). \tag{13}$$

Remark 8. It is asserted by Lemma 4 that the polynomials $p_n(z)$ in Lemma 7 are orthogonal. Hence, we shall call $p_n(z)$ the *associated orthogonal polynomials* for $\{\mu_k\}_{k\geq 0}$ throughout.

3. Big q-Jacobi polynomials

The *big q-Jacobi polynomials* were introduced by Andrews and Askey [3], and they form a family of q-hypergeometric orthogonal polynomials in the basic Askey scheme. For our purpose, the following specialization is required.

For each nonnegative integer ℓ, we define a family of polynomials $\{\mathscr{J}_{\ell,n}(z)\}_{n\geq 0}$ by

$$\mathscr{J}_{\ell,n}(z) := {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+\ell+1}, z \\ q^{\ell+1}, 0 \end{matrix}; q, q\right). \tag{14}$$

Here the *q-hypergeometric series* ${}_{r+1}\phi_r$ is defined by

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_r \end{matrix};q,z\right) := \sum_{n\geq 0} \frac{(a_1,a_2,\ldots,a_{r+1};q)_n z^n}{(q,b_1,b_2,\ldots,b_r;q)_n}. $$

It is stated in [15, p. 438, eq. (14.5.3)] that $\mathscr{J}_{\ell,n}(z)$ satisfies the three-term recursive relation

$$ A_{\ell,n}\mathscr{J}_{\ell,n+1}(z) = (A_{\ell,n}+B_{\ell,n}-1+z)\mathscr{J}_{\ell,n}(z) - B_{\ell,n}\mathscr{J}_{\ell,n-1}(z), \tag{15} $$

where

$$ A_{\ell,n} = \frac{1-q^{2n+2\ell+2}}{(1+q^{2n+\ell+1})(1+q^{2n+\ell+2})}, $$
$$ B_{\ell,n} = -\frac{q^{2n+2\ell+1}(1-q^{2n})}{(1+q^{2n+\ell})(1+q^{2n+\ell+1})}. $$

If we normalize $\mathscr{J}_{\ell,n}(z)$ as monic polynomials

$$ \widetilde{\mathscr{J}}_{\ell,n}(z) := \frac{(q^{\ell+1};q)_n}{(-q^{n+\ell+1};q)_n}\mathscr{J}_{\ell,n}(z), \tag{16} $$

then $\widetilde{\mathscr{J}}_{\ell,0}(z)=1$, $\widetilde{\mathscr{J}}_{\ell,1}(z)=\widetilde{a}_{\ell,0}+z$, and for $n\geq 1$,

$$ \widetilde{\mathscr{J}}_{\ell,n+1}(z) = (\widetilde{a}_{\ell,n}+z)\widetilde{\mathscr{J}}_{\ell,n}(z) - \widetilde{b}_{\ell,n}\widetilde{\mathscr{J}}_{\ell,n-1}(z), \tag{17} $$

where

$$ \widetilde{a}_{\ell,n} = -\frac{q^{2n+\ell+1}(1+q)(1+q^{\ell})}{(1+q^{2n+\ell})(1+q^{2n+\ell+2})}, $$
$$ \widetilde{b}_{\ell,n} = -\frac{q^{2n+2\ell+1}(1-q^{2n})(1-q^{2n+2\ell})}{(1+q^{2n+\ell-1})(1+q^{2n+\ell})^2(1+q^{2n+\ell+1})}. $$

Let us further recall a standard result for orthogonal polynomials presented in [8, p. 25, Exercise 4.4].

Lemma 9. *Suppose that $\{p_n(z)\}_{n\geq 0}$ is a family of polynomials given by a three-term recursive relation for $n\geq 1$,*

$$ p_{n+1}(z) = (a_n+z)p_n(z) - b_n p_{n-1}(z), $$

with initial conditions $p_0(z)=1$ and $p_1(z)=a_0+z$, where $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 1}$ are fixed sequences. Let

$$ r_n(z) := u^{-n}p_n(uz+v) $$

for $n\geq 0$ be a new family of polynomials. Then $r_n(z)$ satisfies the three-term recursive relation

$$ r_{n+1}(z) = \big(u^{-1}(a_n+v)+z\big)r_n(z) - u^{-2}b_n r_{n-1}(z). \tag{18} $$

Now we introduce a family of polynomials $\{\mathscr{P}_{\ell,n}(z)\}_{n\geq 0}$ for each nonnegative integer ℓ:

$$ \mathscr{P}_{\ell,n}(z) := \frac{(-1)^n(q^{\ell+1};q)_n}{q^n(1-q)^n(-q^{n+\ell+1};q)_n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},-q^{n+\ell+1},q\big(1-(1-q)z\big) \\ q^{\ell+1},0 \end{matrix};q,q\right). \tag{19} $$

In other words,

$$ \mathscr{P}_{\ell,n}(z) = \frac{(-1)^n}{q^n(1-q)^n}\widetilde{\mathscr{J}}_{\ell,n}\big((q^2-q)z+q\big). $$

The following result is a direct consequence of (17) and (18).

Theorem 10. *We have* $\mathscr{P}_{\ell,0}(z)=1$, $\mathscr{P}_{\ell,1}(z)=a_{\ell,0}+z$, *and for* $n\geq 1$,

$$\mathscr{P}_{\ell,n+1}(z)=(a_{\ell,n}+z)\mathscr{P}_{\ell,n}(z)-b_{\ell,n}\mathscr{P}_{\ell,n-1}(z), \tag{20}$$

where

$$\begin{aligned}a_{\ell,n}&=\frac{q^{2n+\ell}(1+q)(1+q^{\ell})}{(1-q)(1+q^{2n+\ell})(1+q^{2n+\ell+2})}-\frac{1}{1-q},\\ b_{\ell,n}&=-\frac{q^{2n+2\ell-1}(1-q^{2n})(1-q^{2n+2\ell})}{(1-q)^2(1+q^{2n+\ell-1})(1+q^{2n+\ell})^2(1+q^{2n+\ell+1})}.\end{aligned}$$

4. A linear functional on $\mathbb{Q}(q)[z]$

Define for $0\leq m\leq n$ a family of polynomials in $\mathbb{Q}(q)[z]$,

$$\begin{bmatrix} m,z\\ n\end{bmatrix}_q:=\frac{1}{[n]_q!}\prod_{k=m-n+1}^{m}([k]_q+q^kz).$$

It is clear that $\begin{bmatrix} m,z\\ n\end{bmatrix}_q$ is of degree n in z. Further, $\{\begin{bmatrix} n,z\\ n\end{bmatrix}_q\}_{n\geq 0}$ forms a basis of $\mathbb{Q}(q)[z]$.

Let Φ be the linear functional on $\mathbb{Q}(q)[z]$ given by

$$\Phi\left(\begin{bmatrix} n,z\\ n\end{bmatrix}_q\right):=\frac{1}{(-q^2;q)_n}\qquad (n\geq 0). \tag{21}$$

Lemma 11. *For* $0\leq m\leq n$,

$$\Phi\left(\begin{bmatrix} m,z\\ n\end{bmatrix}_q\right)=\frac{(-1)^{n-m}q^{n-m}}{(-q^2;q)_n}. \tag{22}$$

Proof. It was shown in [5, p. 19, Lemma 1] that for $0\leq m\leq n$,

$$\begin{aligned}\begin{bmatrix} m,z\\ n\end{bmatrix}_q&=\sum_{k=0}^{n}(-1)^{n-k}q^{-n(n-m)+\binom{n-k}{2}}\begin{bmatrix} n-m\\ n-k\end{bmatrix}_q\begin{bmatrix} k,z\\ k\end{bmatrix}_q\\ &=q^{-n(n-m)}\sum_{k=0}^{n-m}(-1)^{n-m-k}q^{\binom{n-m-k}{2}}\begin{bmatrix} n-m\\ n-m-k\end{bmatrix}_q\begin{bmatrix} m+k,z\\ m+k\end{bmatrix}_q.\end{aligned}$$

Here the *q-binomial coefficients* are given by

$$\begin{bmatrix} M\\ N\end{bmatrix}_q:=\begin{cases}\dfrac{(q;q)_M}{(q;q)_N(q;q)_{M-N}} & \text{if } 0\leq N\leq M,\\ 0 & \text{otherwise.}\end{cases}$$

Recalling that Φ is linear, $\Phi\left(\begin{bmatrix} m,z\\ n\end{bmatrix}_q\right)$ equals

$$q^{-n(n-m)}\sum_{k=0}^{n-m}(-1)^{n-m-k}q^{\binom{n-m-k}{2}}\begin{bmatrix} n-m\\ n-m-k\end{bmatrix}_q\Phi\left(\begin{bmatrix} m+k,z\\ m+k\end{bmatrix}_q\right).$$

In light of (21),

$$\begin{aligned}\Phi\left(\begin{bmatrix} m,z\\ n\end{bmatrix}_q\right)&=q^{-n(n-m)}\sum_{k=0}^{n-m}(-1)^{n-m-k}q^{\binom{n-m-k}{2}}\begin{bmatrix} n-m\\ n-m-k\end{bmatrix}_q\frac{1}{(-q^2;q)_{m+k}}\\ &=\frac{(-1)^{n-m}q^{\binom{m+1}{2}-\binom{n+1}{2}}}{(-q^2;q)_m}\,{}_2\phi_1\left(\begin{matrix} q^{-(n-m)},0\\ -q^{m+2}\end{matrix};q,q\right)\\ &=\frac{(-1)^{n-m}q^{\binom{m+1}{2}-\binom{n+1}{2}}}{(-q^2;q)_m}\cdot\frac{q^{(m+2)(n-m)+\binom{n-m}{2}}}{(-q^{m+2};q)_{n-m}},\end{aligned}$$

thereby yielding the desired result. Here for the evaluation of the ${}_2\phi_1$ series, we have applied the *q-Chu–Vandermonde Sum* [13, p. 354, eq. (II.6)]:

$$ {}_2\phi_1\left(\begin{matrix} a, q^{-N} \\ c \end{matrix}; q, q\right) = \frac{a^N (c/a;q)_N}{(c;q)_N} \tag{23} $$

at the $a \to 0$ limiting case. □

Lemma 12. *For $n \geq 0$,*

$$ \Phi\left(\begin{bmatrix} n+1, z \\ n \end{bmatrix}_q\right) = \frac{1+q}{q} - \frac{1}{q(-q^2;q)_n}. \tag{24} $$

Proof. We prove this relation by induction on n. It is clear that the statement is true for $n = 0$ as

$$ \Phi\left(\begin{bmatrix} 1, z \\ 0 \end{bmatrix}_q\right) = \Phi(1) = \Phi\left(\begin{bmatrix} 0, z \\ 0 \end{bmatrix}_q\right) = 1. $$

Assuming that the statement is true for some $n \geq 0$, we shall show that it is also true for $n+1$. We begin by noticing that

$$ \begin{aligned} \begin{bmatrix} n+2, z \\ n+1 \end{bmatrix}_q - q^{n+1} \begin{bmatrix} n+1, z \\ n+1 \end{bmatrix}_q &= \left(([n+2]_q + q^{n+2} z) - q^{n+1}([1]_q + qz)\right) \cdot \frac{1}{[n+1]_q!} \prod_{k=2}^{n+1} ([k]_q + q^k z) \\ &= \frac{[n+2]_q - q^{n+1}}{[n+1]_q} \cdot \begin{bmatrix} n+1, z \\ n \end{bmatrix}_q \\ &= \begin{bmatrix} n+1, z \\ n \end{bmatrix}_q. \end{aligned} $$

Applying the linear functional Φ on both sides gives

$$ \begin{aligned} \Phi\left(\begin{bmatrix} n+2, z \\ n+1 \end{bmatrix}_q\right) &= q^{n+1} \cdot \Phi\left(\begin{bmatrix} n+1, z \\ n+1 \end{bmatrix}_q\right) + \Phi\left(\begin{bmatrix} n+1, z \\ n \end{bmatrix}_q\right) \\ &= q^{n+1} \cdot \frac{1}{(-q^2;q)_{n+1}} + \left(\frac{1+q}{q} - \frac{1}{q(-q^2;q)_n}\right) \\ &= \frac{1+q}{q} - \frac{1}{q(-q^2;q)_{n+1}}, \end{aligned} $$

as desired. Here we have utilized (21) together with the inductive assumption for the second equality. □

Theorem 13. *For any $P(z) \in \mathbb{Q}(q)[z]$,*

$$ q\Phi\big(P(1+qz)\big) + \Phi\big(P(z)\big) = (1+q)P(0). \tag{25} $$

Proof. Since $\{\left[\begin{smallmatrix} n,z \\ n \end{smallmatrix}\right]_q\}_{n\geq 0}$ forms a basis of $\mathbb{Q}(q)[z]$, we may write $P(z)$ as

$$ P(z) = \sum_{n=0}^{N} c_n \begin{bmatrix} n, z \\ n \end{bmatrix}_q, $$

with N the degree of $P(z)$ and $c_n \in \mathbb{Q}(q)$ for each $0 \leq n \leq N$. Note that for $n \geq 0$,

$$ \begin{bmatrix} n, 1+qz \\ n \end{bmatrix}_q = \frac{1}{[n]_q!} \prod_{k=1}^{n} \left([k]_q + q^k(1+qz)\right) = \begin{bmatrix} n+1, z \\ n \end{bmatrix}_q. $$

Therefore,

$$\begin{aligned} q\Phi\big(P(1+qz)\big)+\Phi\big(P(z)\big) &= q\sum_{n=0}^{N} c_n \Phi\left(\begin{bmatrix} n+1,z \\ n \end{bmatrix}_q\right)+\sum_{n=0}^{N} c_n \Phi\left(\begin{bmatrix} n,z \\ n \end{bmatrix}_q\right) \\ \text{(by } {\tiny\substack{(21)\\(24)}}\text{)} \quad &= q\sum_{n=0}^{N} c_n\left(\frac{1+q}{q}-\frac{1}{q(-q^2;q)_n}\right)+\sum_{n=0}^{N} c_n\left(\frac{1}{(-q^2;q)_n}\right) \\ &= (1+q)\sum_{n=0}^{N} c_n. \end{aligned}$$

Finally, we evaluate that

$$P(0)=\sum_{n=0}^{N} c_n \begin{bmatrix} n,0 \\ n \end{bmatrix}_q = \sum_{n=0}^{N} c_n \frac{[n]_q!}{[n]_q!} = \sum_{n=0}^{N} c_n,$$

thereby concluding the required relation. □

Theorem 14. *For $n \ge 0$,*

$$\Phi(z^n)=\epsilon_n. \tag{26}$$

Proof. We first notice that $\Phi(z^0)=\Phi(1)=1=\epsilon_0$. Now for $n\ge 1$, we apply Theorem 13 with $P(z)=z^n$, and derive that

$$q\Phi\big((1+qz)^n\big)+\Phi(z^n)=0,$$

namely,

$$\sum_{k=0}^{n}\binom{n}{k} q^{k+1}\Phi(z^k)+\Phi(z^n)=0.$$

Since this recursive relation for $\Phi(z^n)$ is identical to that for ϵ_n as given in (6), we conclude that $\Phi(z^n)=\epsilon_n$. □

5. Jacobi continued fractions and Hankel determinants

Let $\ell\in\{0,1\}$. We define two linear functionals Φ_ℓ on $\mathbb{Q}(q)[z]$ by

$$\Phi_\ell(z^n) := \Phi(z^{n+\ell}) \qquad (n\ge 0), \tag{27}$$

where the linear functional Φ is as in (21).

Theorem 15. *Let $\ell\in\{0,1\}$. The family of monic polynomials $\{\mathscr{P}_{\ell,n}(z)\}_{n\ge 0}$ given in* (19) *is orthogonal under the linear functional Φ_ℓ.*

Proof. The orthogonality of $\{\mathscr{P}_{\ell,n}(z)\}_{n\ge0}$ is ensured by Lemma 4 since $\mathscr{P}_{\ell,n}(z)$ satisfies a three-term recursive relation as shown in (20). Note that

$$\Phi_0\big(\mathscr{P}_{0,0}(z)\big)=\Phi_0(z^0)=\Phi(z^0)=\epsilon_0\neq 0,$$

and that

$$\Phi_1\big(\mathscr{P}_{1,0}(z)\big)=\Phi_1(z^0)=\Phi(z^1)=\epsilon_1\neq 0,$$

where (26) is invoked. Hence, it suffices to show that for $\ell\in\{0,1\}$, the identity $\Phi_\ell\big(\mathscr{P}_{\ell,n}(z)\big)=0$ holds whenever $n\ge1$.

When $\ell=0$, we start by observing that for $k\ge 0$,

$$(q(1-(1-q)z);q)_k=(q;q)_k\begin{bmatrix} k,z \\ k \end{bmatrix}_q.$$

Thus,

$$\mathscr{P}_{0,n}(z)=\frac{(-1)^n(q;q)_n}{q^n(1-q)^n(-q^{n+1};q)_n}\sum_{k=0}^{n}\frac{q^k(q^{-n},-q^{n+1};q)_k}{(q;q)_k}\begin{bmatrix}k,z\\k\end{bmatrix}_q.$$

Since $\Phi_0\big(\mathscr{P}_{0,n}(z)\big)=\Phi\big(\mathscr{P}_{0,n}(z)\big)$, it follows that for $n\ge 1$,

$$\begin{aligned}\Phi_0\big(\mathscr{P}_{0,n}(z)\big)&=\frac{(-1)^n(q;q)_n}{q^n(1-q)^n(-q^{n+1};q)_n}\sum_{k=0}^{n}\frac{q^k(q^{-n},-q^{n+1};q)_k}{(q;q)_k}\Phi\left(\begin{bmatrix}k,z\\k\end{bmatrix}_q\right)\\ \text{(by (21))}\quad&=\frac{(-1)^n(q;q)_n}{q^n(1-q)^n(-q^{n+1};q)_n}\sum_{k=0}^{n}\frac{q^k(q^{-n},-q^{n+1};q)_k}{(q,-q^2;q)_k}\\ \text{(by (23))}\quad&=\frac{(-1)^n(q;q)_n}{q^n(1-q)^n(-q^{n+1};q)_n}\cdot\frac{(-1)^nq^{n(n+1)}(q^{1-n};q)_n}{(-q^2;q)_n}\\ &=0.\end{aligned}$$

When $\ell=1$, we notice that for $k\ge 0$,

$$z(q(1-(1-q)z);q)_k=(q^2;q)_k\begin{bmatrix}k,z\\k+1\end{bmatrix}_q.$$

Thus,

$$z\cdot\mathscr{P}_{1,n}(z)=\frac{(-1)^n(q^2;q)_n}{q^n(1-q)^n(-q^{n+2};q)_n}\sum_{k=0}^{n}\frac{q^k(q^{-n},-q^{n+2};q)_k}{(q;q)_k}\begin{bmatrix}k,z\\k+1\end{bmatrix}_q.$$

Since $\Phi_1\big(\mathscr{P}_{1,n}(z)\big)=\Phi\big(z\cdot\mathscr{P}_{1,n}(z)\big)$, it follows that for $n\ge 1$,

$$\begin{aligned}\Phi_1\big(\mathscr{P}_{1,n}(z)\big)&=\frac{(-1)^n(q^2;q)_n}{q^n(1-q)^n(-q^{n+2};q)_n}\sum_{k=0}^{n}\frac{q^k(q^{-n},-q^{n+2};q)_k}{(q;q)_k}\Phi\left(\begin{bmatrix}k,z\\k+1\end{bmatrix}_q\right)\\ \text{(by (22))}\quad&=-\frac{(-1)^n(q^2;q)_n}{q^n(1-q)^n(-q^{n+2};q)_n}\cdot\frac{q}{1+q^2}\sum_{k=0}^{n}\frac{q^k(q^{-n},-q^{n+2};q)_k}{(q,-q^3;q)_k}\\ \text{(by (23))}\quad&=-\frac{(-1)^n(q^2;q)_n}{q^n(1-q)^n(-q^{n+2};q)_n}\cdot\frac{q}{1+q^2}\cdot\frac{(-1)^nq^{n(n+2)}(q^{1-n};q)_n}{(-q^3;q)_n}\\ &=0.\end{aligned}$$

The required claim therefore holds. □

Now we are ready to state the J-fraction expressions for the series generated by $\{\epsilon_k\}_{k\ge0}$ and $\{\epsilon_{k+1}\}_{k\ge0}$.

Corollary 16. *Let $\ell\in\{0,1\}$. We have*

$$\sum_{k\ge0}\epsilon_{k+\ell}x^k=\cfrac{\epsilon_\ell}{1+a_{\ell,0}x-\cfrac{b_{\ell,1}x^2}{1+a_{\ell,1}x-\cfrac{b_{\ell,2}x^2}{1+a_{\ell,2}x-\ddots}}}.\tag{28}$$

Further, the associated orthogonal polynomials for $\{\epsilon_{k+\ell}\}_{k\ge0}$ are $\mathscr{P}_{\ell,n}(z)$. Here, $a_{\ell,n}$, $b_{\ell,n}$ and $\mathscr{P}_{\ell,n}(z)$ are as in Theorem 10.

Proof. From (26), we know that when $\ell \in \{0,1\}$, the relation $\epsilon_{k+\ell} = \Phi_\ell(z^k)$ holds for all $k \geq 0$. In view of Theorems 10 and 15, we apply Lemma 6 and obtain

$$\sum_{k\geq 0} \epsilon_{k+\ell} x^k = \sum_{k\geq 0} \Phi_\ell(z^k) x^k = \cfrac{\Phi_\ell(z^0)}{1 + a_{\ell,0}x - \cfrac{b_{\ell,1}x^2}{1 + a_{\ell,1}x - \cfrac{b_{\ell,2}x^2}{1 + a_{\ell,2}x - \ddots}}},$$

where we have noted that $\Phi_0(z^0) = \epsilon_0$ and $\Phi_1(z^0) = \epsilon_1$. □

Finally, we are in a position to complete the proof of Theorem 1.

Proof of Theorem 1. For (7) and (8), we directly apply (12) with Corollary 16 in mind. For (9), we make use of (13) so that

$$\det_{0\leq i,j\leq n}(\epsilon_{i+j+2}) = \det_{0\leq i,j\leq n}(\epsilon_{i+j+1}) \cdot (-1)^{n+1}\mathscr{P}_{1,n+1}(0).$$

Note that by (19),

$$\begin{aligned}
\mathscr{P}_{1,n}(0) &= \frac{(-1)^n (q^2;q)_n}{q^n(1-q)^n(-q^{n+2};q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+2}, q \\ q^2, 0 \end{matrix}; q, q\right) \\
&= \frac{(-1)^n (q^2;q)_n}{q^n(1-q)^n(-q^{n+2};q)_n} \sum_{k\geq 0} \frac{(q^{-n}, -q^{n+2};q)_k q^k}{(q^2;q)_k} \\
&= \frac{(-1)^{n+1} (q;q)_n}{(1-q)^n(-q^{n+1};q)_{n+1}} \sum_{k\geq 1} \frac{(q^{-n-1}, -q^{n+1};q)_k q^k}{(q;q)_k} \\
\text{(by (23))} \quad &= \frac{(-1)^{n+1} (q;q)_n}{(1-q)^n(-q^{n+1};q)_{n+1}} \left(-1 + (-1)^{n+1} q^{(n+1)^2}\right).
\end{aligned}$$

Finally, invoking (8) yields the desired result after simplification. □

6. Conclusion

Noting that Theorem 15 only covers the cases of $\ell = 0$ or 1, it is natural to ask if one could go beyond. Recall that for every $\ell \geq 0$, the orthogonality of $\{\mathscr{P}_{\ell,n}(z)\}_{n\geq 0}$ is ensured by Lemma 4 in light of (20). Now let us reformulate $\mathscr{P}_{\ell,n}(z)$ as

$$\mathscr{P}_{\ell,n}(z) = \frac{(-1)^n (q^{\ell+1};q)_n}{q^n(1-q)^n(-q^{n+\ell+1};q)_n} \sum_{k=0}^{n} \frac{q^k (q^{-n}, -q^{n+\ell+1};q)_k}{(q^{\ell+1};q)_k} \begin{bmatrix} k, z \\ k \end{bmatrix}_q.$$

For each $\ell \geq 0$, define the linear functional Θ_ℓ on $\mathbb{Q}(q)[z]$ by

$$\Theta_\ell\left(\begin{bmatrix} n, z \\ n \end{bmatrix}_q\right) := \frac{(q^{\ell+1};q)_n}{(q, -q^{\ell+2};q)_n} \qquad (n \geq 0). \tag{29}$$

Then $\Theta_\ell\big(\mathscr{P}_{\ell,0}(z)\big) = 1$ and for $n \geq 1$,

$$\begin{aligned}\Theta_\ell\big(\mathscr{P}_{\ell,n}(z)\big) &= \frac{(-1)^n(q^{\ell+1};q)_n}{q^n(1-q)^n(-q^{n+\ell+1};q)_n}\sum_{k=0}^{n}\frac{q^k(q^{-n},-q^{n+\ell+1};q)_k}{(q^{\ell+1};q)_k}\Theta_\ell\left(\begin{bmatrix}k,z\\k\end{bmatrix}_q\right)\\ &= \frac{(-1)^n(q^{\ell+1};q)_n}{q^n(1-q)^n(-q^{n+\ell+1};q)_n}\,{}_2\phi_1\left(\begin{matrix}q^{-n},-q^{n+\ell+1}\\-q^{\ell+2}\end{matrix};q,q\right)\\ \text{(by (23))}\quad &= \frac{(-1)^n(q^{\ell+1};q)_n}{q^n(1-q)^n(-q^{n+\ell+1};q)_n}\cdot\frac{(-1)^n q^{n(n+\ell+1)}(q^{1-n};q)_n}{(-q^{\ell+2};q)_n}\\ &= 0.\end{aligned}$$

Hence, Theorem 15 can be extended as follows.

Theorem 17. *For each nonnegative integer ℓ, the family of monic polynomials $\{\mathscr{P}_{\ell,n}(z)\}_{n\geq 0}$ given in* (19) *is orthogonal under the linear functional Θ_ℓ.*

Remark 18. Given a family of orthogonal polynomials $\{p_n(z)\}_{n\geq 0}$ in $\mathbb{K}[z]$ and two associated linear functionals L_1 and L_2, it is clear by choosing $\{p_n(z)\}_{n\geq 0}$ as a basis of $\mathbb{K}[z]$ that there is a nonzero constant $C \in \mathbb{K}$ such that $L_1 = C \cdot L_2$. In our case, we have

$$\Phi_0 = \epsilon_0\cdot\Theta_0 \qquad \text{and} \qquad \Phi_1 = \epsilon_1\cdot\Theta_1.$$

On the other hand, it is a direct consequence of (12) and (20) that for each nonnegative integer ℓ,

$$\det_{0\leq i,j\leq n}\big(\Theta_\ell(z^{i+j})\big) = \frac{(-1)^{\binom{n+1}{2}}q^{2\binom{n+2}{3}+(2\ell-1)\binom{n+1}{2}}}{(1-q)^{n(n+1)}}\prod_{k=1}^{n}\frac{(q^2,q^{2\ell+2};q^2)_k}{(-q^{\ell+1},-q^{\ell+2},-q^{\ell+2},-q^{\ell+3};q^2)_k}. \tag{30}$$

This identity reduces to (7) and (8) for

$$\begin{aligned}\det_{0\leq i,j\leq n}\big(\Phi_0(z^{i+j})\big) &= \epsilon_0^{n+1}\cdot\det_{0\leq i,j\leq n}\big(\Theta_0(z^{i+j})\big),\\ \det_{0\leq i,j\leq n}\big(\Phi_1(z^{i+j})\big) &= \epsilon_1^{n+1}\cdot\det_{0\leq i,j\leq n}\big(\Theta_1(z^{i+j})\big).\end{aligned}$$

However, for $\ell \geq 2$, the linear functionals Θ_ℓ and Φ are no longer related, thereby resulting in a gap between $\Theta_\ell(z^n)$ and the q-Euler numbers for higher cases of ℓ. In fact, it remains uncertain if there is a closed expression for the moments $\Theta_\ell(z^n)$ when $\ell \geq 2$.

Interestingly, if we directly look at the normalized big q-Jacobi polynomials $\widetilde{\mathscr{J}}_{\ell,n}(z)$, it is possible to deduce an infinite family of nice Hankel determinant evaluations, as demonstrated below. In particular, these evaluations have a quite different nature from the rest because the involved moments, as stated in (32), do have an explicit expression.

Theorem 19. *For each nonnegative integer ℓ, define a sequence $\{\xi_{\ell,n}\}_{n\geq 0}$ by*

$$\xi_{\ell,n} := \frac{q^{(\ell+1)n}(-q;q)_n}{(-q^{\ell+2};q)_n}.$$

Then

$$\det_{0\leq i,j\leq n}(\xi_{\ell,i+j}) = (-1)^{\binom{n+1}{2}}q^{2\binom{n+2}{3}+(2\ell+1)\binom{n+1}{2}}\prod_{k=1}^{n}\frac{(q^2,q^{2\ell+2};q^2)_k}{(-q^{\ell+1},-q^{\ell+2},-q^{\ell+2},-q^{\ell+3};q^2)_k}. \tag{31}$$

Proof. For each $\ell \geq 0$, we introduce the linear functional Ξ_ℓ on $\mathbb{Q}(q)[z]$ by

$$\Xi_\ell(z^n) := \xi_{\ell,n} \qquad (n \geq 0). \tag{32}$$

We then have that for $n \geq 0$,

$$\Xi_\ell\big((z;q)_n\big) = \frac{(q^{\ell+1};q)_n}{(-q^{\ell+2};q)_n}. \tag{33}$$

This is because the *q-binomial theorem* [2, p. 36, eq. (3.3.6)] tells us that

$$(z;q)_n = \sum_{k=0}^{n} (-z)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

so that

$$\begin{aligned}
\Xi_\ell\big((z;q)_n\big) &= \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \cdot \Xi_\ell(z^k) \\
\text{(by (32))} \quad &= \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \cdot \frac{(-1)^k q^{nk-\binom{k}{2}} (q^{-n};q)_k}{(q;q)_k} \cdot \frac{q^{(\ell+1)k}(-q;q)_k}{(-q^{\ell+2};q)_k} \\
&= {}_2\phi_1\left(\begin{matrix} q^{-n}, -q \\ -q^{\ell+2} \end{matrix}; q, q^{n+\ell+1}\right) \\
&= \frac{(q^{\ell+1};q)_n}{(-q^{\ell+2};q)_n},
\end{aligned}$$

as claimed. Here we have made use of the *reverse q-Chu–Vandermonde Sum* [13, p. 354, eq. (II.7)] in the last equality.

Now we consider the orthogonal polynomials $\widetilde{\mathscr{J}}_{\ell,n}(z)$ as defined in (16). To show that Ξ_ℓ is their associated linear functional, we note that $\Xi_\ell\big(\widetilde{\mathscr{J}}_{\ell,0}(z)\big) = \Xi_\ell(1) = 1$ and need to verify that $\Xi_\ell\big(\widetilde{\mathscr{J}}_{\ell,n}(z)\big) = 0$ whenever $n \ge 1$. In fact,

$$\begin{aligned}
\Xi_\ell\big(\widetilde{\mathscr{J}}_{\ell,n}(z)\big) &= \frac{(q^{\ell+1};q)_n}{(-q^{n+\ell+1};q)_n} \sum_{k=0}^{n} \frac{q^k(q^{-n},-q^{n+\ell+1};q)_k}{(q,q^{\ell+1};q)_k} \cdot \Xi_\ell\big((z;q)_k\big) \\
&= \frac{(q^{\ell+1};q)_n}{(-q^{n+\ell+1};q)_n} {}_2\phi_1\left(\begin{matrix} q^{-n}, -q^{n+\ell+1} \\ -q^{\ell+2} \end{matrix}; q, q\right) \\
&= 0.
\end{aligned}$$

Finally, in light of the recursive relation for $\widetilde{\mathscr{J}}_{\ell,n}(z)$ given in (17) with the same $\widetilde{a}_{\ell,n}$ and $\widetilde{b}_{\ell,n}$ therein, we have

$$\sum_{k\ge 0} \xi_{\ell,k} x^k = \cfrac{1}{1+\widetilde{a}_{\ell,0}x - \cfrac{\widetilde{b}_{\ell,1}x^2}{1+\widetilde{a}_{\ell,1}x - \cfrac{\widetilde{b}_{\ell,2}x^2}{1+\widetilde{a}_{\ell,2}x - \ddots}}}. \tag{34}$$

Applying (12) yields the desired Hankel determinant evaluations. □

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