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On the disentanglement of Gaussian quantum states by symplectic rotations

Sur la désintrication des états quantiques Gaussiens par des rotations symplectiques

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Abstract. We show that every Gaussian mixed quantum state can be disentangled by conjugation with a unitary operator corresponding to a symplectic rotation via the metaplectic representation of the symplectic group. The main tools we use are the Werner–Wolf condition for separability on covariance matrices and the symplectic covariance of Weyl pseudo-differential operators.

Résumé. Nous montrons que chaque état quantique Gaussien peut-être rendu séparable (= « désintriqué ») par conjugaison avec un opérateur unitaire associé via le groupe métaplectique à une rotation symplectique. Pour cela nous utilsons la condition de séparabilité de Werner et Wolf sur la matrice de covariance ainsi que la covariance symplectique des opérateurs pseudo-différentiels de Weyl.

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1. Introduction

Gaussian states play an ubiquitous role in quantum information theory and in quantum optics because they are easy to manufacture in the laboratory, and have in addition important extremality properties [12]. Of particular interest are the separability and entanglement properties of Gaussian states; the literature on the topic is immense; two excellent texts whose mathematical setup is rigorous are [1, 2]. It turns out that even if major advances have been made in the study of the separability of Gaussian quantum states in recent years (one of the milestones being Werner and Wolf's paper [10] about the covariance matrices of bipartite states), the topic is still largely open. The aim of this Note is to show that every Gaussian state can be made separable by using a symplectic rotation and of the corresponding metaplectic operator. (We note that physicists use the terminology "passive symplectic transformations" in place of "symplectic rotation"). This result can be viewed as closing a problem originally posed in Wolf et al. [11], who asked which Gaussian states can be entangled by symplectic rotations. A full answer has recently been given in [8] et al. where the Gaussian states that are separable for all symplectic rotations are characterized. Our result (Theorem 1) shows that, conversely, every entangled Gaussian state can be separated ("disentangled") by metaplectic transformations corresponding to symplectic rotations.

We will use the following notation. Let $\mathbb{R}^{2n} = \mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B}$ be the phase space of a bipartite system ($n_A \geq 1$, $n_B \geq 1$). We will use the following phase space variable ordering: $z = (z_A, z_B) = z_A \oplus z_B$ with $z_A = (x_1, p_1, \ldots, x_{n_A}, p_{n_A})$ and $z_B = (x_{n_A+1}, p_{n_A+1}, \ldots, x_n, p_n)$. We equip the symplectic spaces $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$ with their canonical bases. The symplectic structure on $\mathbb{R}^{2n}$ is then $\sigma(z, z') = Jz \cdot z'$ with $J = J_A \oplus J_B$ where

$$J_A = \bigoplus_{k=1}^{n_A} J_k \ , \ J_k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and likewise for J_B. Thus J_A (resp. J_B) determines the symplectic structure on the partial phase space $\mathbb{R}^{2n_A}$ (resp. $\mathbb{R}^{2n_B}$).

2. Result: Statement and Proof

Let Σ be a real positive definite symmetric $2n \times 2n$ matrix (to be called "covariance matrix" from now on) and consider the associated normal probability distribution

$$\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2}\Sigma^{-1} z^2} \ . \tag{1}$$

If the covariance matrix satisfies in addition the condition

$$\Sigma + \frac{i\hbar}{2} J \geq 0 \tag{2}$$

(J the standard symplectic matrix) then ρ is the Wigner distribution of a mixed quantum state, identified with its density operator $\widehat{\rho}$. We notice that property (2) crucially depends on the numerical value of $\hbar$ (see [4, 7]). We will say that $\widehat{\rho}$ is "AB-separable" if there exist sequences of density operators $(\widehat{\rho}_j^A)$ and $(\widehat{\rho}_j^B)$ on $L^2(\mathbb{R}^{n_A})$ and $L^2(\mathbb{R}^{n_B})$, respectively and coefficients $\lambda_j \geq 0$ summing up to one, such that

$$\widehat{\rho} = \sum_j \lambda_j \widehat{\rho}_j^A \otimes \widehat{\rho}_j^B \tag{3}$$

where the convergence is for the trace-class norm. The problem of determining necessary and sufficient conditions for a density operator to be separable is still very largely open; while there exist necessary conditions, no simple sufficient condition for separability is known in the general case; for a recent up to date discussion see Lami et al. [8]. Werner and Wolf [10] have proven that in the Gaussian case $\widehat{\rho}$ is separable if and only if there exists a $2n_A \times 2n_A$ covariance matrix Σ_A and a $2n_B \times 2n_B$ covariance matrix Σ_B such that the following conditions hold:

$$\Sigma_A + \frac{i\hbar}{2} J_A \geq 0 \tag{4}$$

$$\Sigma_B + \frac{i\hbar}{2} J_B \geq 0 \tag{5}$$

$$\Sigma \geq \Sigma_A \oplus \Sigma_B \ . \tag{6}$$

The aim of this Letter is to prove that for every Gaussian density operator there exists a unitary transform $\widehat{U}$ such that $\widehat{U}\widehat{\rho}\widehat{U}^{-1}$ is a separable Gaussian state:

Theorem 1. *Let $\widehat{\rho}$ be a density operator with Gaussian Wigner distribution* (1). *There exists a symplectic rotation $U \in U(n)$ $(= \mathrm{Sp}(n) \cap O(2n,\mathbb{R}))$ such that $\widehat{U}\widehat{\rho}\widehat{U}^{-1}$ is separable where $\widehat{U} \in \mathrm{Mp}(n)$ is any of the two metaplectic operators covering U.*

Proof. We begin by recalling [5,6] that the quantum condition (2) is equivalent to the statement:

$$\text{There exists } S \in \mathrm{Sp}(n) \text{ such that } SB^{2n}(\sqrt{\hbar}) \subset \Omega_\Sigma \tag{7}$$

where $\mathrm{Sp}(n)$ is the symplectic group of the phase space $\mathbb{R}^{2n} \equiv \mathbb{R}^n_x \times \mathbb{R}^n_p$ equipped with the standard symplectic form

$$\sigma = \mathrm{d}p_1 \wedge \mathrm{d}x_1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}x_n ,$$

$B^{2n}(\sqrt{\hbar})$ is the phase space ball defined by $|z| \le \hbar$ and Ω_Σ the covariance ellipsoid of $\widehat{\rho}$:

$$\Omega_\Sigma = \{z \in \mathbb{R}^{2n} : \tfrac{1}{2}\Sigma^{-1}z^2 \le 1\} .$$

Let $S = PR$ $(P = (S^TS)^{1/2}, R = (S^TS)^{-1/2}S)$ be the symplectic polar decomposition [5] of $S \in \mathrm{Sp}(n)$, that is $P \in \mathrm{Sp}(n)$, $P > 0$, and

$$R \in U(n) = \mathrm{Sp}(n) \cap O(2n,\mathbb{R}) .$$

We have $SB^{2n}(\sqrt{\hbar}) = PB^{2n}(\sqrt{\hbar})$ by rotational symmetry of the ball $B^{2n}(\sqrt{\hbar})$. There exists a symplectic rotation $U \in U(n)$ diagonalizing P [5]:

$$P = U^T \Delta U \tag{8}$$

where $\Delta \in \mathrm{Sp}(n)$ is a diagonal matrix whose form will be discussed in a moment. The inclusion $SB^{2n}(\sqrt{\hbar}) \subset \Omega_\Sigma$ in (7) is thus equivalent to $\Delta B^{2n}(\sqrt{\hbar}) \subset U(\Omega_\Sigma)$, that is

$$\Delta B^{2n}(\sqrt{\hbar}) \subset \Omega_{\Sigma_U} \tag{9}$$

where $\Sigma_U = U\Sigma U^T$. This inclusion is equivalent to the matrix inequality

$$\frac{\hbar}{2}\Delta^2 \le \Sigma_U \tag{10}$$

($A \le B$ meaning that $B - A$ is positive semidefinite). We next note that Σ_U is the covariance matrix of the density operator $\widehat{\rho}_U$ with Wigner distribution $\rho_U(z) = \rho(U^Tz)$ that is

$$\rho_U(z) = \frac{1}{(2\pi)^n\sqrt{\det U\Sigma U^T}} e^{-\frac{1}{2}\Sigma^{-1}U^Tz\cdot U^Tz} .$$

Recall now the following symplectic covariance property: if $\widehat{A} = \mathrm{Op}^{\mathrm{W}}(a)$ is a Weyl operator with symbol a and $\widehat{S} \in \mathrm{Mp}(n)$ a metaplectic operator covering $S \in \mathrm{Sp}(n)$ then

$$\widehat{S}\,\mathrm{Op}^{\mathrm{W}}(a)\widehat{S}^{-1} = \mathrm{Op}^{\mathrm{W}}(a \circ S^{-1}) \tag{11}$$

(see for instance [9] or [5, Ch. 7]). Applying this covariance formula to $\widehat{\rho} = (2\pi\hbar)^n \mathrm{Op}^{\mathrm{W}}(\rho)$ yields since $U^T = U^{-1}$,

$$\widehat{\rho}_U = \widehat{U}\widehat{\rho}\widehat{U}^{-1} \tag{12}$$

where $\widehat{U}$ is anyone of the two metaplectic operators $\pm\widehat{U}$ covering U. We claim that $\widehat{\rho}_U$ is separable. To see this, let us come back to the diagonal matrix Δ appearing in the factorization $P = U^T\Delta U$ (8). Its diagonal elements are the eigenvalues $\lambda_1, \dots, \lambda_{2n}$ of the positive definite symplectic matrix P and therefore appear in pairs $(\lambda, 1/\lambda)$ with $\lambda > 0$ [3,5]. In fact, in the AB-ordering we are using, the matrix Δ has the form $\Delta = \Delta_A \oplus \Delta_B$ with

$$\Delta_A = \bigoplus_{k=1}^{n_A} \Delta_k \ , \ \Delta_B = \bigoplus_{k=n_A+1}^{n} \Delta_k$$

and $\Delta_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k^{-1} \end{pmatrix}$ for $k = 1, \dots, n$. Clearly $\Delta_A \in \mathrm{Sp}(n_A)$ and $\Delta_B \in \mathrm{Sp}(n_B)$. The symmetric matrices

$$\Sigma_A = \frac{\hbar}{2}\Delta_A^2 \ , \ \Sigma_B = \frac{\hbar}{2}\Delta_B^2$$

trivially satisfy $\Sigma_A + \frac{i\hbar}{2} J_A \geq 0$ and $\Sigma_B + \frac{i\hbar}{2} J_B \geq 0$. In view of (10) we have

$$\Sigma_A \oplus \Sigma_B \leq \Sigma_U$$

and the theorem now follows using the Werner–Wolf conditions (4)–(6). □

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