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The Pinchuk example revisited

L'exemple de Pinchuk revisité

Filipe Fernandes [a] **and Zbigniew Jelonek** [*, b, c]

[a] Departamento de Matemática, Universidade Federal do Ceará, 60440-900 Fortaleza, Ceará, Brazil

[b] Instytut Matematyczny, Polska Akademia Nauk, Śniadeckich 8, 00-656 Warszawa, Poland

[c] Departamento de Matemática, Universidade Federal do Ceará, Rua Campus do Pici, s/n, Bloco 914, Pici, 60440-900, Fortaleza-CE, Brazil

E-mails: filipematonb@gmail.com, najelone@cyf-kr.edu.pl

Abstract. In this note we provide two special counterexamples to the strong real Jacobian Conjecture; the first one is surjective, the second one has non-dense image.

Résumé. Dans cette note, nous fournissons deux contre-exemples spéciaux à la forte conjecture jacobienne réelle ; le premier est surjectif, le second a une image non dense.

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1. Introduction

The Jacobian conjecture is a famous unsolved problem concerning polynomials in several variables. It states that if the Jacobian of a polynomial function from an $n-$dimensional space to itself is a non-zero constant, then the function has a polynomial inverse (we recall that the Jacobian $J(F)$ of a holomorphic (or smooth) map $F:\mathbb{C}^n\to\mathbb{C}^n$ ($F:\mathbb{R}^n\to\mathbb{R}^n$) is the determinant of the Jacobian matrix of F). It was first conjectured in 1939 by Ott-Heinrich Keller [6] and widely publicized by Shreeram Abhyankar, as an example of a difficult question in algebraic geometry that can be understood using little beyond a knowledge of calculus (see for example [1, 4]). Here we consider it in the following form:

Conjecture 1 (Jacobian Conjecture). *Let $F:\mathbb{C}^2\to\mathbb{C}^2$ be a polynomial mapping with a constant non-zero Jacobian. Then F is a bijection.*

There is also a real version of this conjecture (see [8]):

Conjecture 2 (Strong Real Jacobian Conjecture.). *Let $F:\mathbb{R}^2\to\mathbb{R}^2$ be a polynomial mapping with a non-vanishing Jacobian. Then F is a bijection.*

* Corresponding author.

However, over twenty years ago Pinchuk found an example of locally injective polynomial endomorphism of $\mathbb{R}^2$, which is not injective (see [7] for details of his construction). Pinchuk's example has topological degree two and it is not a surjective mapping, in fact it omits two points. Ronen Peretz asked (unpublished, about 2000) whether the Strong Real Jacobian Conjecture is true whenever F is surjective. A similar question was formulated quite recently by Tiep Dinh (see the end of this paper). We may also ask whether there exists a counterexample F to this conjecture such that the image $F(\mathbb{R}^2)$ is not dense in $\mathbb{R}^2$. The aim of this note is to answer both questions.

2. Construction of the examples

The main idea is to modify an existing non-injective polynomial map with non-vanishing Jacobian by composing with special mappings. Here we use the version of Pinchuk map presented in [2], because it seems to be the simplest known so far, in the sense that its components have the lowest degree (however this construction still can be adapted to any of the known examples, see [5, 7]). Letting

$$\begin{aligned} p &= 4x^6y^3 + 12x^5y^2 + 12x^4y + 4x^3y^2 + 4x^3 + 5x^2y + x + y, \\ q &= -\left(4x^4y^2 + 8x^3y + 4x^2 + 2xy + 1\right) \\ &\qquad \cdot \left(4x^6y^3 + 12x^5y^2 + 12x^4y + 4x^3y^2 + 4x^3 + 7x^2y + 3x + y\right), \end{aligned} \tag{1}$$

we obtain the following:

Theorem 3. *Let p and q be as in* (1) *and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F = (p, q)$. Then F is a non-injective polynomial map with non-vanishing Jacobian and* $(\pm1, 0)$ *are the only points with no inverse image.*

Proof. See [2, Theorem 3.3 and Corollary 4.6]. □

In the following theorem we present a surjective example.

Theorem 4. *Let F be as in Theorem 3 and $\phi : \mathbb{C} \mapsto z^3 - 3z \in \mathbb{C}$, where we treat $\mathbb{C}$ as $\mathbb{R}^2$. Then $f = \phi \circ F$ is a non-injective but surjective polynomial map with non-vanishing Jacobian.*

Proof. The only critical points of the map ϕ are $z = \pm1$. Since $(\pm1, 0) \notin \mathrm{Im}(F)$, f has non-vanishing Jacobian. As ϕ is surjective, f can a priori omit only the points $\phi(\pm1) = \mp2$. However $\phi(\pm2) = \pm2$, therefore f is surjective. □

We present now an example with non-dense image.

Theorem 5. *Let*

$$\psi = \left(\left(xy^2 + x + y\right)(x - y), (xy + 1)^2 + x^2\right). \tag{2}$$

and $\widetilde{F} = (p + 1, q)$, where p and q are as in (1)*. Then $\widetilde{f} = \psi \circ \widetilde{F}$ is a non-injective polynomial map with non-vanishing Jacobian such that $\widetilde{f}(\mathbb{R}^2) \subset \mathbb{R} \times \mathbb{R}^+$.*

Proof. Denote by $D(\psi)$ the Jacobian matrix of ψ. We have

$$\det(D(\psi)) = \left(2y + 2xy^2 + x\right)^2 + x^2\left(2y^2 + 3\right)^2,$$

thus $(0, 0)$ is the only critical point of ψ. From Theorem 3 we have $(0, 0) \notin \mathrm{Im}(\widetilde{F})$. Hence the Jacobian of $\widetilde{f}$ is non-vanishing. Also $(xy + 1)^2 + x^2 > 0$ for every $(x, y) \in \mathbb{R}^2$. Therefore $\mathrm{Im}(\psi) \subset \mathbb{R} \times \mathbb{R}^+$. Consequently also $\mathrm{Im}(\widetilde{f}) \subset \mathbb{R} \times \mathbb{R}^+$. □

We finish this note answering negatively the following question of professor Tiep Dinh (personal comunication, February 2023):

Question. 6. Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a semi-algebraic smooth mapping which is surjective and has a constant non-zero Jacobian. Is then f a diffeomorphism?

Example 7 (Compare with [3]). Let F be as in Theorem 4. Then $G(x,y,z) = (F(x,y), z/J(F(x,y)))$ is a regular non-injective and surjective local diffeomorphism of $\mathbb{R}^3$ with constant non-zero Jacobian.

Proof. G is well defined for all $(x,y,z) \in \mathbb{R}^3$, since $J(F) > 0$. As for $J(G)$, we have

$$J(G) = J\left(F, \frac{z}{J(F)}\right) = J(F, 1/J(F)) + J(F,z)/J(F) = J(F,z)/J(F) = 1.$$

Note that G is surjective. Indeed take any $(a,b,c) \in \mathbb{R}^3$ and consider the equation

$$G(x,y,z) = (a,b,c).$$

Hence $F(x,y) = (a,b)$ and this equation has a solution (x_0,y_0) by Theorem 4. Now it is enough to put $z_0 = cJ(F(x_0,y_0))$ to get the point (x_0,y_0,z_0) such that $G(x_0,y_0,z_0) = (a,b,c)$. Moreover for generic (a,b) there is another pair (x_1,y_1) such that $F(x_1,y_1) = (a,b)$. This means that $(x_0,y_0,z_0) \neq (x_1,y_1,z_1)$ but $G(x_0,y_0,z_0) = G(x_1,y_1,z_1) = (a,b,c)$. Hence G is not a diffeomorphism. □

Remark 8. In fact our example is not only semi-algebraic but regular (in the sense of algebraic geometry, which means that every component f_i of F is locally a quotient of two polynomials P/Q, where Q does not vanish). Moreover in the same way we can show that there exists a regular mapping $F:\mathbb{R}^3 \to \mathbb{R}^3$ with constant non-zero Jacobian such that $F(\mathbb{R}^3) \subset \mathbb{R}^2 \times \mathbb{R}_+$. Indeed, let F be as in Theorem 5. Then $G(x,y,z) = (z/J(F(x,y)), F(x,y))$ is a regular non-injective map with Jacobian equal to one, such that $G(\mathbb{R}^3) \subset \mathbb{R}^2 \times \mathbb{R}_+$. This shows that regular mappings with Jacobian 1 can have non-dense image.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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References

[1] H. Bass, E. H. Connell and D. Wright, "The Jacobian conjecture: reduction of degree and formal expansion of the inverse", *Bull. Am. Math. Soc.* **7** (1982), pp. 287–330.

[2] F. Braun and F. Fernandes, "Very degenerate polynomial submersions and counterexamples to the real Jacobian conjecture", *J. Pure Appl. Algebra* **227** (2023), no. 8, article no. 107345.

[3] L. A. Campbell, "Remarks on the real Jacobian conjecture and Samuelson maps", *Appl. Math. Lett.* **10** (1997), no. 4, pp. 1–3.

[4] A. van der Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Birkhäuser, 2000.

[5] F. Fernandes, "A new class of non-injective polynomial local diffeomorphisms on the plane", *J. Math. Anal. Appl.* **507** (2021), article no. 125736.

[6] O.-H. Keller, "Ganze Cremona–Transformationen", *Monatsh. Math. Phys.* **47** (1939), pp. 299–306.

[7] S. Pinchuk, “A counterexample to the strong real Jacobian Conjecture”, *Math. Z.* **217** (1994), no. 1, pp. 1–4.
[8] J. D. Randall, “The real Jacobian problem”, in *Singularities, Part 2 (Arcata, Calif., 1981)* (P. Orlik, ed.), American Mathematical Society, 1983, pp. 411–414.