Shi-Qiang Chen, Yuchen Ding, Xiaodong Lü and Yuhan Zhang

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Correct order on some certain weighted representation functions

L’ordre correct de certaines fonctions de représentation pondérées

Shi-Qiang Chen \textsuperscript{a}, Yuchen Ding \textsuperscript{a, b}, Xiaodong Lü \textsuperscript{b} and Yuhan Zhang \textsuperscript{b}

\textsuperscript{a} School of Mathematics and Statistics, Anhui Normal University, Wuhu 241002, People’s Republic of China
\textsuperscript{b} School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, People’s Republic of China

\textit{E-mails:} csq20180327@163.com (S.-Q. Chen), ycding@yzu.edu.cn (Y. Ding), xdlv@yzu.edu.cn (X. Lü), Qiaoyuan0804@hotmail.com (Y. Zhang)

\textbf{Abstract.} Let $\mathbb{N}$ be the set of all nonnegative integers. For any positive integer $k$ and any subset $A$ of nonnegative integers, let $r_{1,k}(A,n)$ be the number of solutions $(a_1,a_2)$ to the equation $n = a_1 + ka_2$. In 2016, Qu proved that

$$\liminf_{n \to \infty} r_{1,k}(A,n) = \infty$$

providing that $r_{1,k}(A,n) = r_{1,k}(\mathbb{N} \setminus A, n)$ for all sufficiently large integers, which answered affirmatively a 2012 problem of Yang and Chen. In a very recent article, another Chen (the first named author) slightly improved Qu’s result and obtained that

$$\liminf_{n \to \infty} r_{1,k}(A,n) \log n > 0.$$ 

In this note, we further improve the lower bound on $r_{1,k}(A,n)$ by showing that

$$\liminf_{n \to \infty} \frac{r_{1,k}(A,n)}{n} > 0.$$ 

Our bound reflects the correct order of magnitude of the representation function $r_{1,k}(A,n)$ under the above restrictions due to the trivial fact that $r_{1,k}(A, n) \leq n/k$.

\textbf{Résumé.} Soit $\mathbb{N}$ l’ensemble de tous les entiers non négatifs. Pour tout entier positif $k$ et tout sous-ensemble $A$ d’entiers non négatifs, notons $r_{1,k}(A,n)$ le nombre de solutions $(a_1,a_2)$ de l’équation $n = a_1 + ka_2$. En 2016, Qu a prouvé que

$$\liminf_{n \to \infty} r_{1,k}(A,n) = \infty$$

ce qui signifie que $r_{1,k}(A,n) = r_{1,k}(\mathbb{N} \setminus A, n)$ pour tous les entiers suffisamment grands, ce qui répondait par l’affirmative à un problème de Yang et Chen datant de 2012. Dans un article très récent, un autre Chen (le premier auteur dans notre article) a légèrement amélioré le résultat de Qu et obtenu que

$$\liminf_{n \to \infty} \frac{r_{1,k}(A,n)}{\log n} > 0.$$ 

Dans cette note, nous améliorons encore le minornant de $r_{1,k}(A,n)$ en montrant que

$$\liminf_{n \to \infty} \frac{r_{1,k}(A,n)}{n} > 0.$$ 

Notre limite reflète l’ordre de grandeur correct de la fonction de représentation $r_{1,k}(A,n)$ sous les restrictions ci-dessus en raison du fait trivial que $r_{1,k}(A, n) \leq n/k$.

\textsuperscript{*}Corresponding author
1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers and $A$ a subset of $\mathbb{N}$. For any nonnegative integer $n$, let $R_1(A, n); R_2(A, n)$ and $R_3(A, n)$ be the number of solutions $(a, a')$ to the equations $n = a + a'$ with $a, a' \in A$; $a, a' \in A, a < a'$ and $a, a' \in A, a \leq a'$, respectively. For backgrounds on these representation functions $R_i(A, n), i = 1, 2, 3$, one can refer to an early survey article of Sárközy and Sós [11]. Following Sárközy’s question, Dombi [6], Chen and Wang [5], Lev [7], Sándor [10], Tang [13], Chen and Tang [4] and Chen [3] investigated various properties on values of the representation functions $R_i(A, n)$ and $R_i(\mathbb{N} \setminus A, n), i = 1, 2, 3$.

In an interesting paper, Yang and Chen [15] introduced the following weighted representation function

$$r_{k_1, k_2}(A, n) = \# \{(a_1, a_2) \in A^2 : n = k_1 a_1 + k_2 a_2\},$$

where $A$ is a subset of $\mathbb{N}$ and $k_1, k_2$ are two positive integers. They determined all pairs $(k_1, k_2)$ of positive integers for which there exists a set $A \subseteq \mathbb{N}$ such that

$$r_{k_1, k_2}(A, n) = r_{k_1, k_2}(\mathbb{N} \setminus A, n)$$

for all sufficiently large integers, which would reduce to partial answers to the original question of Sárközy mentioned above for $k_1 = k_2 = 1$ on $R_1(A, n)$. For $1 \leq k_1 < k_2$ with $(k_1, k_2) = 1$, if there exists a set $A \subseteq \mathbb{N}$ such that

$$r_{k_1, k_2}(A, n) = r_{k_1, k_2}(\mathbb{N} \setminus A, n)$$

for all sufficiently large integers, then Yang and Chen proved that $k_1 = 1$. So the studies of the weighted representation function $r_{1,k}(A, n)$ would be of particular interest.

For a positive integer $k \geq 2$, let $\Psi_k$ be the set of all $A \subseteq \mathbb{N}$ such that

$$r_{1,k}(A, n) = r_{1,k}(\mathbb{N} \setminus A, n)$$

for sufficiently large integers $n$. A result of Yang [14] states that if $k, \ell$ are multiplicatively independent (equivalently, $\log k / \log \ell$ is irrational), then $\Psi_k \cap \Psi_\ell = \emptyset$. Qu [9] then gave a complete criteria for which $\Psi_k \cap \Psi_\ell = \emptyset$. It turns out to be that $\Psi_k \cap \Psi_\ell \neq \emptyset$ if and only if $\log k / \log \ell = a/b$ for some odd positive integers $a$ and $b$, which disproved a conjecture of Yang [14]. For related result, see also the article of Li and Ma [8], Shallit [1, 12]. In [15], Yang and Chen posed the following problem:
Problem 1. For any set $A \in \Psi_k$, is it true that $r_{1,k}(A, n) \geq 1$ for all sufficiently large integers $n$? Is it true that $r_{1,k}(A, n) \to \infty$ as $n \to \infty$?

Problem 1 was later answered affirmatively by Qu [9]. Very recently, Chen [2] improved Qu’s result by showing that

$$\liminf_{n \to \infty} \frac{r_{1,k}(A, n)}{\log n} > 0$$

for any $A \in \Psi_k$. In this note, we give the following very much stronger bound.

Theorem 2. Let $k \geq 2$ be a given integer. For any set $A \in \Psi_k$, we have

$$\liminf_{n \to \infty} \frac{r_{1,k}(A, n)}{n} > 0.$$

Remark 3. It can be seen that our new bound is sharp on the order of magnitude of $r_{1,k}(A, n)$ in the sense that

$$\limsup_{n \to \infty} \frac{r_{1,k}(A, n)}{n} \leq \frac{1}{k}.$$

Perhaps, it should be pointed out that the original argument of Qu [9] with necessary adjustments can also lead to the bound given by Chen [2]. It is also worth mentioning that our argument here leading to the sharp bound above in Theorem 2 is different and simplified comparing with the somewhat complicated ones taken by Qu and Chen.

2. Proofs

Following Qu [9], we may write $A$ as the following union of the “blocks”

$$A = \bigcup_{i=0}^{\infty} (t_{2i}, t_{2i+1}),$$

where $0 \leq t_0 < t_1 < t_2 < \cdots$ is an increasing sequence of integers. The proof of our theorem is based on the following lemma of Qu [9, Lemma 2.1].

Lemma 4. Let $k \geq 2$ be a given integer. For any $A \in \Psi_k$ with

$$A = \bigcup_{i=0}^{\infty} (t_{2i}, t_{2i+1}),$$

there exist an odd positive integer $a$ and a nonnegative integer $i_0$ such that $t_{i+a} = kt_i$ for all $i \geq i_0$.

Proof of Theorem 2. By Lemma 4 for $A \in \Psi_k$ with

$$A = \bigcup_{i=0}^{\infty} (t_{2i}, t_{2i+1}),$$

there exists an odd positive integer $a$ such that $t_{i+a} = kt_i$ for all $i \geq i_0$. Without loss of generality, we can assume that $t_0 = 0$, otherwise one can consider $A = A \setminus (0, t_{i_0})$ instead of $A$ (this can be seen from the proofs below). Let $T = 4(t_{i_0+2} - t_0)$. Then there exists some odd integer $g \in \mathbb{N}$ such that $k^g > T$.

From now on, let $n$ be a sufficiently large number. It is clear that there are nonnegative integers $m$ and $r$ with $0 \leq r < (k^g + 1)$ such that

$$n = (k^g + 1) m + r.$$

We can assume that $m \in [k^s t_\ell, k^s t_{\ell+1})$ for two nonnegative integers $s$ and $\ell$ with

$$0 \leq \ell \leq a - 1.$$
Recall that the integer $n$ is assumed to be sufficiently large, it follows that both $m$ and $s$ are sufficiently large. We will prove that

$$r_{1,k}(A, n) \geq \frac{n}{k^5 t_a(k^8 + 2)} - (k^8 + 1),$$

from which it follows clearly that

$$\liminf_{n \to \infty} \frac{r_{1,k}(A, n)}{n} > 0.$$  

Since $r_{1,k}(A, n) = r_{1,k}(\mathbb{N} \setminus A, n)$ for all sufficiently large $n$, we can also assume, without loss of generality, that

$$[k^s t_\ell, k^s t_{\ell+1}) \subseteq A.$$  

Since $g$ is an odd integer, we make the observation that each of the following three intervals

$$[k^s t_{\ell-2}, k^s t_{\ell-1}), [k^s t_{\ell+2}, k^s t_{\ell+3}) \quad \text{and} \quad [k^{s+g-1} t_\ell, k^{s+g-1} t_{\ell+1})$$

contains in $A$ as well, which is crucial in the following arguments. Before the continuation of the proof, we make the following notice that for brevity we write, for example,

$$k^3 t_2, k^4 t_{2-a} \quad \text{and} \quad k^2 t_{2+a}$$

as the same number at different occasions. The proofs are divided into three cases:

**Case I.** $k^s t_\ell + k^{s-4} \leq m < k^s t_{\ell+1} - k^{s-4}$. Noting that for any $0 \leq q < k^{s-5} - r$, we have

$$n = (m + kq + r) + k(k^8 + 1 - m - q),$$

where both $m + kq + r$ and $k^8 + 1 - m - q$ belong to $A$ since

$$m + kq + r \in [k^s t_\ell, k^s t_{\ell+1}) \quad \text{and} \quad k^8 + 1 - m - q \in [k^{s+g-1} t_\ell, k^{s+g-1} t_{\ell+1}).$$

Thus, we deduce that

$$r_{1,k}(A, n) \geq k^{s-5} - r$$

$$\geq \frac{n}{k^5 t_{\ell+1}} - (k^8 + 1)$$

$$\geq \frac{n - r}{k^5 t_a(k^8 + 1)} - (k^8 + 1)$$

$$\geq \frac{n}{k^5 t_a(k^8 + 2)} - (k^8 + 1).$$

**Case II.** $k^s t_\ell \leq m < k^s t_\ell + k^{s-4}$. For any

$$k^{s-1}(t_\ell - t_{\ell-1}) + k^{s-5} + r < q \leq k^{s-1}(t_\ell - t_{\ell-2}),$$

it can be seen that

$$m - kq + r \in [k^s t_{\ell-2}, k^s t_{\ell-1}) \subseteq A$$

and

$$k^{s-1} m + q \in [k^{s+g-1} t_\ell, k^{s+g-1} t_{\ell+1}) \subseteq A,$$

where the latter inclusion relation comes from the observation that $[k^{s+g-1} t_\ell, k^{s+g-1} t_{\ell+1})$ contains in $A$ made previously and the facts that

$$k^{s-1} m + q < k^{s-1} (k^s t_\ell + k^{s-4}) + k^{s-1}(t_\ell - t_{\ell-2}) \leq k^{s+g-1} t_{\ell+1}$$

since $k^8 > T \geq 2(t_\ell - t_{\ell-2})$. Note that

$$n = (m - kq + r) + k(k^{g-1} m + q)$$

for all these $q$, from which we conclude that

$$r_{1,k}(A, n) \geq k^{s-1}(t_\ell - t_{\ell-2}) - k^{s-1}(t_\ell - t_{\ell-1}) - k^{s-5} - r$$

$$\geq \frac{1}{2} k^{s-1} - r.$$
\[
\geq \frac{n-r}{2kt_a(k^g+1)} - (k^g + 1) \\
\geq \frac{n}{2kt_a(k^g+2)} - (k^g + 1).
\]

**Case III.** \[ k^s t_{\ell+1} - k^s - 4 \leq m < k^s t_{\ell+1}. \] It can be verified directly that
\[
m + kq + r \in \left[ k^s t_{\ell+2}, k^s t_{\ell+3} \right] \subseteq A
\]
and
\[
k^{g-1}m - q \in \left[ k^{s+g-1}t_\ell, k^{s+g-1}t_{\ell+1} \right] \subseteq A,
\]
for any
\[
k^{s-1}(t_{\ell+2} - t_{\ell+1}) + k^{g-5} \leq q \leq k^{s-1}(t_{\ell+3} - t_{\ell+1}) - r
\]
via similar arguments in **Case II.** In fact,
\[
k^{g-1}m - q \geq k^{g-1}(k^s t_{\ell+1} - k^{s-4}) - k^{s-1}(t_{\ell+3} - t_{\ell+1}) \geq k^{s+g-1}t_\ell
\]
since \( k^g > T \geq 2(t_{\ell+3} - t_{\ell+1}) \). Note that
\[
n = (m + kq + r) + k\left(k^{g-1}m - q\right)
\]
for all these \( q \), from which we conclude that
\[
r_{1,k}(A,n) \geq k^{s-1}(t_{\ell+3} - t_{\ell+1}) - k^{s-1}(t_{\ell+2} - t_{\ell+1}) - k^{s-5} - r
\]
\[
\geq \frac{1}{2} k^{s-1} - r
\]
\[
\geq \frac{n-r}{2kt_a(k^g+1)} - (k^g + 1)
\]
\[
\geq \frac{n}{2kt_a(k^g+2)} - (k^g + 1).
\]

This completes the proof of Theorem 2. \( \square \)

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**Declaration of interests**

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