



ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus

Mathématique


Patrick Gérard

The zero dispersion limit for the Benjamin–Ono equation on the line

Volume 362 (2024), p. 619-634

Online since: 9 July 2024

<https://doi.org/10.5802/crmath.575>

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



The Comptes Rendus. Mathématique are a member of the
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569



Research article / *Article de recherche*

Partial differential equations, Mathematical physics / *Equations aux dérivées partielles, Physique mathématique*

The zero dispersion limit for the Benjamin–Ono equation on the line

Limite à faible dispersion pour l'équation de Benjamin–Ono sur la droite

Patrick Gérard ^a

^a Université Paris–Saclay, Laboratoire de Mathématiques d'Orsay, CNRS, UMR 8628, 91405 Orsay, France

E-mail: patrick.gerard@universite-paris-saclay.fr

Abstract. We identify the zero dispersion limit of a solution of the Benjamin–Ono equation on the line corresponding to every initial datum in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We infer a maximum principle and a local smoothing property for this limit. The proof is based on an explicit formula for the Benjamin–Ono equation and on the combination of calculations in the special case of rational initial data, with approximation arguments. We also investigate the special case of an initial datum equal to the characteristic function of a finite interval, and prove the lack of semigroup property for this zero dispersion limit.

Résumé. Nous identifions la limite à faible dispersion d'une solution de l'équation de Benjamin–Ono sur la droite correspondant à toute donnée initiale de carré intégrable et bornée. Nous en déduisons un principe du maximum et une propriété de régularisation locale pour cette limite. La démonstration est fondée sur une formule explicite pour l'équation de Benjamin–Ono et sur la combinaison de calculs dans le cas particulier de données initiales rationnelles avec des arguments d'approximation. Nous étudions également le cas particulier d'une donnée initiale égale à la fonction caractéristique d'un intervalle de longueur finie, et démontrons que cette limite à faible dispersion ne vérifie pas la propriété de semi-groupe.

2020 Mathematics Subject Classification. 37K15, 47B35.

Manuscript received 1 August 2023, accepted 5 October 2023.

1. Introduction

We consider the Benjamin–Ono equation on the line with a small dispersion $\varepsilon > 0$,

$$\partial_t u^\varepsilon + \partial_x \left[(u^\varepsilon)^2 \right] = \varepsilon \partial_x |D| u^\varepsilon, \quad u^\varepsilon(0, x) = u_0(x), \quad (1)$$

where $u_0 \in L^2(\mathbb{R})$ and is real valued. It is well known [18, 25] that equation (1) has a global solution $u^\varepsilon \in C(\mathbb{R}, L^2(\mathbb{R}))$, which is characterized as the strong limit of smooth solutions with any initial data approximating the initial datum u_0 strongly in $L^2(\mathbb{R})$, and that the L^2 norm $\|u^\varepsilon(t)\|_{L^2(\mathbb{R})}$ is independent of the time variable t .

Our purpose is the description of the weak limit of $u^\varepsilon(t)$ in $L^2(\mathbb{R})$ as $\varepsilon \rightarrow 0$. Such a problem is delicate even for smooth data u_0 , because the formal limit of equation (1) as ε tends to 0 is the inviscid Burgers–Hopf equation [5, 17]

$$\partial_t u + \partial_x (u^2) = 0,$$

which is well known to display singularities in finite time. In fact, numerical simulations — see e.g. [8] — show that, near such singularities, the solution u^ε display very strong oscillations as ε tends to 0, so that the weak limit of u^ε , if it exists, is very difficult to identify.

This zero–dispersion limit problem was first addressed by Lax and Levermore [20–22] in the case of the Korteweg–de Vries equation,

$$\partial_t v^\varepsilon + \partial_x \left[(v^\varepsilon)^2 \right] = \varepsilon \partial_x^3 v^\varepsilon, \quad v^\varepsilon(0, x) = v_0(x), \quad (2)$$

using the integrability of this equation and the inverse scattering transform for the Lax operator. Lax–Levermore’s pioneering work initiated a series of remarkable results describing v^ε for special initial data v_0 , under various assumptions on the spectral theory of the corresponding Lax operator, see in particular [6, 7, 16, 28] and references therein. All these references have in common the use of the inverse scattering theory and the study of a related Riemann–Hilbert problem, for describing the solution v^ε , and consequently depend on strong assumptions on the initial data through the spectral theory of its Lax operator — here, a Schrödinger operator.

Let us also mention recent approaches to similar problems for the Fermi–Pasta–Ulam chain [9] on a wide range of time and energy scales. We refer to [10] for a connection to the zero–dispersion limit for the KdV equation (2).

In the case of the Benjamin–Ono equation (1) on the line, much less is known, with the notable exceptions of [23, 24]. Again, these references use the inverse scattering theory for the Lax operator inherited from the integrability of the Benjamin–Ono equation. The case of the Benjamin–Ono equation with periodic boundary conditions was studied in [11, 12] where the inverse spectral theory of [14] was used to establish an explicit description of the zero dispersion limit in the case of bell-shaped data. Furthermore, in [11], an explicit formula proved in [13] was used to prove the existence of a zero–dispersion limit for every initial datum in $L^\infty(\mathbb{T})$. This suggests that the results of [13] could also be used for the zero–dispersion limit on the line. This is the purpose of this paper, where we prove general results on the zero–dispersion limit of (1) without using the inverse spectral theory of the Lax operator.

1.1. Statement of the results

Our first main result is the following.

Theorem 1. *As $\varepsilon \rightarrow 0$, the solution $u^\varepsilon(t)$ of (1) converges weakly in $L^2(\mathbb{R})$ to some function $ZD[u_0](t)$ for every $t \in \mathbb{R}$. The mapping*

$$u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \mapsto ZD[u_0]$$

has the following properties.

- (1) *For every $t \in \mathbb{R}$, $\|ZD[u_0](t)\|_{L^2} \leq \|u_0\|_{L^2}$, and $ZD[u_0]$ is continuous on \mathbb{R} with values in $L^2(\mathbb{R})$ endowed with the weak topology.*
- (2) *If u_0^δ converges strongly to u_0 in $L^2(\mathbb{R})$ as $\delta \rightarrow 0$, with a uniform bound on $\|u_0^\delta\|_{L^\infty}$, then, for every $t \in \mathbb{R}$, $ZD[u_0^\delta](t)$ converges to $ZD[u_0](t)$ weakly in $L^2(\mathbb{R})$.*

The above theorem is already quite surprising compared to what is known for the KdV equation. Indeed, as far as we know, the existence of a weak limit for every time and for every initial datum $v_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ is an open problem for equation (2). Coming back to the

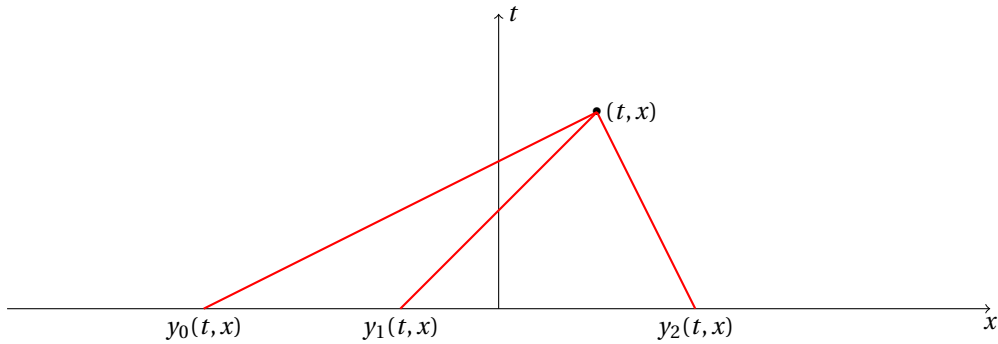


Figure 1. The crossing of characteristics

Benjamin–Ono equation, recall that a similar result was proved on the torus in [11]. In fact, on the line, one can go much further in the description of the zero dispersion limit. Under a slight additional assumption on the regularity of u_0 , we can relate the zero dispersion limit $ZD[u_0]$ to the multivalued solution of the inviscid Burgers–Hopf equation with initial datum u_0 .

Theorem 2. *In the conditions of Theorem 1, assume moreover that u_0 is a C^1 function tending to 0 at infinity as well as its first derivative. For every $t \in \mathbb{R}$, the set $K_t(u_0)$ of critical values of the function*

$$y \in \mathbb{R} \mapsto y + 2tu_0(y)$$

is a compact subset of measure 0. For every connected component Ω of $K_t(u_0)^c$, there exists a nonnegative integer ℓ such that, for every $x \in \Omega$, the equation

$$y + 2tu_0(y) = x$$

has $2\ell + 1$ simple real solutions

$$y_0(t, x) < y_1(t, x) < \dots < y_{2\ell}(t, x),$$

and the zero dispersion limit is given by

$$ZD[u_0](t, x) = \sum_{k=0}^{2\ell} (-1)^k u_0(y_k(t, x)). \tag{3}$$

Remarks 3.

- (1) Formula (3) was proved by Miller–Wetzel [23] – see also Miller–Xu [24] — in the special case of a rational Klaus–Shaw initial potential, and by Gassot [11, 12] in the special case of a general bell shaped initial potential with periodic boundary conditions, using inverse spectral theory for the Lax operator. Here we use a different approach and we have essentially no constraint on the initial datum u_0 .
- (2) As soon as several characteristics are crossing at a point (t, x) , the function $u = ZD[u_0]$ given by (3) is not a weak solution of the inviscid Burgers–Hopf equation

$$\partial_t u + \partial_x(u^2) = 0$$

in a neighborhood of (t, x) , hence the convergence of u^ε to u cannot be strong in L^2_{loc} , confirming the strong oscillations detected by numerical simulations [8]. We refer to the end of Section 3 for more detail about this lack of compactness.

(3) Formula (3) was introduced by Y. Brenier in [1–3] to approximate the entropic solution [19, 27] of the inviscid Burgers–Hopf equation. This formula has the following weak version. For every test function φ on \mathbb{R} ,

$$\int_{\mathbb{R}} ZD[u_0](t, x) \varphi(x) dx = \int_{\mathbb{R}} \varphi(y + 2tu_0(y)) u_0(y) (1 + 2tu'_0(y)) dy. \tag{4}$$

as can be easily deduced by making the change of variable $y = y_k(t, x)$ on each branch of the solutions $y + 2tu_0(y) = x$.

A striking consequence of Theorem 2 is the following maximum principle.

Corollary 4. *For every $u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, for every $t \in \mathbb{R}$,*

$$\text{ess inf } u_0 \leq \text{ess inf } ZD[u_0](t) \leq \text{ess sup } ZD[u_0](t) \leq \text{ess sup } u_0.$$

Contrary to what is known about the inviscid limit of the Burgers–Hopf equation [19, 27], where the maximum principle is inherited from the parabolic structure, the above maximum principle is surprising, because similar estimates do not hold for the solution u^ε of the Benjamin–Ono equation for a given $\varepsilon > 0$. In fact, for periodic boundary conditions, one can even prove that the Benjamin–Ono flow map may lose up to 1/2 derivatives in the C^α regularity (see [15, Section 4]).

Another consequence of Theorem 2 is an alternative characterization of the weak limit $ZD[u_0](t)$, which leads to a local smoothing property.

Corollary 5. *For every $u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, for every $t \in [-T, T]$, the distributional derivative $2t\partial_x ZD[u_0](t)$ is the measure*

$$2t\partial_x ZD[u_0](t, x) = 1 - \mu_{2tu_0},$$

where we have set, for every real valued function $f \in L^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi(x) d\mu_f(x) := \int_{\mathbb{R}} \varphi(y + f(y)) dy.$$

This measure has a bounded total variation on every finite interval of \mathbb{R} . In particular, for every $t \neq 0$, for every $s < \frac{1}{2}$, $ZD[u_0](t) \in H^s_{\text{loc}}(\mathbb{R})$.

Finally, the combination of Theorem 1 and Theorem 2 allows us to calculate $ZD[u_0]$ for discontinuous functions u_0 , for instance if u_0 is the characteristic function of a finite interval. See section 5 below.

1.2. Structure of the paper

This paper is organized as follows. In Section 2, we prove Theorem 1 by using an explicit formula recently derived for the initial value problem of the Benjamin–Ono on the line [13]. Assuming the initial datum u_0 to be L^∞ , it is possible to pass to the limit in this formula as ε tends to 0 and to obtain a formula for the zero–dispersion limit $ZD[u_0](t)$, using the Toeplitz operator of symbol u_0 acting on the Hardy space $L^2_+(\mathbb{R})$, see formula (7) below. In Section 3, we transform formula (7) in the case of a rational initial datum u_0 into a linear system on \mathbb{C}^{N+1} , where $2N$ is the degree of the denominator of u_0 , and we obtain the formula (3) in this case. We also prove that the weak limit to $ZD[u_0](t)$ is not strong in the case of three crossing characteristics. In Section 4, we prove Theorem 2 in its whole generality, and we infer Corollary 4 and Corollary 5. Finally, in Section 5, we describe the zero–dispersion limit $ZD[u_0]$ when u_0 is the characteristic function of the interval $] - 1, 1[$, and we prove that the map $t \mapsto ZD[.](t)$ is not a semigroup of transformations of $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

2. Proof of Theorem 1

We first recall the notation from [13]. Here D denotes $-i\partial_x$, Π denotes the orthogonal projector $\mathbf{1}_{D \geq 0}$ from $L^2(\mathbb{R})$ onto the Hardy space

$$L^2_+(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \forall \xi < 0, \widehat{f}(\xi) = 0\},$$

where \widehat{f} denotes the Fourier transform. We recall that $L^2_+(\mathbb{R})$ identifies to holomorphic functions f in the upper half plane with

$$\sup_{y>0} \int_{\mathbb{R}} |f(x + iy)|^2 dx < +\infty.$$

We denote by G the adjoint operator, on $L^2_+(\mathbb{R})$, of the multiplication by x . The domain of G consists of those functions f in $L^2_+(\mathbb{R})$ such that the restriction of \widehat{f} to $]0, +\infty[$ belongs to the Sobolev space H^1 . Furthermore, for any function $f \in L^2_+(\mathbb{R})$ such that the restriction of \widehat{f} to $]0, 1[$ belongs to $H^1(]0, 1[)$, we set

$$I_+(f) = \widehat{f}(0^+).$$

Finally, for $b \in L^\infty(\mathbb{R})$ T_b denotes the Toeplitz operator defined on $L^2_+(\mathbb{R})$ by $T_b(f) = \Pi(bf)$. This operator is bounded, with $\|T_b\| = \|b\|_{L^\infty}$, and is selfadjoint if b is real valued.

From [13], we know that, if $w_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ and is real valued, the solution of the initial value problem for the Benjamin–Ono equation,

$$\partial_s w + \partial_x(w^2) = \partial_x |D|w, \quad w(0, x) = w_0(x) \tag{5}$$

is given by

$$w(s, x) = \Pi w(s, x) + \overline{\Pi w(s, x)}, \quad x \in \mathbb{R},$$

where, for every complex number z with $\text{Im}(z) > 0$,

$$\Pi w(s, z) = \frac{1}{2i\pi} I_+ \left[(G - 2s(D - T_{w_0}) - z\text{Id})^{-1} (\Pi w_0) \right]. \tag{6}$$

We recall [13] that the resolvent in the right hand side of formula (6) is well defined. Indeed, on the other hand, the operator $G - 2sD$ has the following dissipation property,

$$\forall f \in \text{Dom}(G - 2sD), \quad \text{Im} \langle (G - 2sD)f | f \rangle \leq 0,$$

and is maximal, in the sense that $G - 2sD - z\text{Id} : \text{Dom}(G - 2sD) \rightarrow L^2_+(\mathbb{R})$ is invertible for every complex number z with $\text{Im}(z) > 0$. On the other hand, the operator $2sT_{w_0}$ is bounded and self-adjoint. The claim therefore follows from standard perturbation theory.

From a simple scaling argument, the solution u^ε of (1) is given by

$$u^\varepsilon(t, x) = \varepsilon w(\varepsilon t, x),$$

where $w(s, x)$ is the solution of (5) with $w_0(x) = \varepsilon^{-1} u_0(x)$. Therefore we can write

$$u^\varepsilon(t, x) = \Pi u^\varepsilon(t, x) + \overline{\Pi u^\varepsilon(t, x)}, \quad x \in \mathbb{R},$$

where, for every complex number z with $\text{Im}(z) > 0$,

$$\Pi u^\varepsilon(t, z) = \frac{1}{2i\pi} I_+ \left[(G - 2\varepsilon t D + 2t T_{u_0} - z\text{Id})^{-1} (\Pi u_0) \right].$$

On the other hand, recall that the L^2 norm of $u^\varepsilon(t)$ is independent of t , equal to the L^2 norm of u_0 , so there exists a subsequence ε_j tending to 0 such that $u^{\varepsilon_j}(t)$ has a weak limit in $L^2(\mathbb{R})$. We claim that all these weak limits coincide.

Lemma 6. For every $t \in \mathbb{R}$, $u^\varepsilon(t)$ converges weakly in $L^2(\mathbb{R})$ to $u(t)$, with

$$u(t, x) = \Pi u(t, x) + \overline{\Pi u(t, x)}, \quad x \in \mathbb{R},$$

where, for every complex number z with $\text{Im}(z) > 0$,

$$\Pi u(t, z) = \frac{1}{2i\pi} I_+ \left[(G + 2tT_{u_0} - z\text{Id})^{-1} (\Pi u_0) \right]. \tag{7}$$

Proof. Since $u \in L^2(\mathbb{R}) \mapsto \Pi u(z) \in \mathbb{C}$ is a continuous linear form for every complex number z with $\text{Im}(z) > 0$, it is enough to characterize $\Pi u(t, z)$ for $\text{Im}(xz) > 0$, and even for $\text{Im}(z)$ big enough, by analytic continuation. We observe that, for $\text{Im}(z) > 2|t|\|u_0\|_{L^\infty}$, the resolvent of $G - 2\varepsilon tD + 2tT_{u_0}$ converges strongly to the resolvent of $G + 2tT_{u_0}$ as $\varepsilon \rightarrow 0$. Indeed, by standard perturbation theory, it is enough to check that the resolvent of $G - 2\varepsilon tD$ converges to G , with a uniform bound by $(\text{Im}(z))^{-1}$ if $\text{Im}(z) > 0$, and this is immediate from the following explicit formula,

$$\widehat{h}_\varepsilon(\xi) = i \int_\xi^\infty \widehat{f}(\eta) e^{i\varepsilon t(\eta^2 - \xi^2) + iz(\eta - \xi)} d\eta.$$

for $h_\varepsilon := (G - 2\varepsilon tD - z\text{Id})^{-1} f$.

Furthermore, from the equation

$$i\partial_\xi \widehat{f}_\varepsilon(\xi) - (2\varepsilon t\xi + z)\widehat{f}_\varepsilon(\xi) = \widehat{g}_\varepsilon(\xi), \quad g_\varepsilon := \Pi u_0 - 2tT_{u_0} f_\varepsilon, \quad \xi > 0,$$

we observe that the Fourier transform of

$$f_\varepsilon = (G - 2\varepsilon tD + 2tT_{u_0} - z\text{Id})^{-1} (\Pi u_0)$$

is uniformly bounded in $H^1(]0, 1])$ as ε tends to 0.

Consequently, if $\text{Im}(z) > 2|t|\|u_0\|_{L^\infty}$, we have

$$\Pi u^\varepsilon(t, z) \longrightarrow \frac{1}{2i\pi} I_+ \left[(G + 2tT_{u_0} - z\text{Id})^{-1} (\Pi u_0) \right].$$

Therefore, by analytic continuation, for every z such $\text{Im}(z) > 0$,

$$\Pi u(t, z) = \frac{1}{2i\pi} I_+ \left[(G + 2tT_{u_0} - z\text{Id})^{-1} (\Pi u_0) \right].$$

which is (7). From formula (7), a similar argument allows us to prove properties (1) and (2) in Theorem 1. □

Remark 7.

- (1) Notice that it is not so easy to estimate the L^2 norm of $u(t)$ from formula (7). However, we know that it is bounded by the L^2 norm of u_0 because of the L^2 conservation law for the Benjamin–Ono equation.
- (2) The above arguments also show that, for any convergent sequence t_ε of real numbers, $u^\varepsilon(t_\varepsilon)$ is weakly convergent in L^2 to $u(\lim_{\varepsilon \rightarrow 0} t_\varepsilon)$. Consequently, for every $T > 0$, u^ε is uniformly convergent to u in $C([-T, T], L^2_w(\mathbb{R}))$, where $L^2_w(\mathbb{R})$ is $L^2(\mathbb{R})$ endowed with the weak topology.
- (3) If the initial datum u_0 is replaced by a sequence u_0^ε strongly convergent to u_0 , with a uniform bound in L^∞ , the above arguments show that $u^\varepsilon(t)$ is still weakly convergent to $ZD[u_0](t)$.

3. Proof of Theorem 2 for a rational initial datum

In this section, we prove formula (3) if u_0 is a rational function with real coefficients, with no pole on the real line,

$$u_0(y) = \frac{P_0(y)}{Q_0(y)},$$

where, for some positive integer N , Q_0 is a monic real polynomial of degree $2N$, and P_0 is a real polynomial of degree at most $2N - 1$. With no loss of generality, we may assume — by an approximation argument in formula (7), thanks to property (2) of Theorem 1—, that Q_0 has only simple poles in the complex domain, so that

$$u_0(y) = \sum_{j=1}^N \frac{c_j}{y - p_j} + \frac{\bar{c}_j}{y - \bar{p}_j}, \text{Im}(p_j) > 0, \Pi u_0(y) = \sum_{j=1}^N \frac{\bar{c}_j}{y - \bar{p}_j}.$$

Set $u(t) := ZD[u_0](t)$. We calculate directly $u(t, x) = \Pi u(t, x) + \overline{\Pi u(t, x)}$ for $x \in \mathbb{R}$. Recall that, according to formula (7), for $\text{Im}(z) > 0$,

$$\Pi u(t, z) = \frac{1}{2i\pi} I_+ \left[(G + 2tT_{u_0} - z\text{Id})^{-1} (\Pi u_0) \right].$$

Notice that the algebraic equation $y + 2tu_0(y) = z$ is equivalent to

$$yQ_0(y) - zQ_0(y) + 2tP_0(y) = 0,$$

which is an equation of degree $2N + 1$. Hence, if $z = x \in \mathbb{R}$, it has an odd number $2\ell(t, x) + 1$ of real zeroes. Discarding only a finite set of points x for a given t , we may consider the case where $\ell(t, x) = \ell \in \mathbb{Z}_{\geq 0}$ and where these zeroes are simple, ordered as

$$y_0(t, x) < y_1(t, x) < \dots < y_{2\ell}(t, x),$$

so that the derivative of $y + 2tu_0(y)$ at $y = y_k(t, x)$ has the sign of $(-1)^k$. Denote the other zeroes of this equation in the complex domain by

$$y_m(t, x), m = 2\ell + 1, \dots, 2N,$$

with $\text{Im}(y_{2p}(t, x)) > 0$ and $y_{2p-1}(t, x) = \overline{y_{2p}(t, x)}$ if $p = \ell + 1, \dots, N$. The Cauchy–Riemann equations and the implicit function theorem show that, at $z = x \in \mathbb{R}$,

$$\frac{\partial \text{Im}(y_k(t, z))}{\partial \text{Im}(z)} = \frac{\partial \text{Re}(y_k(t, z))}{\partial \text{Re}(z)} = \frac{1}{1 + 2tu'_0(y_k(t, x))}$$

which has the sign of $(-1)^k$. If x is shifted into the upper half plane to a complex number z with a small positive imaginary part, we infer

$$\text{Im}(y_{2k}(t, z)) > 0, \text{Im}(y_{2p-1}(t, z)) < 0, k = 0, \dots, N, p = 1, \dots, N.$$

For such a complex number z , let us calculate

$$f_{t,z} := (G + 2tT_{u_0} - z\text{Id})^{-1} (\Pi u_0).$$

It turns out that $f_{t,z}$ is a rational function. Indeed, for any function f in $L^2_+(\mathbb{R})$, we have

$$Gf(y) = yf(y) + \frac{1}{2i\pi} I_+(f), T_{u_0}f(y) = u_0(y)f(y) - \sum_{j=1}^N c_j \frac{f(p_j)}{y - p_j}.$$

The first identity is a consequence of the Fourier representation of G and of the jump formula at $\xi = 0$ for distributional derivatives. The second identity comes from the following standard property of Toeplitz operators with rational symbols,

$$T_{(y-p)^{-1}}(f)(y) = \frac{f(y) - f(p)}{y - p}, \text{Im}(p) > 0.$$

Hence the equation on $f_{t,z}$ can be reformulated as

$$(y - z + 2tu_0(y)) f_{t,z}(y) = u_0(y) + \lambda(t, z) + \sum_{j=1}^N \frac{\mu_j(t, z)}{y - p_j},$$

where $\lambda(t, z), \mu_j(t, z), j = 1, \dots, N$, are complex numbers such that the rational function

$$f_{t,z}(y) = \frac{u_0(y) + \lambda(t, z) + \sum_{j=1}^N \frac{\mu_j(t, z)}{y - p_j}}{y - z + 2t u_0(y)},$$

belongs to the Hardy space. This imposes the cancellation of the numerator if y is any of the zeroes of the denominator in the upper half plane, and therefore leads to the linear system

$$u_0(y_{2k}(t, z)) + \lambda(t, z) + \sum_{j=1}^N \frac{\mu_j(t, z)}{y_{2k}(t, z) - p_j} = 0, \quad k = 0, \dots, N,$$

or, if $t \neq 0$,

$$\lambda(t, z) + \sum_{j=1}^N \frac{\mu_j(t, z)}{y_{2k}(t, z) - p_j} = \frac{y_{2k}(t, z) - z}{2t}, \quad k = 0, \dots, N. \tag{8}$$

Then we have

$$\Pi u(t, z) = \frac{1}{2i\pi} I_+ (f_{t,z}) = - \lim_{y \rightarrow \infty} y f_{t,z}(y),$$

since $f_{t,z}(y)$ is a rational function of y . In view of the expression of $f_{t,z}$, we conclude

$$\overline{\Pi u(t, z)} = -\lambda(t, z).$$

From Cramer's formulae for the system (8), we have

$$\begin{aligned} \lambda(t, z) &= \frac{N(t, z)}{D(t, z)}, \\ D(t, z) &:= \begin{vmatrix} 1 & \frac{1}{y_0 - p_1} & \cdots & \frac{1}{y_0 - p_N} \\ 1 & \frac{1}{y_2 - p_1} & \cdots & \frac{1}{y_2 - p_N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{y_{2N} - p_1} & \cdots & \frac{1}{y_{2N} - p_N} \end{vmatrix}, \\ N(t, z) &:= \frac{1}{2t} \begin{vmatrix} y_0 - z & \frac{1}{y_0 - p_1} & \cdots & \frac{1}{y_0 - p_N} \\ y_2 - z & \frac{1}{y_2 - p_1} & \cdots & \frac{1}{y_2 - p_N} \\ \vdots & \vdots & \ddots & \vdots \\ y_{2N} - z & \frac{1}{y_{2N} - p_1} & \cdots & \frac{1}{y_{2N} - p_N} \end{vmatrix}. \end{aligned}$$

The following lemma follows from elementary manipulations on Cauchy and Vandermonde determinants.

Lemma 8. *Given complex numbers $z_0, \dots, z_N, p_1, \dots, p_N$ pairwise distinct, we have*

$$\frac{\begin{vmatrix} z_0 & \frac{1}{z_0 - p_1} & \cdots & \frac{1}{z_0 - p_N} \\ z_1 & \frac{1}{z_1 - p_1} & \cdots & \frac{1}{z_1 - p_N} \\ \vdots & \vdots & \ddots & \vdots \\ z_N & \frac{1}{z_N - p_1} & \cdots & \frac{1}{z_N - p_N} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1}{z_0 - p_1} & \cdots & \frac{1}{z_0 - p_N} \\ 1 & \frac{1}{z_1 - p_1} & \cdots & \frac{1}{z_1 - p_N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{z_N - p_1} & \cdots & \frac{1}{z_N - p_N} \end{vmatrix}} = \sum_{\alpha=0}^N z_\alpha - \sum_{j=1}^N p_j.$$

Proof. Using the formula for the Cauchy determinants, we have

$$A := \begin{vmatrix} z_0 & \frac{1}{z_0-p_1} & \cdots & \frac{1}{z_0-p_N} \\ z_1 & \frac{1}{z_1-p_1} & \cdots & \frac{1}{z_1-p_N} \\ \vdots & \vdots & \ddots & \vdots \\ z_N & \frac{1}{z_N-p_1} & \cdots & \frac{1}{z_N-p_N} \end{vmatrix} = \sum_{\alpha=0}^N (-1)^\alpha z_\alpha D_\alpha$$

$$B := \begin{vmatrix} 1 & \frac{1}{z_0-p_1} & \cdots & \frac{1}{z_0-p_N} \\ 1 & \frac{1}{z_1-p_1} & \cdots & \frac{1}{z_1-p_N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{z_N-p_1} & \cdots & \frac{1}{z_N-p_N} \end{vmatrix} = \sum_{\alpha=0}^N (-1)^\alpha D_\alpha$$

with

$$D_\alpha := \prod_{\substack{\beta \neq \alpha, \gamma \neq \alpha \\ \beta < \gamma}} (z_\beta - z_\gamma) \prod_j (z_\alpha - p_j) \Delta,$$

$$\Delta := \frac{\prod_{j < k} (p_k - p_j)}{\prod_{\beta, j} (z_\beta - p_j)}.$$

Consequently, we are led to evaluate the following quotient of Vandermonde determinants,

$$\frac{A}{B} = \frac{V(R)}{V(Q)}, \quad R(z) := zQ(z), \quad Q(z) := \prod_{j=1}^N (z - p_j),$$

$$V(P) := \begin{vmatrix} P(z_0) & 1 & z_0 & \cdots & z_0^{N-1} \\ P(z_1) & 1 & z_1 & \cdots & z_1^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P(z_N) & 1 & z_N & \cdots & z_N^{N-1} \end{vmatrix}.$$

We notice that the linear form V cancels on polynomials of degree at most $N - 1$, and on the polynomial \tilde{P} defined as

$$\tilde{P}(z) := \prod_{\alpha=0}^N (z - z_\alpha).$$

Since

$$R(z) - \tilde{P}(z) = zQ(z) - \tilde{P}(z) = \left(\sum_{\alpha=0}^N z_\alpha - \sum_{j=1}^N p_j \right) z^N + \mathbb{C}_{\leq N-1}[z]$$

$$= \left(\sum_{\alpha=0}^N z_\alpha - \sum_{j=1}^N p_j \right) Q(z) + \mathbb{C}_{\leq N-1}[z],$$

we infer that

$$\frac{V(R)}{V(Q)} = \frac{V(R - \tilde{P})}{V(Q)} = \sum_{\alpha=0}^N z_\alpha - \sum_{j=1}^N p_j.$$

□

Using Lemma 8, we have

$$\lambda(t, z) = \frac{1}{2t} \left(\sum_{\alpha=0}^N y_{2\alpha}(t, z) - \sum_{j=1}^N p_j - z \right). \tag{9}$$

In view of the algebraic equation

$$yQ_0(y) - zQ_0(y) + 2tP_0(y) = 0, \quad Q_0(y) = \prod_{j=1}^N (y - p_j) (y - \bar{p}_j),$$

we have

$$z + \sum_{j=1}^N (p_j + \bar{p}_j) = \sum_{\alpha=0}^{2N} y_\alpha(t, z). \tag{10}$$

Now we make z tend to x on the real line, so that $y_k(t, x)$ is real for $k = 0, 1, \dots, 2\ell$, and $y_{2p-1}(t, x) = \overline{y_{2p}(t, x)}$ if $p = \ell + 1, \dots, N$. Consequently,

$$\begin{aligned} u(t, x) &= -\left(\lambda(t, x) + \overline{\lambda(t, x)}\right) \\ &= \frac{1}{2t} \left(2x + \sum_{j=1}^N (p_j + \bar{p}_j) - 2 \sum_{\alpha=0}^{\ell} y_{2\alpha}(t, x) - \sum_{\beta=2\ell+1}^{2N} y_\beta(t, x) \right), \\ &= \frac{1}{2t} \left(x + \sum_{\gamma=1}^{\ell} y_{2\gamma-1}(t, x) - \sum_{\alpha=0}^{\ell} y_{2\alpha}(t, x) \right), \\ &= \sum_{k=0}^{2\ell} (-1)^k u_0(y_k(t, x)). \end{aligned}$$

The proof of formula (3) when u_0 is rational is complete.

Let us end this section by an elementary observation about formula (3). We claim that, if $\ell = 1$, this alternate sum of solutions of the inviscid Burgers equation is not a solution of the inviscid Burgers–Hopf equation. Indeed, let us argue by contradiction. Assume $\ell = 1$ and that u satisfies the inviscid Burgers–Hopf equation in some neighborhood V of (t, x) in \mathbb{R}^2 . Writing $v_k(t, x) := u_0(y_k(t, x))$ for $k = 0, 1, 2$, we observe that $\partial_t v_k + \partial_x(v_k^2) = 0$, hence

$$\partial_x [(v_0 - v_1 + v_2)^2 - v_0^2 + v_1^2 - v_2^2] = 0$$

in V . We can reformulate the above identity as

$$\partial_x [(v_1 - v_0)(v_1 - v_2)] = 0$$

on V . Since we are dealing with real analytic functions, this identity extends to the whole connected component of V in the domain W characterized by $\ell = 1$. In other words, the function $(v_1 - v_0)(v_1 - v_2)$ is not depending on x in this domain. Since this function tends to 0 as (t, x) tends to the boundary of W , we conclude that $(v_1 - v_0)(v_1 - v_2)$ is identically 0, which is a contradiction since $y_k(t, x) + 2tv_k(t, x) = x$, while $y_0(t, x) < y_1(t, x) < y_2(t, x)$.

4. Proof of Theorem 2, of the maximum principle and of the local smoothing property

We now consider the case where $u_0 \in L^2(\mathbb{R})$ is continuously differentiable, with $|u(x)| + |u'(x)| \rightarrow 0$ as $x \rightarrow \infty$. Using a standard mollifier, we may approximate u_0 in $L^2(\mathbb{R}) \cap C_b^1(\mathbb{R})$ by a sequence u_0^δ of functions in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. Since L^2 rational functions are dense in $H^2(\mathbb{R})$, we may assume that u_0^δ is rational. Let $t \in \mathbb{R}$, and let $K_t(u_0)$ be the set of critical values of the function

$$f_t : y \in \mathbb{R} \mapsto y + 2tu_0(y) \in \mathbb{R}.$$

By the Sard theorem, $K_t(u_0)$ has zero Lebesgue measure. Furthermore, since $1 + 2tu_0'(y) \rightarrow 1$ as $y \rightarrow \infty$, the set of critical points of f_t is compact, so that its image $K_t(u_0)$ by f_t is compact too. Let Ω be any connected component of $K_t(u_0)^c$. Let ℓ be the nonnegative integer such that, for every $x \in \Omega$, the equation $y + 2tu_0(y) = x$ has $2\ell + 1$ solutions, denoted by

$$y_0(t, x) < \dots < y_{2\ell}(t, x).$$

Let ω be any open subinterval of Ω such that $\bar{\omega}$ is compact. For δ small enough, ω does not meet $K_t(u_0^\delta)^c$, and, for every $x \in \omega$, the equation

$$y + 2tu_0^\delta(y) = x$$

has exactly $2\ell + 1$ solutions,

$$y_0^\delta(t, x) < \dots < y_{2\ell}^\delta(t, x).$$

Furthermore, for every $k \in \{0, 1, \dots, 2\ell\}$, $y_k^\delta(t, x) \rightarrow y_k(t, x)$ as $\delta \rightarrow 0$. From Section 3, we know that, for $x \in \omega$,

$$ZD[u_0^\delta](t, x) = \sum_{k=0}^{2\ell} (-1)^k u_0^\delta(y_k^\delta(t, x)).$$

Passing to the weak limit in $L^2(\omega)$ as $\delta \rightarrow 0$, we infer, for almost every $x \in \omega$,

$$ZD[u_0](t, x) = \sum_{k=0}^{2\ell} (-1)^k u_0(y_k(t, x)).$$

Since ω is an arbitrary relatively compact subinterval of Ω , this proves formula (3).

We now come to the proof of Corollary 4. First consider the case where u_0 is continuously differentiable with $|u_0(x)| + |u_0'(x)| \rightarrow 0$ as $x \rightarrow \infty$, as in Section 4. Let us come back to formula (3),

$$u(t, x) = \sum_{k=0}^{2\ell} (-1)^k u_0(y_k(t, x))$$

where $y_0(t, x) < \dots < y_{2\ell}(t, x)$ are the solutions of $y + 2tu_0(y) = x$. In view of the monotonicity of the sequence $k \mapsto y_k(t, x)$, we infer the monotonicity of the sequence $k \mapsto u_0(y_k(t, x))$. Consequently, we have

$$\min_{0 \leq k \leq 2\ell} u_0(y_k(t, x)) \leq \sum_{k=0}^{2\ell} (-1)^k u_0(y_k(t, x)) \leq \max_{0 \leq k \leq 2\ell} u_0(y_k(t, x))$$

and finally

$$\min_{y \in \mathbb{R}} u_0(y) \leq u(t, x) \leq \max_{y \in \mathbb{R}} u_0(y),$$

which is the claimed maximum principle.

The general case of $u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ follows by applying a standard mollifier to u_0 and using property (2) in Theorem 1.

Finally, let us prove Corollary 5. Let us first assume that u_0 is continuously differentiable with $|u_0(x)| + |u_0'(x)| \rightarrow 0$ as $x \rightarrow \infty$. Using the weak version (4) of formula (3), we have, for every $\varphi \in C^1(\mathbb{R})$ with compact support,

$$\begin{aligned} \langle 2t\partial_x ZD[u_0](t), \varphi \rangle &= -2t \int_{\mathbb{R}} u_0(y) \varphi'(y + 2tu_0(y))(1 + 2tu_0'(y)) dy \\ &= 2t \int_{\mathbb{R}} u_0'(y) \varphi(y + 2tu_0(y)) dy \\ &= \int_{\mathbb{R}} \left[\frac{d}{dy} \left(\int_{-\infty}^{y+2tu_0(y)} \varphi(s) ds \right) - \varphi(y + 2tu_0(y)) \right] dy \\ &= \int_{\mathbb{R}} \varphi(s) ds - \int_{\mathbb{R}} \varphi(y + 2tu_0(y)) dy. \end{aligned}$$

By mollifying u_0 , this statement still holds if $u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. We infer that, for every $u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$2t\partial_x ZD[u_0](t) = 1 - \mu_{2tu_0}, \quad \int_{\mathbb{R}} \varphi(y) d\mu_f(y) := \int_{\mathbb{R}} \varphi(y + f(y)) dy. \tag{11}$$

Notice that $\mu_f([-R, R]) \leq 2(R + \|f\|_\infty)$. This proves the first claim of Corollary 5. The second claim is a consequence of the Sobolev embedding

$$BV_{\text{loc}}(\mathbb{R}) \subset H_{\text{loc}}^s(\mathbb{R})$$

for every $s < \frac{1}{2}$.

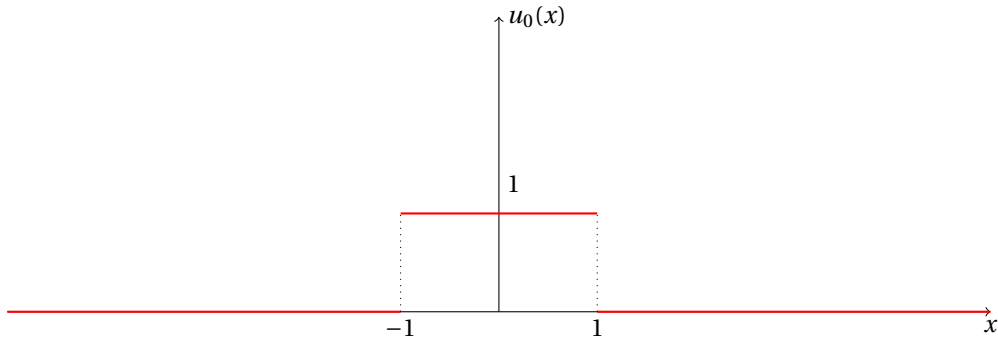


Figure 2. The datum at $t = 0$

Remark 9. Equation (11) implies a one-sided Lipschitz condition for $ZD[u_0](t)$. This property is well-known for entropic solutions of the inviscid Burgers–Hopf equation [4, 26].

5. A special case and the lack of semigroup property

In this section, we consider the case where u_0 is the characteristic function of a finite interval. We are going to calculate $ZD[u_0]$ by approximating u_0 by a sequence of compactly supported C^1 functions. An alternative method would be to use formula (11) above.

Theorem 10. *Let u_0 be the characteristic function of $] - 1, 1[$. Then $ZD[u_0]$ can be described as follows. If $t \in]0, 1[$,*

$$ZD[u_0](t, x) = \begin{cases} 0 & \text{if } x \in]-\infty, -1[\cup]2t + 1, +\infty[\\ \frac{x+1}{2t} & \text{if } x \in]-1, 2t - 1[\\ 1 & \text{if } x \in]2t - 1, 1[\\ 1 - \frac{x-1}{2t} & \text{if } x \in]1, 2t + 1[. \end{cases}$$

If $t \in]1, +\infty[$,

$$ZD[u_0](t, x) = \begin{cases} 0 & \text{if } x \in]-\infty, -1[\cup]2t + 1, +\infty[\\ \frac{x+1}{2t} & \text{if } x \in]-1, 1[\\ \frac{1}{t} & \text{if } x \in]1, 2t - 1[\\ 1 - \frac{x-1}{2t} & \text{if } x \in]2t - 1, 2t + 1[. \end{cases}$$

Furthermore, $ZD[u_0](-t, -x) = ZD[u_0](t, x)$.

Proof. We use property (2) of Theorem 1 and Theorem 2. We approximate u_0 by the following smooth function u_0^δ as $\delta \rightarrow 0^+$.

$$u_0^\delta(x) = \begin{cases} 0 & \text{if } x \in]-\infty, -1 - \delta[\cup]1 + \delta, +\infty[\\ 1 & \text{if } x \in [-1, 1] \end{cases}$$

and u_0^δ is strictly increasing from 0 to 1 on the interval $[-1 - \delta, -1]$, strictly decreasing from 1 to 0 on the interval $[1, 1 + \delta]$. Let assume $t \in]0, 1[$. Given $x \in \mathbb{R}$, the solutions of the equation $y + 2tu_0^\delta(y) = x$ are as follows

If $x < -1 - \delta$, the only solution is $y_0^\delta(t, x) = x$ and therefore

$$ZD[u_0^\delta](t, x) = u_0^\delta(y_0^\delta(t, x)) = 0.$$

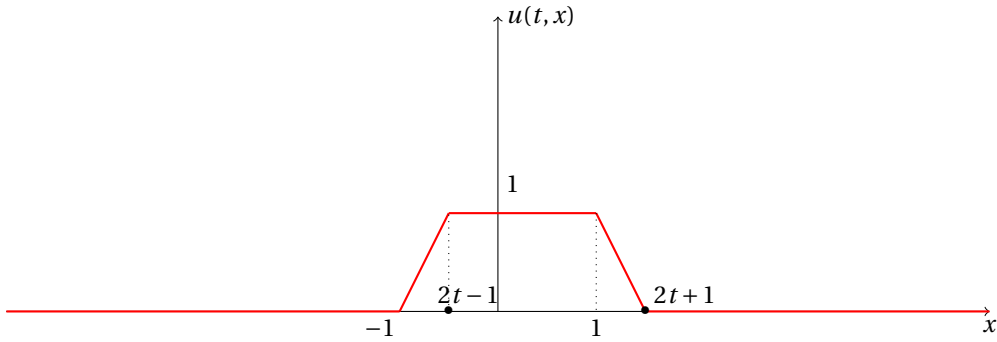


Figure 3. The weak limit at $t \in]0, 1[$

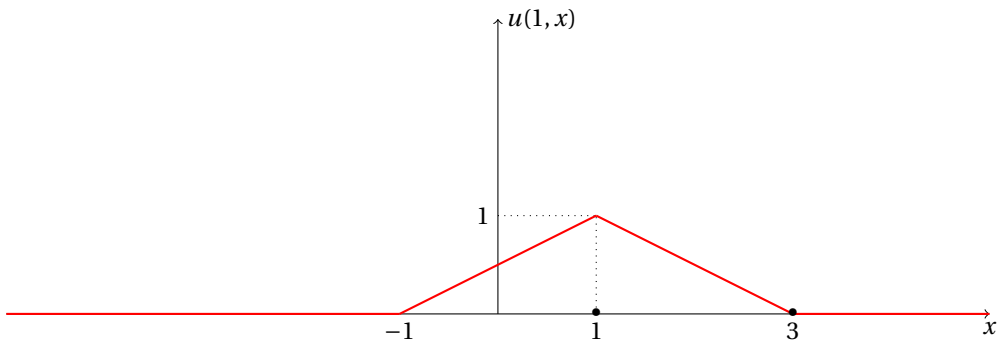


Figure 4. The weak limit at $t = 1$

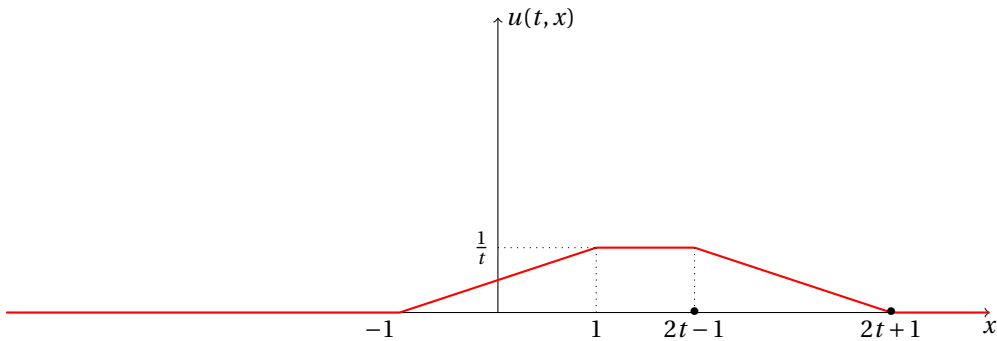


Figure 5. The weak limit at $t \in]1, +\infty[$

If $-1 < x < 2t - 1$, there exists only one solution y_0^δ , it is simple and it belongs to $] -1 - \delta, -1[$. Then

$$ZD[u_0^\delta](t, x) = u_0^\delta(y_0^\delta(t, x)) = \frac{x - y_0^\delta}{2t}.$$

If $2t - 1 < x < 1$, there exists only one solution $y_0^\delta(t, x) = x - 2t$, it is simple and it belongs to $] -1, 1[$. Then

$$ZD[u_0^\delta](t, x) = u_0^\delta(y_0^\delta(t, x)) = 1.$$

If $1 + \delta < x < 1 + 2t$ is outside a set of measure 0, there exist three simple solutions $y_0^\delta(t, x) = x - 2t$, $y_1^\delta(t, x) \in]1, 1 + \delta[$ and $y_2^\delta(t, x) = x$. Then

$$ZD \left[u_0^\delta \right] (t, x) = \sum_{k=0}^2 (-1)^k u_0^\delta \left(y_k^\delta(t, x) \right) = 1 - \frac{x - y_1^\delta(t, x)}{2t}.$$

If $1 + 2t < x$, then there exists only one solution $y_0\delta(t, x) = x$ and

$$ZD \left[u_0^\delta \right] (t, x) = u_0^\delta \left(y_0^\delta(t, x) \right) = 0.$$

Finally, passing to the limit as $\delta \rightarrow 0$ by keeping in mind that

$$0 \leq ZD \left[u_0^\delta \right] \leq 1$$

in view of the maximum principle, we obtain the result.

The case $t > 1$ can be handled similarly. Finally, the formula

$$ZD[u_0](-t, -x) = ZD[u_0](t, x)$$

follows from the fact that u_0 is even and therefore that the solution u^ε of (1) satisfies $u^\varepsilon(t, x) = u^\varepsilon(-t, -x)$. □

We conclude this section by observing that the map ZD does not satisfy the semigroup property. More precisely, we show on the above example that, in general,

$$ZD[ZD[u_0](t)](s) \neq ZD[u_0](t + s).$$

Let us choose $t = 1$ and $s > 0$ in the above example. Then, according to Theorem 10,

$$ZD[u_0](1, x) = \begin{cases} 0 & \text{if } x \in]-\infty, -1[\cup]3, +\infty[\\ \frac{x+1}{2} & \text{if } x \in]-1, 1[\\ 1 - \frac{x-1}{2} & \text{if } x \in]1, 3[. \end{cases}$$

Set $u_1(x) := ZD[u_0](1, x)$. This is a continuous, piecewise linear function, and we can easily approximate it by C^1 functions by smoothing it in neighbourhoods of $x = -1, 1, 3$. Given $s \in]0, 1[$, the function $y \mapsto y + 2su_1(y)$ is continuous and strictly increasing, sending $]-\infty, -1[$ onto itself, $]-1, 1[$ onto $]-1, 2s + 1[$, $]1, 3[$ onto $]2s + 1, 3[$ and $]3, +\infty[$ onto itself. Consequently, we obtain, after passing to the limit,

$$ZD[u_1](s, x) = \begin{cases} 0 & \text{if } x \in]-\infty, -1[\cup]3, +\infty[\\ \frac{x+1}{2(s+1)} & \text{if } x \in]-1, 2s + 1[\\ \frac{3-x}{2(1-s)} & \text{if } x \in]2s + 1, 3[\\ 0 & \text{if } x \in]3, +\infty[. \end{cases}$$

Comparing with the expression of $ZD[u_0](1 + s, x)$ from Theorem 10, we conclude that $ZD[ZD[u_0](1)](s) \neq ZD[u_0](1 + s)$.

Remark 11. As pointed to us by Y. Brenier, this lack of semigroup property can also be proved by observing that the semigroup associated to the entropic solution [19, 27] of the inviscid Burgers–Hopf equation is deduced in [1–3] from the map $ZD[.]$ through a Trotter formula. If the latter was a semigroup, it would coincide with the former, which is obviously wrong.

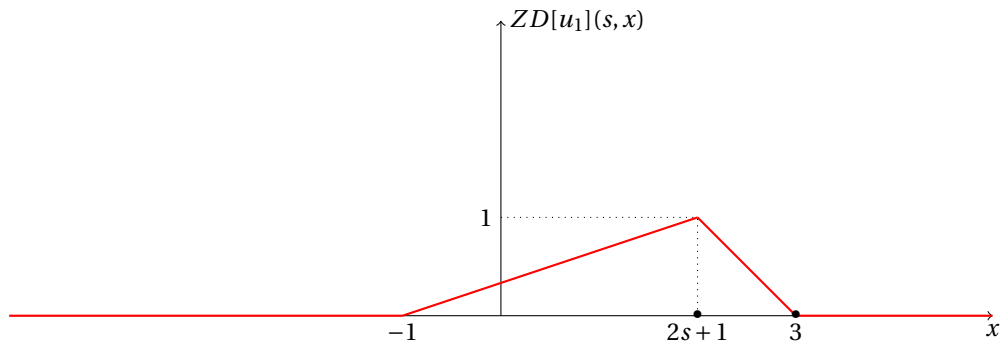


Figure 6. The iteration $ZD[ZD[u_0](1)](s)$ for $s \in]0, 1[$.

Acknowledgments

It is a pleasure to thank R. Badreddine, D. Bambusi, Y. Brenier, M. Gallone, L. Gassot, H. Holden, E. Lenzmann, P. Miller and N. Tzvetkov for fruitful discussions about this problem.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

- [1] Y. Brenier, “Une application de la symétrisation de Steiner aux équations hyperboliques : la méthode de transport et écroulement”, *C. R. Math.* **292** (1981), pp. 563–566.
- [2] Y. Brenier, “Résolution d’équations d’évolution quasilinéaires en dimension N d’espace à l’aide d’équations linéaires en dimension $N+1$ ”, *J. Differ. Equations* **50** (1983), pp. 375–390.
- [3] Y. Brenier, “Averaged multivalued solutions for scalar conservation laws”, *SIAM J. Numer. Anal.* **21** (1984), pp. 1013–1037.
- [4] Y. Brenier and S. Osher, “The discrete one-sided Lipschitz condition for convex scalar conservation laws”, *SIAM J. Numer. Anal.* **25** (1988), no. 1, pp. 8–23.
- [5] J. M. Burgers, “A mathematical model illustrating the theory of turbulence”, in *Advances in Applied Mechanics. Vol. 1*, Academic Press Inc., 1948, pp. 171–199.
- [6] T. Claeys and T. Grava, “Universality of the break-up profile for the KdV equation in the small Dispersion limit using the Riemann–Hilbert approach”, *Commun. Math. Phys.* **286** (2009), pp. 979–1009.
- [7] P. Deift, S. Vanakides and X. Zhou, “New results in small dispersion KdV by an extension of the steepest descent method for Riemann–Hilbert problems”, *Int. Math. Res. Not.* **1997** (1997), pp. 286–299.
- [8] G. M. Fiksdal, *Numerical methods for the Benjamin–Ono equation*, PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2013. Master Thesis.
- [9] M. Gallone, M. Marian, A. Ponno and S. Ruffo, “Burgers turbulence in the Fermi–Pasta–Ulam–Tsingou chain”, *Phys. Rev. Lett.* **129** (2022), article no. 114101.
- [10] M. Gallone and A. Ponno, “Hamiltonian field theory close to the wave equation: from Fermi–Pasta–Ulam to water waves”, 2022. <https://arxiv.org/abs/2202.13454>.

- [11] L. Gassot, “Lax eigenvalues in the zero-dispersion limit for the Benjamin-Ono equation on the torus”, *SIAM J. Math. Anal.* **55** (2023), no. 5, pp. 5782–5822.
- [12] L. Gassot, “Zero-dispersion limit for the Benjamin–Ono equation on the torus with single well initial data”, *Commun. Math. Phys.* **401** (2023), pp. 2793–2843.
- [13] P. Gérard, “An explicit formula for the Benjamin–Ono equation”, *Tunis. J. Math.* **5** (2023), no. 3, pp. 593–603.
- [14] P. Gérard and T. Kappeler, “On the integrability of the Benjamin–Ono equation on the torus”, *Commun. Pure Appl. Math* **74** (2021), no. 8, pp. 1685–1747.
- [15] P. Gérard, T. Kappeler and P. Topalov, “On smoothing properties and Tao’s gauge transform for the Benjamin–Ono equation on the torus”, 2021. <https://arxiv.org/abs/2109.00610>, to appear in *Annales Scientifiques de l’ÉNS*.
- [16] T. Grava and C. Klein, “Numerical solution of the small dispersion limit of Korteweg–de Vries and Whitham equations”, *Appl. Math.* **60** (2007), no. 11, pp. 1623–1664.
- [17] E. Hopf, “The partial differential equation $u_t + uu_x = \mu u_x x$ ”, *Commun. Pure Appl. Math.* **3** (1950), pp. 201–230.
- [18] A. D. Ionescu and C. E. Kenig, “Global well-posedness of the Benjamin–Ono equation in low regularity spaces”, *J. Am. Math. Soc.* **20** (2007), no. 3, pp. 753–798.
- [19] S. N. Kruzhkov, “Generalized solutions of the Cauchy problem in the large for first order nonlinear equations”, *Dokl. Akad. Nauk SSSR* **187** (1969), pp. 29–32.
- [20] P. D. Lax and C. D. Levermore, “The small dispersion limit of the Korteweg–de Vries equation. I”, *Commun. Pure Appl. Math.* **36** (1983), pp. 253–290.
- [21] P. D. Lax and C. D. Levermore, “The small dispersion limit of the Korteweg–de Vries equation. II”, *Commun. Pure Appl. Math.* **36** (1983), pp. 571–593.
- [22] P. D. Lax and C. D. Levermore, “The small dispersion limit of the Korteweg–de Vries equation. III”, *Commun. Pure Appl. Math.* **36** (1983), pp. 809–829.
- [23] P. D. Miller and A. N. Wetzal, “The scattering transform for the Benjamin–Ono equation in the small dispersion limit”, *Phys. D: Nonlinear Phenom.* **333** (2016), pp. 185–199.
- [24] P. D. Miller and Z. Xu, “On the zero–dispersion limit of the Benjamin–Ono Cauchy problem for positive initial data”, *Commun. Pure Appl. Math.* **64** (2011), no. 2, pp. 205–270.
- [25] L. Molinet and D. Pilod, “The Cauchy problem for the Benjamin–Ono equation in L^2 revisited”, *Anal. PDE* **5** (2012), no. 2, pp. 365–395.
- [26] O. Oleinik, “Discontinuous solutions of nonlinear differential equations”, *Usp. Mat. Nauk* **12** (1957), pp. 3–73.
- [27] D. Serre, *Systems of conservation laws. I: Hyperbolicity, entropies, shock waves*, Cambridge University Press, 1999.
- [28] S. Venakides, “The zero dispersion limit of the Korteweg–de Vries equation for initial potentials with non–trivial reflection coefficient”, *Appl. Math.* **38** (1985), pp. 125–155.