ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Sébastien Boucksom and Mattias Jonsson

Non-Archimedean Green's functions and Zariski decompositions

Volume 362, Special Issue S1 (2024), p. 5-42

Online since: 6 June 2024
Issue date: 6 June 2024

Part of Special Issue: Complex algebraic geometry, in memory of Jean-Pierre Demailly
Guest editor: Claire Voisin (CNRS, Institut de Mathématiques de Jussieu-Paris rive gauche, France)

https://doi.org/10.5802/crmath.579

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2024, Vol. 362, Special issue S1, p. 5-42
https://doi.org/10.5802/crmath.579

Research article / *Article de recherche*
Algebraic geometry / *Géométrie algébrique*

Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Non-Archimedean Green's functions and Zariski decompositions

Fonctions de Green non-archimédiennes et décompositions de Zariski

Sébastien Boucksom [*,a] **and Mattias Jonsson** [b]

[a] Sorbonne Université and Université Paris Cité, CNRS, IMJ-PRG, F-75005 Paris, France
[b] Dept of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA
E-mails: sebastien.boucksom@imj-prg.fr, mattiasj@umich.edu

To the memory of Jean-Pierre Demailly, with admiration

Abstract. We study the non-Archimedean Monge–Ampère equation on a smooth projective variety over a discretely or trivially valued field. First, we give an example of a Green's function, associated to a divisorial valuation, which is not $\mathbb{Q}$-PL (i.e. not a model function in the discretely valued case). Second, we produce an example of a function whose Monge–Ampère measure is a finite atomic measure supported in a dual complex, but which is not invariant under the retraction associated to any snc model. This answers a question by Burgos Gil et al. in the negative. Our examples are based on geometric constructions by Cutkosky and Lesieutre, and arise via base change from Green's functions over a trivially valued field; this theory allows us to efficiently encode the Zariski decomposition of a pseudoeffective numerical class.

Résumé. Nous étudions l'équation de Monge–Ampère non-archimédienne sur une variété projective lisse sur un corps de valuation discrète ou triviale. Tout d'abord, nous donnons un exemple de fonction de Green, associée à une valuation divisorielle, qui n'est pas $\mathbb{Q}$-PL (i.e. pas une fonction modèle, dans le cas de valuation discrète). Ensuite, nous produisons un exemple de fonction dont la mesure de Monge–Ampère est à support dans un complexe dual, mais qui n'est invariante par la rétraction associée à aucun modele snc. Ceci répond négativement à une question de Burgos Gil et al. Nos exemples sont basés sur des constructions géométriques de Cutkosky et Lesieutre, et sont produits par changement de base à partir de fonctions de Green sur un corps trivialement valué ; cette théorie nous permet d'encoder de façon efficace la décomposition de Zariski de toute classe pseudo-effective.

Funding. The second author was partially supported by NSF grants DMS-1900025 and DMS-2154380.

Manuscript received 31 March 2023, accepted 28 September 2023.

* Corresponding author

Introduction

In the seminal paper [43], Yau studied the Monge–Ampère equation

$$(\omega + dd^c\varphi)^n = \mu \tag{MA}$$

on a compact n-dimensional Kähler manifold (X, ω), where μ is a smooth, strictly positive measure on X of mass $\int \omega^n$, and φ a smooth function on X such that the $(1,1)$-form $\omega + dd^c\varphi$ is positive. Yau proved that there exists a smooth solution φ, unique up to a constant. If ω is a rational class, say $\omega = c_1(L)$ for an ample line bundle L, then φ can be viewed as a positive metric on L, and $(\omega + dd^c\varphi)^n$ its the curvature measure.

As observed by Kontsevich, Soibelman, and Tschinkel [31, 32], when studying degenerating 1-parameter families of Kähler manifolds, it can be fruitful to use non-Archimedean geometry in the sense of Berkovich over the field $\mathbb{C}((\varpi))$ of complex Laurent series. In this context, a Monge–Ampère operator was introduced by Chambert–Loir [19], and a version of (MA) was solved by the authors and Favre [11]; see below. Uniqueness of solutions was proved earlier by Yuan and Zhang [44].

Now, the method in [11] is variational in nature, inspired by [4] in the complex case. It has the advantage of being able to deal with more general measures μ, but the drawback of providing less regularity information on the solution. In fact, [11] only gives a continuous solution, and is thus closer in spirit to the work of Kołodziej [30] than to [43].

It is therefore interesting to ask whether we can say more about the regularity of φ in (MA), at least for special measures μ. In the non-Archimedean setting, there are many possible regularity notions; to describe the one we are focusing on, we first need to make the non-Archimedean version of (MA) more precise, following [10, 11].

Let X be a smooth projective variety over $K = \mathbb{C}((\varpi))$ of dimension n. Consider a simple normal crossing (snc) model $\mathscr{X}$ of X, over the valuation ring $K^\circ = \mathbb{C}[[\varpi]]$. The dual complex $\Delta_{\mathscr{X}}$ embeds in the Berkovich analytification X^{an}, and there is a continuous retraction $p_{\mathscr{X}}\colon X^{\mathrm{an}} \to \Delta_{\mathscr{X}}$.

A semipositive closed (1,1)-form on X^{an} in the sense of loc. cit. is represented by a nef relative numerical class $\omega \in \mathrm{N}^1(\mathscr{X}/\operatorname{Spec} K^\circ)$ for some snc model $\mathscr{X}$. We assume that the image $[\omega]$ of ω in $\mathrm{N}^1(X)$ is ample. In this case, there is a natural space $\mathrm{CPSH}(\omega) = \mathrm{CPSH}(X, \omega)$ of continuous ω-plurisubharmonic (psh) functions, and a Monge–Ampère operator taking a function $\varphi \in \mathrm{CPSH}(\omega)$ to a positive Radon measure $\varphi \to (\omega + dd^c\varphi)^n$ on X^{an} of mass $[\omega]^n$; see also [20] for a local theory. When $[\omega]$ is rational, so that $[\omega] = c_1(L)$ for an ample ($\mathbb{Q}$-)line bundle L on X, we can view any $\varphi \in \mathrm{CPSH}(\omega)$ as a semipositive continuous metric on L^{an}, with curvature measure $(\omega + dd^c\varphi)^n$.

As in [11], let us normalize the Monge–Ampère operator and write

$$\mathrm{MA}_\omega(\varphi) := \tfrac{1}{[\omega]^n}(\omega + dd^c\varphi)^n.$$

The main result in [11] is that if μ is a Radon probability measure on X^{an} supported in some dual complex, then there exists $\varphi \in \mathrm{CPSH}(\omega)$, unique up to an additive real constant, such that $\mathrm{MA}_\omega(\varphi) = \mu$. More precisely, this was proved assuming that X is defined over an algebraic curve, an assumption that was later removed in [18]. Here we want to study whether for special measures μ, the solution is regular in some sense.

We first consider the class of *piecewise linear* (PL) functions. A function $\varphi \in \mathrm{C}^0(X^{\mathrm{an}})$ is ($\mathbb{Q}$-)PL if it is associated to a vertical $\mathbb{Q}$-divisor on some snc model, and PL functions are also known as *model functions*. The set $\mathrm{PL}(X)$ of PL functions is a dense $\mathbb{Q}$-linear subspace of $\mathrm{C}^0(X^{\mathrm{an}})$, and it is closed under taking finite maxima and minima.

If $\varphi \in \mathrm{PL}(X) \cap \mathrm{CPSH}(\omega)$, then the measure $\mu = \mathrm{MA}_\omega(\varphi)$ is a rational divisorial measure, i.e. a rational convex combination of Dirac masses at divisorial valuations. For example, when $[\omega] = c_1(L)$ is rational, the space $\mathrm{PL}(X) \cap \mathrm{CPSH}(\omega)$ can be identified with the space of semipositive *model metrics* on L^{an}, represented by a nef model $\mathscr{L}$ of L, and $\mathrm{MA}_\omega(\varphi)$ can be computed in terms of intersection numbers of $\mathscr{L}$.

Assuming ω rational, one may ask whether, conversely, the solution to $\mathrm{MA}_\omega(\varphi) = \mu$, with μ a rational divisorial measure, is necessarily PL. Here we focus on the case when $\mu = \delta_x$ is a Dirac measure, where $x \in X^{\mathrm{div}}$ is a divisorial valuation. In this case, it was proved in [11] that the solution $\varphi_x \in \mathrm{CPSH}(\omega)$ to the Monge–Ampère equation

$$\mathrm{MA}_\omega(\varphi_x) = \delta_x, \quad \varphi_x(x) = 0 \tag{$\star$}$$

is the *Green's function* of x, given by $\varphi_x = \sup\{\psi \in \mathrm{CPSH}(\omega) \mid \psi(x) \le 0\}$.

Theorem A. *Assume that ω is a rational semipositive closed (1,1)-form with* $[\omega]$ *ample, and that* $x \in X^{\mathrm{div}}$ *is a divisorial valuation. Let* $\varphi_x \in \mathrm{CPSH}(\omega)$ *be the Green's function satisfying* $(\star)$ *above. Then:*

- (i) *in dimension 1,* $\varphi_x \in \mathrm{PL}(X)$*;*
- (ii) *in dimension* ≥ 2*, it may happen that* $\varphi_x \notin \mathrm{PL}(X)$.

Writing $[\omega] = c_1(L)$, Theorem A says that the metric on L^{an} corresponding to φ_x is a model metric in dimension 1, but not necessarily in dimension 2 and higher. This answers a question in [11], see Remark 8.8 in loc. cit.

Here (i) is well known, for example from the work of Thuillier [42]; see Section 8.5. As for (ii), we present one example where X is an abelian surface, and another one where $X = \mathbb{P}^3$; see Examples 99 and 100.

We will discuss the structure of these examples shortly, but mention here that they are both $\mathbb{R}$-*PL*, i.e. they belong to the smallest $\mathbb{R}$-linear subspace $\mathbb{R}\mathrm{PL}(X)$ of $\mathrm{C}^0(X^{\mathrm{an}})$ containing $\mathrm{PL}(X)$ and stable under max and min. The question then arises whether also in higher dimension, the solution φ_x to $(\star)$ is $\mathbb{R}$-PL for any divisorial valuation x. While we don't have a counterexample to this exact question (with ω rational, but see Example 67), we prove that the situation can be quite complicated in dimensions three and higher.

Namely, let us say that a function $\varphi \in \mathrm{C}^0(X^{\mathrm{an}})$ is *invariant under retraction* if $\varphi = \varphi \circ p_{\mathscr{X}}$ for some (and hence any sufficiently high) snc model $\mathscr{X}$. For example, a function on X^{an} is $\mathbb{R}$-PL iff it is invariant under retraction and its restriction to any dual complex $\Delta_{\mathscr{X}}$ is $\mathbb{R}$-PL in the sense that it is affine on the cells of some subdivision of $\Delta_{\mathscr{X}}$ into real simplices.

If $\varphi \in \mathrm{CPSH}(\omega)$ is invariant under retraction, say $\varphi = \varphi \circ p_{\mathscr{X}}$, then the Monge–Ampère measure $\mathrm{MA}_\omega(\varphi)$ is supported in $\Delta_{\mathscr{X}}$. However, if μ is supported in $\Delta_{\mathscr{X}}$, then the solution φ to $\mathrm{MA}_\omega(\varphi) = \mu$ may not satisfy $\varphi = \varphi \circ p_{\mathscr{X}}$, see [25, Appendix A]. Still, one may ask whether φ is invariant under retraction, that is, $\varphi = \varphi \circ p_{\mathscr{X}'}$ for any sufficiently high snc model $\mathscr{X}'$, see Question 2 in loc. cit.. A version of this question (see Remark 77) in the context of Calabi–Yau varieties plays a key role in the recent work of Yang Li [36], see also [1, 28, 37]. Our next result provides a negative answer in general.

Theorem B. *Let* $X = \mathbb{P}^3_K$*, with* $K = \mathbb{C}((\varpi))$*, and let ω be the closed (1,1)-form associated to the numerical class of* $\mathscr{O}(1)$ *on* $\mathbb{P}^3_{K^\circ}$*. Then there exists* $\varphi \in \mathrm{CPSH}(\omega)$ *such that* $\mathrm{MA}_\omega(\varphi)$ *has finite support in some dual complex, but φ is not invariant under retraction. In particular,* $\varphi \notin \mathbb{R}\mathrm{PL}(X)$.

Let us now say more about the examples underlying Theorem B and Theorem A (ii). They all arise in the *isotrivial case*, when the variety X over K is the base change of a smooth projective variety Y over $\mathbb{C}$, and the $(1,1)$-form ω is defined by the pullback of an ample numerical class $\theta \in \mathrm{N}^1(Y)$ to the trivial (snc) model Y_{K° of $X = Y_K$. In this case, we can draw on the global pluripotential theory over a trivially valued field developed in [13], a theory which interacts well with algebro-geometric notions such as diminished base loci and Zariski decompositions of pseudoeffective classes.

Specifically, given a smooth projective complex variety Y, and an ample numerical class $\theta \in \mathrm{N}^1(Y)$, we have a convex set $\mathrm{CPSH}(\theta) = \mathrm{CPSH}(Y,\theta) \subset \mathrm{C}^0(Y^{\mathrm{an}})$ of continuous θ-psh functions, where Y^{an} now denotes the Berkovich analytification of Y with respect to the *trivial* absolute value on $\mathbb{C}$. A *divisorial valuation* on Y is of the form $v = t\,\mathrm{ord}_E$, where $t \in \mathbb{Q}_{\geq 0}$ and $E \subset Y'$ is a prime divisor on a smooth projective variety Y' with a proper birational morphism $Y' \to Y$. When instead $t \in \mathbb{R}_{\geq 0}$, we say that v is a *real divisorial valuation.* If $\Sigma \subset Y^{\mathrm{an}}$ is a finite set of real divisorial valuations, then we consider the Green's function of Σ, defined as

$$\varphi_\Sigma := \sup\{\varphi \in \mathrm{CPSH}(Y,\theta) \mid \varphi|_\Sigma \leq 0\}.$$

By [13], $\varphi_\Sigma \in \mathrm{CPSH}(Y,\theta)$, and the Monge–Ampère measure of φ_Σ is supported in Σ.

The base change $X = Y_{\mathbb{C}((\varpi))} \to Y$ induces a surjective map $\pi\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$, and this map admits a canonical section $\sigma\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$, called *Gauss extension*, and whose image consists of all $\mathbb{C}^\times$-invariant points in X^{an}. For any $\varphi \in \mathrm{CPSH}(Y,\theta)$ we have $\pi^\star\varphi \in \mathrm{CPSH}(X,\omega)$, and

$$\mathrm{MA}_\omega(\pi^\star\varphi) = \sigma_\star \mathrm{MA}_\theta(\varphi).$$

In particular, if $v \in Y^{\mathrm{div}}$, then $\pi^\star\varphi_{\{v\}}$ is the Green's function for $x := \sigma(v) \in X^{\mathrm{div}}$. As both $\pi^\star$ and $\sigma^\star$ preserve the classes of $\mathbb{Q}$-PL and $\mathbb{R}$-PL functions, we see that in order to prove Theorem A (ii), it suffices to find a surface Y and $v \in Y^{\mathrm{div}}$, such that $\varphi_v := \varphi_{\{v\}}$ is not $\mathbb{Q}$-PL.

Further, to prove Theorem B, it suffices to find a finite set Σ of real divisorial valuations on $Y = \mathbb{P}^3_{\mathbb{C}}$ such that $\pi^\star\varphi_\Sigma$ fails to be invariant under retraction. Indeed, the Gauss extension map σ takes real divisorial valuations to Abhyankar valuations, and these are exactly the ones that lie in a dual complex. We then use the following criterion. Define the *center* of any function $\varphi \in \mathrm{PSH}(Y,\theta)$ by

$$Z_Y(\varphi) := c_Y\{\varphi < \sup\varphi\},$$

where $c_Y\colon Y^{\mathrm{an}} \to Y$ is the center map, see Section 3. We show that if $\pi^\star\varphi$ is invariant under retraction, then $Z_Y(\varphi) \subset Y$ is a strict Zariski closed subset, see Corollary 97. It therefore suffices to find a Green's function φ_Σ whose center is Zariski dense.

Our analysis of the Green's functions φ_Σ is based on a relation between θ-psh functions and families of b-divisors. Namely, we can pick a proper birational morphism $\rho\colon Y' \to Y$, with Y' smooth, prime divisors $E_i \subset Y'$, and $c_i \in \mathbb{R}_{>0}$, such that $\Sigma = \{c_i^{-1}\,\mathrm{ord}_{E_i}\}$. If we set $D := \sum_i c_i^{-1} E_i$, then we can express φ_Σ in terms of the *b-divisorial Zariski decomposition* of the numerical class $\rho^\star\theta - \lambda[D]$, for $\lambda \in (-\infty, \lambda_{\mathrm{psef}}]$, where $\lambda_{\mathrm{psef}} \in \mathbb{R}$ is the largest λ such that this class is pseudoeffective (psef), see Theorem 57. The analysis of the Zariski decomposition of a psef class θ in terms of θ-psh functions is of independent interest.

Let us first consider the case of dimension two. The Zariski decomposition of $\rho^\star\theta - \lambda D$ is then an $\mathbb{R}$-PL function of λ, and this implies that the Green's function φ_Σ is $\mathbb{R}$-PL. On the other hand, φ_Σ need not be $\mathbb{Q}$-PL. In fact, we prove in Theorem 60 that φ_Σ is $\mathbb{Q}$-PL iff the pseudoeffective threshold λ_{psef} is a rational number. To prove Theorem A (ii), it therefore suffices to find a divisorial valuation v on a surface Y such that λ_{psef} is irrational, and such examples can be found with Y an abelian surface, and $v = \mathrm{ord}_E$ for a prime divisor E on Y.

Using a geometric construction by Cutkosky [21], we also give an example of a divisorial valuation v on $Y = \mathbb{P}^3$ such that φ_v is $\mathbb{R}$-PL but not $\mathbb{Q}$-PL for $\theta = c_1(\mathcal{O}(1))$, see Example 65. Being $\mathbb{R}$-PL, this example is invariant under retraction. As explained above, in order to prove Theorem B, it suffices to find Σ such that the center $c_Y(\varphi_\Sigma)$ is a Zariski dense subset of Y. Using the notation above, we show that the center contains the image on Y of the diminished base locus of the pseudoeffective class $\rho^\star\theta - \lambda_{\mathrm{psef}}[D]$ on Y'. We can then use a construction of Lesieutre [35], who showed that if $Y = \mathbb{P}^3$, $\theta = c_1(\mathcal{O}(1))$, and $\rho\colon Y' \to Y$ is the blowup at nine very general points, then there exists an effective $\mathbb{R}$-divisor D on Y' supported on the exceptional locus on ρ, such that the diminished base locus of $\rho^\star\theta - D$ is Zariski dense. If we write $D = \sum_{i=1}^{9} c_i E_i$, then we can take $\Sigma = \{c_i^{-1} \operatorname{ord}_{E_i}\}$.

Structure of the paper

The article is organized as follows. In Section 1 we recall some facts from birational geometry and pluripotential theory over a trivially valued field. This is used in Section 2 to relate θ-psh functions and suitable families of b-divisors, after which we study the center of a θ-psh function in Section 3. In Section 4 we define the extremal function $V_\theta \in \mathrm{PSH}(\theta)$ associated to a psef class: by evaluating this function at divisorial valuations we recover the minimal vanishing order of θ along a valuation. The extremal function is also closely related to various notions of Zariski decomposition of a psef class, as explored in Section 5. After all this, we are finally ready to study Green's functions in Section 6 and Section 7. Finally, in Section 8 and Section 9 we turn to the discretely valued case and prove Theorems A and B.

Notation and conventions

A *variety* over a field F is a geometrically integral F-scheme of finite type. We use the abbreviations *usc* for "upper semicontinuous", *lsc* for "lower semicontinuous", and *iff* for "if and only if".

Acknowledgement

The authors would like to thank José Burgos Gil, Antoine Ducros, Gerard Freixas, Walter Gubler, John Lesieutre and Milan Perera for useful exchanges related to this work. This article is dedicated to the memory of Jean-Pierre Demailly, whose extraordinary contributions to complex analytic and algebraic geometry have had a tremendous influence on our own research.

1. Preliminaries

Throughout the paper (except in Section 8) X denotes a smooth projective variety over an algebraically closed field k of characteristic 0.

1.1. *Positivity of numerical classes and base loci*

We denote by $\mathrm{N}^1(X)$ the (finite dimensional) vector space of numerical equivalence classes $\theta = [D]$ of $\mathbb{R}$-divisors D on X. It contains the following convex cones, corresponding to various positivity notions for numerical classes:

- the *pseudoeffective cone* $\mathrm{Psef}(X)$, defined as the closed cone generated by all classes of effective divisors;
- the *big cone* $\mathrm{Big}(X)$, the interior of $\mathrm{Psef}(X)$;
- the *nef cone* $\mathrm{Nef}(X)$, equal to the closed convex cone generated by all classes of basepoint free line bundles;
- the *ample cone* $\mathrm{Amp}(X)$, the interior of $\mathrm{Nef}(X)$;
- the *movable cone* $\mathrm{Mov}(X)$, the closed convex cone generated by all classes of line bundles with base locus of codimension at least 2.

These cones satisfy

$$\mathrm{Nef}(X) \subset \mathrm{Mov}(X) \subset \mathrm{Psef}(X),$$

where the first (resp. second) inclusion is an equality when $\dim X \le 2$ (resp. $\dim X \le 1$), but is in general strict for $\dim X > 2$ (resp. $\dim X > 1$). We will make use of the following simple property:

Lemma 1. *If $\theta \in \mathrm{N}^1(X)$ is movable, then $\theta|_E \in \mathrm{N}^1(E)$ is pseudoeffective for any prime divisor $E \subset X$.*

The *asymptotic base locus* $\mathbb{B}(D) \subset X$ of a $\mathbb{Q}$-divisor D is defined as the base locus of $\mathcal{O}_X(mD)$ for any $m \in \mathbb{Z}_{>0}$ sufficiently divisible. The *diminished* (or *restricted*) *base locus* and the *augmented base locus* of an $\mathbb{R}$-divisor D are respectively defined as

$$\mathbb{B}_-(D) := \bigcup_A \mathbb{B}(D+A) \quad \text{and} \quad \mathbb{B}_+(D) := \bigcap_A \mathbb{B}(D-A),$$

where A ranges over all ample $\mathbb{R}$-divisors such that $D-A$ (resp. $D+A$) is a $\mathbb{Q}$-divisor. Since ampleness is a numerical property, these loci only depend on the numerical class $\theta = [D] \in \mathrm{N}^1(X)$, and will be denoted by $\mathbb{B}_-(\theta) \subset \mathbb{B}_+(\theta)$.

The augmented base locus $\mathbb{B}_+(\theta)$ is Zariski closed, and satisfies

$$\theta \in \mathrm{Big}(X) \Longleftrightarrow \mathbb{B}_+(\theta) \neq X \quad \text{and} \quad \theta \in \mathrm{Amp}(X) \Longleftrightarrow \mathbb{B}_+(\theta) = \emptyset.$$

The diminished base locus satisfies

$$\mathbb{B}_-(\theta) = \bigcup_{\varepsilon \in \mathbb{Q}_{>0}} \mathbb{B}_+(\theta + \varepsilon\omega) \tag{1}$$

for any $\omega \in \mathrm{Amp}(X)$. It is thus an at most countable union of subvarieties, which is not Zariski closed in general, and can even be Zariski dense (see [35]). We further have

$$\begin{aligned} \theta \in \mathrm{Psef}(X) &\Longleftrightarrow \mathbb{B}_-(\theta) \neq X; \\ \theta \in \mathrm{Nef}(X) &\Longleftrightarrow \mathbb{B}_-(\theta) = \emptyset; \\ \theta \in \mathrm{Mov}(X) &\Longleftrightarrow \operatorname{codim} \mathbb{B}_-(\theta) \geq 2. \end{aligned}$$

1.2. *The Berkovich space*

We use [13, §1] as a reference. The *Berkovich space* X^{an} is defined as the Berkovich analytification of X with respect to the trivial absolute value on k [3]. We view it as a compact (Hausdorff) topological space, whose points are *semivaluations*, i.e. valuations $v\colon k(Y)^\times \to \mathbb{R}$ for some subvariety $Y \subset X$. We denote by $v_{Y,\mathrm{triv}} \in X^{\mathrm{an}}$ the trivial valuation on $k(Y)$, and set $v_{\mathrm{triv}} := v_{X,\mathrm{triv}}$. These trivial semivaluations are precisely the fixed points of the scaling action $\mathbb{R}_{>0} \times X^{\mathrm{an}} \to X^{\mathrm{an}}$ given by $(t, v) \mapsto tv$.

We denote $X^{\mathrm{div}} \subset X^{\mathrm{an}}$ the (dense) subset of *divisorial valuations*, of the form $v = t\,\mathrm{ord}_E$ with $t \in \mathbb{Q}_{\geq 0}$ and E a prime divisor on a birational model $\pi\colon Y \to X$ (the case $t = 0$ corresponding to $v = v_{\mathrm{triv}}$, by convention). In the present work, where $\mathbb{R}$-divisors arise naturally, it will be convenient to allow t to be real, in which case we will say that $v = t\,\mathrm{ord}_E$ is a *real divisorial valuation*. We denote by

$$X^{\mathrm{div}}_{\mathbb{R}} = \mathbb{R}_{>0} X^{\mathrm{div}}$$

the set of real divisorial valuations. It is contained in the space $X^{\mathrm{lin}} \subset X^{\mathrm{an}}$ of *valuations of linear growth* (see [17] and [13, §1.5]).

1.3. *Rational and real piecewise linear functions*

In [13], various classes of $\mathbb{Q}$-PL functions on X^{an} were introduced, and the purpose of what follows is to discuss their $\mathbb{R}$-PL counterparts.

First, any ideal $\mathfrak{b} \subset \mathcal{O}_X$ defines a homogeneous function

$$\log|\mathfrak{b}|\colon X^{\mathrm{an}} \longrightarrow [-\infty, 0]$$

such that $\log|\mathfrak{b}|(v) := -v(\mathfrak{b})$ for $v \in X^{\mathrm{an}}$.

Second, any *flag ideal* $\mathfrak{a}$, i.e. a coherent fractional ideal sheaf on $X \times \mathbb{A}^1$ invariant under the $\mathbb{G}_m$-action on $\mathbb{A}^1$ and trivial on $X \times \mathbb{G}_m$, defines a continuous function

$$\varphi_{\mathfrak{a}} \colon X^{\mathrm{an}} \longrightarrow \mathbb{R}$$

given by $\varphi_{\mathfrak{a}}(v) = -\sigma(v)(\mathfrak{a})$, where $\sigma \colon X^{\mathrm{an}} \to (X \times \mathbb{A}^1)^{\mathrm{an}}$ is the *Gauss extension*, defined as follows. If v is a valuation on $k(Y)$ for some subvariety $Y \subset X$, then $\sigma(v)$ is the unique valuation on $k(Y \times \mathbb{A}^1) = k(Y)(\varpi)$ with the following property: if $f = \sum_j f_j \varpi^j \in k(Y)[\varpi]$, then $\sigma(v)(f) = \min_j\{v(f_j) + j\}$.

Concretely, any flag ideal can be written $\mathfrak{a} = \sum_{\lambda \in \mathbb{Z}} \mathfrak{a}_\lambda \varpi^{-\lambda}$ for a decreasing sequence of ideals $\mathfrak{a}_\lambda \subset \mathcal{O}_X$ such that $\mathfrak{a}_\lambda = \mathcal{O}_X$ for $\lambda \ll 0$ and $\mathfrak{a}_\lambda = 0$ for $\lambda \gg 0$, and then $\varphi_{\mathfrak{a}} = \max_\lambda(\log|\mathfrak{a}_\lambda| + \lambda)$.

We denote by:

- $\mathrm{PL}^+_{\mathrm{hom}}(X)$ the set of $\mathbb{Q}_+$-linear combinations of functions of the form $\log|\mathfrak{b}|$ with $\mathfrak{b} \subset \mathcal{O}_X$ a nonzero ideal;
- $\mathrm{PL}^+(X)$ the set of functions $\varphi \in \mathrm{C}^0(X^{\mathrm{an}})$ of the form $\varphi = \max_i\{\psi_i + \lambda_i\}$ for a finite family $\psi_i \in \mathrm{PL}^+_{\mathrm{hom}}(X)$ and $\lambda_i \in \mathbb{Q}$; equivalently, functions of the form $\varphi = \frac{1}{m}\varphi_{\mathfrak{a}}$ for a flag ideal $\mathfrak{a}$ and $m \in \mathbb{Z}_{>0}$;
- $\mathrm{PL}(X)$ the set of differences of functions in $\mathrm{PL}^+(X)$, called *rational piecewise linear functions* ($\mathbb{Q}$*-PL functions* for short).

The sets $\mathrm{PL}^+_{\mathrm{hom}}(X)$ are stable under addition and max, while $\mathrm{PL}(X)$ is a $\mathbb{Q}$-vector space, stable under max, and is dense in $\mathrm{C}^0(X^{\mathrm{an}})$.

As in [13, §3.1], we denote by $\mathrm{PL}(X)_{\mathbb{R}}$ the $\mathbb{R}$-vector space generated by $\mathrm{PL}(X)$. It is not stable under max anymore; to remedy this, we further introduce:

- the set $\mathrm{PL}^+(X)_{\mathbb{R}}$ of $\mathbb{R}_+$-linear combinations of functions in $\mathrm{PL}^+(X)$;
- the set $\mathbb{R}\mathrm{PL}^+(X)$ of finite maxima of functions in $\mathrm{PL}^+(X)_{\mathbb{R}}$;
- the set $\mathbb{R}\mathrm{PL}(X)$ of differences of functions in $\mathbb{R}\mathrm{PL}^+(X)$; we call its elements *real piecewise linear functions* ($\mathbb{R}$*-PL functions* for short).

As one immediately sees, the sets $\mathrm{PL}^+(X)_{\mathbb{R}}$ and $\mathbb{R}\mathrm{PL}^+(X)$ are convex cones in $\mathrm{C}^0(X^{\mathrm{an}})$, and $\mathbb{R}\mathrm{PL}(X)$ is thus an $\mathbb{R}$-vector space. Further, $\mathbb{R}\mathrm{PL}^+(X)$, and hence $\mathbb{R}\mathrm{PL}(X)$, are clearly stable under max. Thus $\mathbb{R}\mathrm{PL}(X)$ is the smallest $\mathbb{R}$-linear subspace of $\mathrm{C}^0(X^{\mathrm{an}})$ that is stable under max and contains $\mathrm{PL}(X)$.

Finally, introduce the convex cone $\mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$ of $\mathbb{R}_+$-linear combinations of functions in $\mathrm{PL}^+_{\mathrm{hom}}(X)$ (this is again not stable under max anymore). We then have:

Lemma 2. *A function $\varphi \in \mathrm{C}^0(X^{\mathrm{an}})$ lies in $\mathbb{R}\mathrm{PL}^+(X)$ iff $\varphi = \max_i\{\psi_i + \lambda_i\}$ for a finite family $\psi_i \in \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$ and $\lambda_i \in \mathbb{R}$.*

Proof. Since any function in $\mathbb{R}\mathrm{PL}^+(X)$ is a finite max of functions $\varphi \in \mathrm{PL}^+(X)_{\mathbb{R}}$, it suffices to show that φ is of the desired form. Write $\varphi = \sum_{i=1}^r t_i \varphi_i$ with $t_i \in \mathbb{R}_{>0}$ and $\varphi_i \in \mathrm{PL}^+(X)$, i.e. $\varphi_i = \max_j\{\psi_{ij} + \lambda_{ij}\}$ with $\psi_{ij} \in \mathrm{PL}^+_{\mathrm{hom}}(X)$ and $\lambda_{ij} \in \mathbb{Q}$. Then

$$\varphi = \max_{j_1,\dots,j_r} \sum_{i=1}^r t_i\left(\psi_{ij_i} + \lambda_{ij_i}\right).$$

Since each $\sum_i t_i \psi_{ij_i}$ lies in $\mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$, this shows that φ is of the desired form.

Conversely, assume $\varphi = \max_i\{\psi_i + \lambda_i\}$ for a finite family $\psi_i \in \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$ and $\lambda_i \in \mathbb{R}$. For each i, write $\psi_i = \sum_j t_{ij}\psi_{ij}$ with $\psi_{ij} \in \mathrm{PL}^+_{\mathrm{hom}}(X) \le 0$. Pick $v \in X^{\mathrm{an}}$ and i such that $\varphi(v) = \psi_i(v) + \lambda_i$. Since φ is bounded, we can find $c \in \mathbb{Q}$ such that $\psi_{ij}(v) \ge c$ for all j. This shows that $\varphi = \max_i \varphi_i$ with $\varphi_i := \sum_j t_{ij} \max\{\psi_{ij}, c\} + \lambda_i$. For all i, j, $\max\{\psi_{ij}, c\}$ lies in $\mathrm{PL}^+(X)$, thus $\varphi_i \in \mathrm{PL}^+(X)_{\mathbb{R}}$, and hence $\varphi \in \mathbb{R}\mathrm{PL}^+(X)$. □

1.4. *Homogeneous functions vs. b-divisors*

We use [7, §1] and [13, §6.4] as references for what follows. Recall that

- a *(real) b-divisor over* X is a collection $B = (B_Y)$ of $\mathbb{R}$-divisors on all (smooth) birational models $Y \to X$, compatible under push-forward as cycles, i.e. an element of the $\mathbb{R}$-vector space
$$\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}} := \varprojlim_Y \mathrm{Z}^1(Y)_{\mathbb{R}};$$
- a b-divisor $B = (B_Y)$ is *effective* if B_Y is effective for all Y; if B, B' are b-divisors, then we write $B \le B'$ iff $B' - B$ is effective;
- a b-divisor $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is said to be $\mathbb{R}$-*Cartier* if there exists a model Y, called a *determination* of B, such that $B_{Y'}$ is the pullback of B_Y for all higher birational models Y'; thus the space of $\mathbb{R}$-Cartier b-divisors is given by
$$\mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}} := \varinjlim_Y \mathrm{Z}^1(Y)_{\mathbb{R}}.$$

Example 3. Any $\mathbb{R}$-divisor D on a model $Y \to X$ determines an $\mathbb{R}$-Cartier b-divisor $\overline{D} \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$, obtained by pulling back D to all higher models, and any $\mathbb{R}$-Cartier b-divisor is of this form.

For any $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ and $v \in X^{\mathrm{div}}$, we define $v(B) \in \mathbb{R}$ as follows: pick a prime divisor E on a birational model $Y \to X$ and $t \in \mathbb{Q}_{\ge 0}$ such that $v = t\,\mathrm{ord}_E$, and set
$$v(B) := t\,\mathrm{ord}_E(B_Y).$$
This is independent of the choices made, and the function $\psi_B \colon X^{\mathrm{div}} \to \mathbb{R}$ defined by
$$\psi_B(v) := v(B)$$
is homogeneous (with respect to the scaling action of $\mathbb{Q}_{>0}$).

Definition 4. *We say that a homogeneous function* $\psi\colon X^{\mathrm{div}} \to \mathbb{R}$ *is of* divisorial type *if* $\psi(\mathrm{ord}_E) = 0$ *for all but finitely many prime divisors* $E \subset X$.

The next result is straightforward:

Lemma 5. *The map* $B \mapsto \psi_B$ *sets up a vector space isomorphism between* $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ *and the space of homogeneous functions of divisorial type on* X^{div}. *Moreover,* $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ *is effective iff* $\psi_B \ge 0$.

We endow $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ with the topology of pointwise convergence on X^{div}. If Ω is a topological space, then a map $f\colon \Omega \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is thus continuous iff $v \circ f\colon \Omega \to \mathbb{R}$ is continuous for all $v \in X^{\mathrm{div}}$. We will also say that $f\colon \Omega \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is lsc (resp. usc) iff $v \circ f\colon \Omega \to \mathbb{R}$ is lsc (resp. usc) for all $v \in X^{\mathrm{div}}$.

If Ω is a convex subset of a real vector space, then we say that $f\colon \Omega \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is convex if $v \circ f$ is convex for all $v \in X^{\mathrm{div}}$. This amounts to $f((1-t)x_0 + t x_1) \le (1-t) f(x_0) + t f(x_1)$ for $x_0, x_1 \in \Omega$, $0 \le t \le 1$. We say that f is concave if $-f$ is convex.

Finally, if $\Omega \subset \mathbb{R}$ is an interval, then $f\colon \Omega \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is increasing (resp. decreasing) if $v \circ f$ is increasing (resp. decreasing) for each $v \in X^{\mathrm{div}}$.

Next we will generalize [13, Theorem 6.32] to real coefficients.

Definition 6. *We denote by* $\mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{R}}$ *the convex cone of divisors* $B \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$ *that are antieffective and relatively semiample over* X. *We also set* $\mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{Q}} := \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}} \cap \mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{R}}$.

Proposition 7. *The map* $B \mapsto \psi_B$ *induces an isomorphism between* $\mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$ *and the* $\mathbb{R}$*-vector space generated by (the restrictions to* X^{div} *of) all functions* $\log|\mathfrak{b}|$ *with* $\mathfrak{b} \subset \mathcal{O}_X$ *a nonzero ideal. This isomorphism restricts to a bijection*
$$\mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{R}} \xrightarrow{\sim} \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}.$$

Proof. The first point is a consequence of [13, Theorem 6.32], which also yields

$$\mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{PL}^+_{\mathrm{hom}}(X).$$

Since the right-hand side generates the convex cone $\mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$, it suffices to show that the convex cone of antieffective and relatively semiample divisors in $\mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$ is generated by antieffective and semiample divisors in $\mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}}$. By definition of a relatively semiample $\mathbb{R}$-Cartier b-divisor, we have $B = \sum_i t_i B_i$ with $t_i > 0$ and $B_i \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}}$ relatively semiample. By the Negativity Lemma (see [7, Proposition 2.12]), $B'_i := B_i - \overline{B_{i,X}}$ is antieffective, and still relatively semiample. Denoting by $B_X = -\sum_\alpha c_\alpha E_\alpha$ the irreducible decomposition of the antieffective $\mathbb{R}$-divisor B_X, we infer

$$B = \sum_i t_i B'_i + \sum_\alpha c_\alpha(-\overline{E_\alpha})$$

where $-\overline{E_\alpha} \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}}$ is antieffective and relatively semiample. The result follows. □

1.5. *Numerical b-divisor classes*

The space of *numerical b-divisor classes* is defined as

$$\mathrm{N}^1_{\mathrm{b}}(X) := \varprojlim_Y \mathrm{N}^1(Y),$$

equipped with the inverse limit topology (each finite dimensional $\mathbb{R}$-vector space $\mathrm{N}^1(Y)$ being endowed with its canonical topology).

Any b-divisor defines a numerical b-divisor class. This yields a natural quotient map

$$\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}} \longrightarrow \mathrm{N}^1_{\mathrm{b}}(X) \quad B \longmapsto [B].$$

One should be wary of the fact this map is *not* continuous with respect to the topology of pointwise convergence of $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$. However, we observe:

Lemma 8. *For any finite set $\mathscr{E}$ of prime divisors on X, the quotient map $B \mapsto [B]$ is continuous on the subspace $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R},\mathscr{E}}$ of b-divisors B such that B_X is supported by $\mathscr{E}$.*

Proof. For any model $Y \to X$, each B_Y with $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R},\mathscr{E}}$ lives in the finite dimensional vector space generated by the strict transforms of the elements of $\mathscr{E}$ and the π-exceptional prime divisors. Thus $B \mapsto [B_Y] \in \mathrm{N}^1(Y)$ is continuous on $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R},\mathscr{E}}$, and the result follows. □

The set of numerical classes of $\mathbb{R}$-Cartier b-divisors can be identified with the direct limit

$$\varinjlim_Y \mathrm{N}^1(Y) \subset \mathrm{N}^1_{\mathrm{b}}(X).$$

In particular, any numerical class $\theta \in \mathrm{N}^1(X)$ defines a numerical b-divisor class $\overline{\theta} = (\theta_Y)_Y \in \mathrm{N}^1_{\mathrm{b}}(X)$, where θ_Y is the pullback of θ to Y.

Definition 9. *The cone of* nef b-divisor classes

$$\mathrm{Nef}_{\mathrm{b}}(X) \subset \mathrm{N}^1_{\mathrm{b}}(X)$$

is defined as the closed convex cone generated by all classes of nef $\mathbb{R}$-Cartier b-divisors.

Here an $\mathbb{R}$-Cartier b-divisor B is said to be nef if B_Y is nef for some (hence any) determination Y of B.

The following characterization is essentially formal (see [7, Lemma 2.10]).

Lemma 10. *A b-divisor $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is nef iff B_Y is movable for all birational models $Y \to X$. In other words,* $\mathrm{Nef}_{\mathrm{b}}(X) = \varprojlim_Y \mathrm{Mov}(Y)$.

We finally record the following version of the Negativity Lemma (see [7, Proposition 2.12]).

Lemma 11. *If $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is nef, then $B \le \overline{B_Y}$ for any birational model $Y \to X$.*

1.6. *Plurisubharmonic functions*

We use [13, §4] as a reference. Given a $\mathbb{Q}$-line bundle $L \in \operatorname{Pic}(X)_{\mathbb{Q}}$ and a numerical class $\theta \in \mathrm{N}^1(X)$, we denote by

- $\mathcal{H}^{\mathrm{gf}}(L) = \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ the set of *generically finite Fubini–Study* functions for L, i.e. functions $\varphi\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ of the form
$$\varphi = m^{-1} \max_i \{\log|s_i| + \lambda_i\},$$
where $m \in \mathbb{Z}_{>0}$ is sufficiently divisible, (s_i) is a (nonempty) finite set of nonzero sections of mL, and $\lambda_i \in \mathbb{Q}$;
- $\mathcal{H}_{\mathrm{hom}}(L) \subset \mathcal{H}^{\mathrm{gf}}(L)$ the set of *homogeneous Fubini–Study functions*, for which the λ_i can be chosen to be 0;
- $\mathrm{PSH}(\theta)$ the set of θ*-psh* functions $\varphi\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$, $\varphi \not\equiv -\infty$, obtained as limits of decreasing nets (φ_i) of generically finite Fubini–Study functions φ_i for $\mathbb{Q}$-line bundles L_i such that $c_1(L_i) \to \theta$ in $\mathrm{N}^1(X)$. We also write $\mathrm{PSH}(L) := \mathrm{PSH}(c_1(L))$;
- $\mathrm{CPSH}(\theta) \subset \mathrm{PSH}(\theta)$ the subset of continuous θ-psh functions;
- $\mathrm{PSH}_{\mathrm{hom}}(\theta) \subset \mathrm{PSH}(\theta)$ the subset of homogeneous θ-psh functions, that is, functions $\varphi \in \mathrm{PSH}(\theta)$ such that $\varphi(tv) = t\varphi(v)$ for $v \in X^{\mathrm{an}}$ and $t \in \mathbb{R}_{>0}$.

All functions in $\mathrm{PSH}(\theta)$ are finite valued on the set $X^{\mathrm{div}} \subset X^{\mathrm{an}}$ of divisorial valuations, and we endow $\mathrm{PSH}(\theta)$ with the topology of pointwise convergence on X^{div}. For all $\varphi, \psi \in \mathrm{PSH}(\theta)$, we further have
$$\varphi \le \psi \text{ on } X^{\mathrm{div}} \Longleftrightarrow \varphi \le \psi \text{ on } X^{\mathrm{an}}.$$
In particular, the topology of $\mathrm{PSH}(\theta)$ is Hausdorff. The set of θ-psh functions is preserved by the action of $\mathbb{R}_{>0}$ given by $(t,\varphi) \mapsto t \cdot \varphi$, where $(t \cdot \varphi)(v) := t\varphi(t^{-1}v)$.

Lemma 12. *For any $\theta \in \mathrm{N}^1(X)$ we have:*

(i) $\mathrm{PSH}(\theta) \neq \emptyset \Rightarrow \theta \in \operatorname{Psef}(X)$*;*
(ii) $0 \in \mathrm{PSH}(\theta) \Leftrightarrow \theta \in \operatorname{Nef}(X)$*;*
(iii) $\theta \in \operatorname{Big}(X) \Rightarrow \mathrm{PSH}(\theta) \neq \emptyset$*.*

As we shall see in Proposition 27, (i) is in fact an equivalence, rendering (iii) redundant.

Proof. For (i) and (ii) see [13, (4.1), (4.3)]. If θ is big, we find a big $\mathbb{Q}$-line bundle L such that $\theta - c_1(L)$ is nef. Then $\mathrm{PSH}(\theta) \supset \mathrm{PSH}(L) \supset \mathcal{H}^{\mathrm{gf}}(L) \neq \emptyset$, which proves (iii). □

Example 13. For any effective $\mathbb{R}$-divisor D, $\psi_D := \psi_{\overline{D}}$ (see Lemma 5) satisfies $-\psi_D \in \mathrm{PSH}_{\mathrm{hom}}([D])$.

Our assumption that X is smooth and k is of characteristic zero implies that the *envelope property* holds for any class $\theta \in \mathrm{N}^1(X)$, see [16, Theorem A]. This means that if $(\varphi_\alpha)_\alpha$ is any family in $\mathrm{PSH}(\theta)$ that is uniformly bounded above, and $\varphi := \sup_\alpha \varphi_\alpha$, then the usc regularization $\varphi^\star$, given by $\varphi^\star(x) = \limsup_{y \to x} \varphi(y)$, is θ-psh.

The envelope property has many favorable consequences, as discussed in [13, §5]. For example, for any birational model $\pi\colon Y \to X$ and any $\theta \in \mathrm{N}^1(X)$ we have
$$\mathrm{PSH}(\pi^\star\theta) = \pi^\star \mathrm{PSH}(\theta), \tag{2}$$
see [13, Lemma 5.13].

1.7. *The homogeneous decomposition of a psh function*

We refer to [13, §6.3] for details on what follows. Fix $\theta \in \mathrm{N}^1(X)$. For any $\varphi \in \mathrm{PSH}(\theta)$ and $\lambda \le \sup\varphi$, setting

$$\widehat{\varphi}^\lambda := \inf_{t>0}\{t\cdot\varphi - t\lambda\} \tag{3}$$

defines a homogeneous θ-psh function $\widehat{\varphi}^\lambda \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$. The family $(\widehat{\varphi}^\lambda)_{\lambda\le\sup\varphi}$ is further concave, decreasing, and continuous for the topology of $\mathrm{PSH}_{\mathrm{hom}}(\theta)$ (i.e. pointwise convergence on X^{div}), and it gives rise to the *homogeneous decomposition*

$$\varphi = \sup_{\lambda\le\sup\varphi}\{\widehat{\varphi}^\lambda + \lambda\}. \tag{4}$$

For $\lambda = \sup\varphi = \varphi(v_{\mathrm{triv}})$, the function $\widehat{\varphi}^{\max} := \widehat{\varphi}^{\sup\varphi}$ computes the directional derivatives of φ at v_{triv}, i.e.

$$\widehat{\varphi}^{\max}(v) = \lim_{t\to 0_+}\frac{\varphi(tv)-\varphi(v_{\mathrm{triv}})}{t} \tag{5}$$

for $v \in X^{\mathrm{an}}$. The limit exists as the function $t \mapsto \varphi(tv)$ on $(0,\infty)$ is convex and decreasing, see [13, Proposition 4.12].

Example 14. Assume $\varphi = \varphi_{\mathfrak{a}}$ for a flag ideal $\mathfrak{a} = \sum_{\lambda\in\mathbb{Z}}\mathfrak{a}_\lambda\varpi^{-\lambda}$ on $X\times\mathbb{A}^1$. Then $\widehat{\varphi}^{\max} = \log|\mathfrak{a}_{\lambda_{\max}}|$ where $\lambda_{\max} := \max\{\lambda\in\mathbb{Z} \mid \mathfrak{a}_\lambda \ne 0\}$ (see [13, Example 6.28]).

2. Psh functions and families of b-divisors

We work with a fixed numerical class $\theta \in \mathrm{N}^1(X)$.

2.1. *Homogeneous psh functions and b-divisors*

Recall that a function $\psi \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is uniquely determined by its values on X^{div}. We say that ψ is of divisorial type if its restriction to X^{div} is of divisorial type, that is, $\psi(\mathrm{ord}_E) = 0$ for all but finitely many prime divisors $E \subset X$.

Slightly generalizing [13, Theorem 6.40], we show:

Proposition 15. *The map $B \mapsto \psi_B$ in Section 1.4 sets up a 1–1 correspondence between:*

(i) *the set of b-divisors $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ such that $B \le 0$ and $\overline{\theta} + [B] \in \mathrm{N}^1_{\mathrm{b}}(X)$ is nef;*

(ii) *the set of θ-psh homogeneous functions $\psi \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ of divisorial type.*

Proof. Pick B as in (i). On the one hand, $\psi_{\overline{B_X}} \in \mathrm{PSH}_{\mathrm{hom}}(-B_X)$, see Example 13. On the other hand, since $\overline{\theta} + [B] = \overline{(\theta + [B_X])} + ([B] - \overline{[B_X]})$ is nef, it follows from [13, Theorem 6.40] that $\psi_{B-\overline{B_X}} = \psi_B - \psi_{\overline{B_X}}$ lies in $\mathrm{PSH}_{\mathrm{hom}}(\theta + B_X)$. Thus

$$\psi_B \in \mathrm{PSH}(\theta + B_X) + \mathrm{PSH}(-B_X) \subset \mathrm{PSH}(\theta).$$

Conversely, pick ψ as in (ii), so that $\psi = \psi_B$ with $0 \ge B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$. By [13, Corollary 6.17], we can write ψ as the pointwise limit of a decreasing net (ψ_i) such that $\psi_i \in \mathscr{H}_{\mathrm{hom}}(L_i)$ with $L_i \in \mathrm{Pic}(X)_{\mathbb{Q}}$ and $\lim_i c_1(L_i) = \theta$. Then $\psi_i = \psi_{B_i}$ for a unique Cartier b-divisor $0 \ge B_i \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}}$ such that $\overline{L_i} + B_i$ is semiample (see [13, Lemma 6.34]), and hence $\overline{c_1(L_i)} + [B_i] \in \mathrm{N}^1_{\mathrm{b}}(X)$ is nef. Further, $B_i \searrow B$ in $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$, and hence $[B_i] \to [B]$ in $\mathrm{N}^1_{\mathrm{b}}(X)$ (see Lemma 8). Since $\overline{c_1(L_i)} + [B_i]$ is nef for all i, we conclude, as desired, that $\overline{\theta} + [B]$ is nef. □

2.2. *Rees valuations*

In order to formulate a version of Proposition 15 for general θ-psh functions, the following notion will be useful.

Definition 16. *Given any effective $\mathbb{R}$-divisor D on X with irreducible decomposition $D = \sum_\alpha c_\alpha E_\alpha$ on X, we denote by $\Gamma_D \subset X^{\mathrm{div}}_{\mathbb{R}}$ the set of* Rees valuations *of D, defined as the real divisorial valuations $v_\alpha := c_\alpha^{-1} \operatorname{ord}_{E_\alpha}$.*

Note that $v_\alpha(D) = 1$ for all α. We can now state a variant of [13, Theorem 6.21]:

Proposition 17. *Pick $\psi \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$, and an effective $\mathbb{R}$-divisor D on X. Then*

$$\max_{\Gamma_D} \psi \le -1 \Longleftrightarrow \psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D).$$

Recall that $0 \ge -\psi_D \in \mathrm{PSH}_{\mathrm{hom}}([D])$.

Proof. If $\psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D)$, then $\psi \le -\psi_D$, and hence $\max_\Gamma \psi \le -1$, since $\psi_D \equiv 1$ on Γ_D. Conversely, assume $\max_{\Gamma_D} \psi \le -1$. Consider first the case where $\theta = c_1(L)$ for a $\mathbb{Q}$-line bundle and $\psi \in \mathscr{H}_{\mathrm{hom}}(L)$. For any m sufficiently divisible we thus have $\psi = \frac{1}{m} \max_i \log|s_i|$ for a finite set of nonzero section $s_i \in \mathrm{H}^0(X, mL)$. Using the notation of Definition 16, we get for all i and all α

$$c_\alpha^{-1} \operatorname{ord}_{E_\alpha}(s_i) = -\log|s_i|(v_\alpha) \ge m,$$

and hence $\operatorname{ord}_{E_\alpha}(s_i) \ge \lceil m c_\alpha \rceil$. This implies $s_i = s_i' s_{D_m}$ for some $s_i' \in \mathrm{H}^0(X, m(L - D_m))$, where

$$D_m := m^{-1} \lceil mD \rceil = \sum_\alpha m^{-1} \lceil m c_\alpha \rceil E_\alpha$$

and $s_{D_m} \in \mathrm{H}^0(X, D_m)$ is the canonical section. Since $\psi_{D_m} = -\log|s_{D_m}|$, we infer

$$\psi + \psi_{D_m} = \frac{1}{m} \max_i \log|s_i'| \in \mathscr{H}_{\mathrm{hom}}(L - D_m) \subset \mathrm{PSH}_{\mathrm{hom}}(L - D_m).$$

When $m \to \infty$, ψ_{D_m} decreases to ψ_D, and $[D_m] \to [D]$ in $\mathrm{N}^1(X)$, and we infer $\psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(L - D)$.

In the general case, ψ can be written as the pointwise limit of a decreasing net $\psi_i \in \mathscr{H}_{\mathrm{hom}}(L_i)$, where $L_i \in \mathrm{Pic}(X)_{\mathbb{Q}}$ satisfies that $c_1(L_i) - \theta$ is nef and tends to 0 (see [13, Corollary 6.17]). Pick $t \in (0,1)$. For all i large enough and all α, we then have $c_\alpha^{-1} \psi_i(\operatorname{ord}_{E_\alpha}) \le -t$, and hence

$$\psi_i + t\psi_D \in \mathscr{H}_{\mathrm{hom}}(L_i - tD) \subset \mathrm{PSH}_{\mathrm{hom}}(L_i - tD)$$

by the previous step of the proof. Since $\psi_i + t\psi_D$ decreases to $\psi + t\psi_D$ and $L_i - tD \to \theta - tD$ in $\mathrm{N}^1(X)$, we infer $\psi + t\psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - tD)$ (see [13, Theorem 4.5]). Pick any $\omega \in \mathrm{Amp}(X)$. Then $\psi + t\psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D + \omega)$ for all $t \in (0,1)$ close to 1, so by the envelope property (see [13, Theorem 5.11]), we get $\psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D + \omega)$. As this is true for all $\omega \in \mathrm{Amp}(X)$, we conclude $\psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D)$ (again see [13, Theorem 4.5]). □

2.3. *Psh functions and families of b-divisors*

We now extend Proposition 15 to general θ-psh functions. We say that $\varphi \in \mathrm{PSH}(\theta)$ is of divisorial type if the homogeneous psh function $\widehat{\varphi}^{\max} \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is of divisorial type, see Section 1.7. By (5), this is equivalent to $\varphi(\operatorname{ord}_E) = \sup \varphi$ for all but finitely many prime divisors $E \subset X$.

Theorem 18. *There is a 1–1 correspondence between:*

(i) *the set of θ-psh functions $\varphi \in \mathrm{PSH}(\theta)$ of divisorial type;*
(ii) *the set of continuous, concave, decreasing families $(B_\lambda)_{\lambda \le \lambda_{\max}}$ of b-divisors, for some $\lambda_{\max} \in \mathbb{R}$, such that $B_\lambda \le 0$ and $\overline{\theta} + [B_\lambda] \in \mathrm{N}^1_{\mathrm{b}}(X)$ is nef for all $\lambda \le \lambda_{\max}$.*

The correspondence is given by

$$\varphi = \sup_{\lambda\le\lambda_{\max}} \{\psi_{B_\lambda} + \lambda\}, \quad \psi_{B_\lambda} = \widehat{\varphi}^{\lambda}. \tag{6}$$

In particular, we have $\lambda_{\max} = \sup\varphi$ *and* $\widehat{\varphi}^{\max} = \psi_{B_{\lambda_{\max}}}$.

Proof. Pick a family $(B_\lambda)_{\lambda\le\lambda_{\max}}$ as in (ii). By Proposition 15, setting $\psi_\lambda := \psi_{B_\lambda}$ defines a continuous, concave and decreasing family $(\psi_\lambda)_{\lambda\le\lambda_{\max}}$ in $\mathrm{PSH}_{\mathrm{hom}}(\theta)$. Since θ has the envelope property, the usc upper envelope $\varphi := \sup^{\star}_{\lambda\le\lambda_{\max}}(\psi_\lambda + \lambda)$ lies in $\mathrm{PSH}(\theta)$. On X^{div}, φ coincides with $\sup_{\lambda\le\lambda_{\max}}(\psi_\lambda + \lambda)$ (see [13, Theorem 5.6]). By Legendre duality, we further have $\psi_\lambda = \widehat{\varphi}^{\lambda}$ for $\lambda < \lambda_{\max}$ (see [13, Theorem 6.24]), and hence also for $\lambda = \lambda_{\max}$, by continuity of both sides on $(-\infty, \lambda_{\max}]$.

Conversely, pick φ as in (i), so that $\widehat{\varphi}^{\max} \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is of divisorial type. For each $\lambda \le \sup\varphi$ we then have $0 \ge \widehat{\varphi}^{\lambda} \ge \widehat{\varphi}^{\max}$, which shows that $\widehat{\varphi}^{\lambda} \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is also of divisorial type. By Proposition 15, we thus have $\widehat{\varphi}^{\lambda} = \psi_{B_\lambda}$ for a b-divisor $B_\lambda \le 0$ such that $\overline{\theta} + [B_\lambda]$ is nef, and the family $(B_\lambda)_{\lambda\le\sup\varphi}$ is concave, decreasing and continuous, since so is $(\widehat{\varphi}^{\lambda})_{\lambda\le\sup\varphi}$. ☐

Remark 19. Not every θ-psh function is of divisorial type. For example, assume $\dim X = 1$, and pick a sequence $(p_j)_{j\in\mathbb{N}}$ of closed points on X, with corresponding ideals $\mathfrak{m}_j \subset \mathcal{O}_X$, and a sequence ε_j in $\mathbb{R}_{>0}$ such that $\sum_j \varepsilon_j \le \deg\theta$. Then $\varphi := \sum_j \varepsilon_j \log|\mathfrak{m}_j| \in \mathrm{PSH}(\theta)$, and $-\varepsilon_j = \varphi(\mathrm{ord}_{p_j}) < \sup\varphi = 0$ for all j (see [13, Example 4.13]).

3. The center of a θ-psh function

In this section we introduce the notion of the center of a θ-psh function. This is a subset of X defined in terms of the locus on X^{an} where φ is smaller than its maximum.

3.1. *The center map*

For any $v \in X^{\mathrm{an}}$, we denote by $c_X(v) \in X$ its center, and by

$$Z_X(v) := \overline{\{c_X(v)\}} \subset X$$

the corresponding subvariety. The center map $c_X\colon X^{\mathrm{an}} \to X$ is surjective and anticontinuous, i.e. the preimage of a closed subset is open. In particular, any subvariety $Z \subset X$ is of the form $Z = Z_X(v)$ for some v; we can simply take $v = \mathrm{ord}_Z$.

More generally, for any subset $S \subset X^{\mathrm{an}}$ we set

$$Z_X(S) := \bigcup_{v\in S} Z_X(v). \tag{7}$$

This is smallest subset of X that contains $c_X(S)$ and is closed under specialization.

3.2. *The center of a θ-psh function*

We can now introduce

Definition 20. *We define the* center *on X of any θ-psh function $\varphi \in \mathrm{PSH}(\theta)$ as*

$$Z_X(\varphi) := Z_X(\{\varphi < \sup\varphi\}) \subset X.$$

Example 21. For any nonzero ideal $\mathfrak{b} \subset \mathcal{O}_X$, the function $\psi = \log|\mathfrak{b}|$ is θ-psh if θ is sufficiently ample, and then $Z_X(\varphi) = V(\mathfrak{b})$. More generally, if $\varphi = \sum_i t_i \log|\mathfrak{b}_i|$ with $t_i \in \mathbb{R}_{>0}$ and $\mathfrak{b}_i \subset \mathcal{O}_X$ a nonzero ideal, then $Z_X(\varphi) = \bigcup_i V(\mathfrak{b}_i)$.

Recall that to any θ-psh function $\varphi \in \mathrm{PSH}(\theta)$ we can associate a homogeneous θ-psh function $\widehat{\varphi}^{\max} \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$, see Section 1.7.

Lemma 22. *For any $\varphi \in \mathrm{PSH}(\theta)$ we have $\{\varphi < \sup\varphi\} = \{\widehat{\varphi}^{\max} < 0\}$. As a consequence, $Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max})$. Moreover, the following conditions are equivalent:*

(i) *φ is of divisorial type;*
(ii) *$\widehat{\varphi}^{\max}$ is of divisorial type;*
(iii) *$Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max})$ contains at most finitely many prime divisors $E \subset X$.*

Proof. Pick any $v \in X^{\mathrm{an}}$. By (5) and the fact that $t \mapsto \varphi(tv)$ is decreasing and convex, it follows that $\varphi(v) < \sup\varphi$ iff $\widehat{\varphi}^{\max}(v) < 0$. Thus $Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max})$ since $\sup\widehat{\varphi}^{\max} = 0$.

Now the equivalence (i) $\Leftrightarrow$ (ii) is definitional, and (ii) $\Leftrightarrow$ (iii) is clear since a prime divisor $E \subset X$ belongs to $Z_X(\widehat{\varphi}^{\max})$ iff $\widehat{\varphi}^{\max}(\mathrm{ord}_E) < 0$. □

Together with Example 14, Lemma 22 implies

Corollary 23. *If $\varphi = \varphi_{\mathfrak{a}}$ for a flag ideal $\mathfrak{a} = \sum_{\lambda\in\mathbb{Z}} \mathfrak{a}_\lambda \varpi^{-\lambda}$ on $X \times \mathbb{A}^1$, then $Z_X(\varphi_{\mathfrak{a}}) = V(\mathfrak{a}_{\lambda_{\max}})$, where* $\lambda_{\max} := \max\{\lambda \in \mathbb{Z} \mid \mathfrak{a}_\lambda \neq 0\}$.

Theorem 24. *For any $\varphi \in \mathrm{PSH}(\theta)$, the center $Z_X(\varphi)$ is a strict subset of X, and an at most countable union of (strict) subvarieties. Moreover, we have $c_X^{-1}(Z_X(\varphi)) = \{\varphi < \sup\varphi\}$.*

Proof. Note that $Z_X(\varphi)$ does not contain the generic point of X, so $Z_X(\varphi) \neq X$. Also note that by Lemma 22 we may assume that φ is homogeneous.

If $\varphi \in \mathcal{H}_{\mathrm{hom}}(L)$ for a $\mathbb{Q}$-line bundle L, so that $\varphi = \frac{1}{m}\max_i \log|s_i|$ for a finite set of nonzero sections $s_i \in \mathrm{H}^0(X, mL)$, then $Z_X(\varphi) = \bigcap_i (s_i = 0)$, which is Zariski closed. In general, φ can be written as the limit of a decreasing sequence $\varphi_m \in \mathcal{H}_{\mathrm{hom}}(L_m)$ with $L_m \in \mathrm{Pic}(X)_{\mathbb{Q}}$ such that $c_1(L_m) \to \theta$ (see [13, Remark 6.18]). For any $v \in X^{\mathrm{div}}$ we then have

$$c_X(v) \in Z_X(\varphi) \Longleftrightarrow \varphi(v) < 0 \Longleftrightarrow \varphi_m(v) < 0 \text{ for some } m,$$

i.e. $Z_X(\varphi) = \bigcup_m Z_X(\varphi_m)$, an at most countable union of strict subvarieties.

Next pick $v \in X^{\mathrm{an}}$, and set $Z = Z_X(v)$. By [13, Proposition 4.12], $\varphi(tv) = t\varphi(v)$ converges to $\varphi(v_{Z,\mathrm{triv}}) = \sup_{Z^{\mathrm{an}}}\varphi$ as $t \to +\infty$, and hence $\varphi(v) < 0 \Leftrightarrow \varphi \equiv -\infty$ on Z^{an}. By definition of the center, if $c_X(v)$ lies in $Z_X(\varphi)$, then we can find $w \in X^{\mathrm{an}}$ such that $\varphi(w) < 0$ and $c_X(v) \in Z_X(w)$, i.e. $Z \subset Z_X(w)$. Then $\varphi \equiv -\infty$ on $Z_X(w)^{\mathrm{an}} \supset Z^{\mathrm{an}}$, which yields $\varphi(v) < 0$. Conversely, assume $\varphi(v) < 0$, and hence $\varphi \equiv -\infty$ on Z^{an}. We can find $w \in X^{\mathrm{div}}$ such that $Z = Z_X(w)$. Since $\varphi \equiv -\infty$ on $Z^{\mathrm{an}} = Z_X(w)^{\mathrm{an}}$, we get $\varphi(w) < 0$, and hence $c_X(v) \in Z_X(w) \subset Z_X(\varphi)$. □

For later use we record

Lemma 25. *If $\varphi_i \in \mathrm{PSH}(\theta_i)$, $i = 1, 2$, then $Z_X(\varphi_1 + \varphi_2) = Z_X(\varphi_1) \cup Z_X(\varphi_2)$.*

3.3. *Centers of PL functions*

The following result will play a crucial role in what follows.

Lemma 26. *If $\varphi \in \mathrm{PSH}(\theta)$ lies in $\mathbb{R}\mathrm{PL}^+(X)$ (resp. $\mathbb{R}\mathrm{PL}(X)$), then $Z_X(\varphi)$ is Zariski closed (resp. not Zariski dense) in X.*

Proof. Assume first $\varphi \in \mathbb{R}\mathrm{PL}^+(X)$, and write $\varphi = \max_i\{\psi_i + \lambda_i\}$ for a finite set $\psi_i \in \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$ and $\lambda_i \in \mathbb{R}$. As in Example 14, we then have $\max_i \lambda_i = \sup\varphi$, and $\widehat{\varphi}^{\max} = \max_{\lambda_i = \sup\varphi} \psi_i$. This shows that

$$Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max}) = \bigcap_{\lambda_i = \sup\varphi} Z_X(\psi_i)$$

is Zariski closed (see Example 21). Assume next $\varphi \in \mathbb{R}\mathrm{PL}(X)$ and write $\varphi = \varphi_1 - \varphi_2$ with $\varphi_1, \varphi_2 \in \mathbb{R}\mathrm{PL}^+(X)$. After replacing θ with a sufficiently ample class, we may assume that φ_1, φ_2 are θ-psh. By (5) we have $\widehat{\varphi}^{\max} = \widehat{\varphi}_1^{\max} - \widehat{\varphi}_2^{\max}$, and hence

$$Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max}) \subset Z_X(\widehat{\varphi}_1^{\max}) \cup Z_X(\widehat{\varphi}_2^{\max}) = Z_X(\varphi_1) \cup Z_X(\varphi_2),$$

which cannot be Zariski dense, since $Z_X(\varphi_1)$ and $Z_X(\varphi_2)$ are both Zariski closed strict subsets by the first part of the proof. □

4. Extremal functions and minimal vanishing orders

Next we define a trivially valued analogue of an important construction in the complex analytic case.

4.1. *Extremal functions*

For any $\theta \in \mathrm{N}^1(X)$, we define the *extremal function* $V_\theta \colon X^{\mathrm{an}} \to [-\infty, 0]$ as the pointwise envelope

$$V_\theta := \sup\left\{\varphi \in \mathrm{PSH}(\theta) \mid \varphi \le 0\right\}. \tag{8}$$

Proposition 27. *For any* $\theta \in \mathrm{N}^1(X)$ *we have*

$$\theta \in \mathrm{Psef}(X) \implies V_\theta \in \mathrm{PSH}_{\mathrm{hom}}(\theta);$$
$$\theta \notin \mathrm{Psef}(X) \implies V_\theta \equiv -\infty;$$
$$\theta \in \mathrm{Nef}(X) \iff V_\theta \equiv 0.$$

In particular, $\mathrm{PSH}(\theta)$ *is nonempty iff* θ *is pseudoeffective. For any* $\omega \in \mathrm{Amp}(X)$, *we further have*

$$V_{\theta+\varepsilon\omega} \searrow V_\theta \text{ as } \varepsilon \searrow 0. \tag{9}$$

Proof. Since the action $(t, \varphi) \mapsto t \cdot \varphi$ of $\mathbb{R}_{>0}$ preserves the set of candidate functions φ in (8), V_θ is necessarily fixed by the action, and hence homogeneous. If θ is not psef, then $\mathrm{PSH}(\theta)$ is empty (see Lemma 12), and hence $V_\theta \equiv -\infty$. By Lemma 12, we also have $V_\theta \equiv 0$ iff θ is nef.

Next, assume $\theta \in \mathrm{Big}(X)$. Then $\mathrm{PSH}(\theta)$ is non-empty (see Lemma 12), and the envelope property implies that $V_\theta^\star$ is θ-psh and nonpositive. It is thus a candidate in (8), and hence $V_\theta^\star \le V_\theta$, which shows that $V_\theta^\star = V_\theta$ is θ-psh.

Assume now $\theta \in \mathrm{Psef}(X)$, and pick $\omega \in \mathrm{Amp}(X)$. For each $\varepsilon > 0$, the previous step yields $V_\varepsilon := V_{\theta+\varepsilon\omega} \in \mathrm{PSH}_{\mathrm{hom}}(\theta + \varepsilon\omega)$. For $0 < \varepsilon < \delta$ we further have $\mathrm{PSH}(\theta) \subset \mathrm{PSH}(\theta + \varepsilon\omega) \subset \mathrm{PSH}(\theta + \delta\omega)$, and hence $V_\delta \ge V_\varepsilon \ge V_\theta$. Set $V := \lim_\varepsilon V_\varepsilon$. For any $\delta > 0$ fixed, we have $V_\varepsilon \in \mathrm{PSH}_{\mathrm{hom}}(\theta + \delta\omega)$ for $\varepsilon \le \delta$, and $V_\varepsilon \searrow V$ as $\varepsilon \to 0$. Thus $V \in \mathrm{PSH}_{\mathrm{hom}}(\theta + \delta\omega)$ for all $\delta > 0$, and hence $V \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$. Since V is a candidate in (8), we get $V \le V_\theta$, and hence $V_\theta = V = \lim_\varepsilon V_\varepsilon$. This proves that V_θ is θ-psh, as well as (9). □

4.2. *Minimal vanishing orders*

For $\theta \in \mathrm{Psef}(X)$, the function $V_\theta \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is uniquely determined by its restriction to X^{div}, where it is furthermore finite valued. For any $v \in X^{\mathrm{div}}$ we set

$$v(\theta) := -V_\theta(v) = \inf\{-\varphi(v) \mid \varphi \in \mathrm{PSH}(\theta), \varphi \le 0\} \in \mathbb{R}_{\ge 0}. \tag{10}$$

Note that

$$v(\theta) = \sup_{\varepsilon > 0} v(\theta + \varepsilon\omega) \tag{11}$$

for any $\omega \in \mathrm{Amp}(X)$, by (9). As we next show, these invariants coincide with the minimal/asymptotic vanishing orders studied in [6, 22, 40].

Proposition 28. *Pick $v \in X^{\mathrm{div}}$. Then:*

(i) *the function $\theta \mapsto v(\theta)$ is homogeneous, convex and lsc on* $\mathrm{Psef}(X)$*; in particular, it is continuous on* $\mathrm{Big}(X)$*;*

(ii) *for any $\theta \in \mathrm{Psef}(X)$ we have*

$$v(\theta) \le \inf\left\{v(D) \mid D \equiv \theta \text{ effective } \mathbb{R}\text{-divisor}\right\}, \tag{12}$$

and equality holds when θ is big.

Note that equality in (12) fails in general for θ is not big, as there might not even exist any effective $\mathbb{R}$-divisor D in the class of θ.

Proof. Using $\mathrm{PSH}(\theta) + \mathrm{PSH}(\theta') \subset \mathrm{PSH}(\theta + \theta')$ and $\mathrm{PSH}(t\theta) = t\,\mathrm{PSH}(\theta)$ for $\theta, \theta' \in \mathrm{Psef}(X)$ and $t > 0$, it is straightforward to see that $\theta \mapsto v(\theta)$ is convex and homogeneous on $\mathrm{Psef}(X)$. Being also finite valued, it is automatically continuous on the interior $\mathrm{Big}(X)$. For any $\omega \in \mathrm{Amp}(X)$ and $\varepsilon > 0$, $\theta \mapsto v(\theta + \varepsilon\omega)$ is thus continuous on $\mathrm{Psef}(X)$, and (11) thus shows that $\theta \mapsto v(\theta)$ is lsc, which proves (i).

Next pick $\theta \in \mathrm{Psef}(X)$. For each effective $\mathbb{R}$-divisor $D \equiv \theta$, the function $-\psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$, see Example 13, is a competitor in (8). Thus $-v(D) = \psi_D(v) \le V_\theta(v) = -v(\theta)$, which proves the first half of (ii). Now assume θ is big, and denote by $v'(\theta)$ the right-hand side of (12). Both $v(\theta)$ and $v'(\theta)$ are (finite valued) convex function of $\theta \in \mathrm{Big}(X)$. They are therefore continuous, and it is thus enough to prove the equality $v(\theta) = v'(\theta)$ when $\theta = c_1(L)$ with $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ big. To this end, pick an ample $\mathbb{Q}$-line bundle A, and set $\omega := c_1(A)$. By [13, Theorem 4.15], for any $\varepsilon > 0$ we can find $\varphi \in \mathcal{H}^{\mathrm{gf}}(L + A)$ such that $\varphi \ge V_\theta$ and $\varphi(v_{\mathrm{triv}}) = \sup\varphi \le \varepsilon$. By definition, we have $\varphi = m^{-1}\max_i\{\log|s_i| + \lambda_i\}$ with m sufficiently divisible and a finite family of nonzero sections $s_i \in \mathrm{H}^0(X, m(L + A))$ and constants $\lambda_i \in \mathbb{Q}$. Then $\max_i \lambda_i = m\sup\varphi \le m\varepsilon$, and $m^{-1}v(s_i) = v(D_i)$ with $D_i := m^{-1}\mathrm{div}(s_i) \equiv \theta + \omega$, and hence $m^{-1}v(s_i) \ge v'(\theta + \omega)$. Thus

$$-v(\theta) = V_\theta(v) \le \varphi(v) = m^{-1}\max_i\{v(s_i) + \lambda_i\} \le -v'(\theta + \omega) + \varepsilon.$$

This shows $v'(\theta) \ge v(\theta) \ge v'(\theta + \omega)$, and hence $v'(\theta) = v(\theta)$, since $\lim_{\omega\to 0} v'(\theta + \omega) = v'(\theta)$ by continuity on the big cone. □

Remark 29. If $L \in \mathrm{Pic}(X)$ is big, then [22, Corollary 2.7] (or, alternatively, a small variant of the above argument) shows that $v(c_1(L))$ is also equal to the asymptotic vanishing order

$$\begin{aligned} v(\|L\|) &:= \lim_{m\to\infty} \tfrac{1}{m}\min\left\{v(s) \mid s \in \mathrm{H}^0(X, mL) \setminus \{0\}\right\} \\ &= \inf\left\{v(D) \mid D \sim_{\mathbb{Q}} L \text{ effective } \mathbb{Q}\text{-divisor}\right\}. \end{aligned}$$

Remark 30. Continuity of minimal vanishing orders beyond the big cone is a subtle issue. For any $v \in X^{\mathrm{div}}$, the function $\theta \mapsto v(\theta)$, being convex and lsc on $\mathrm{Psef}(X)$, is automatically continuous on any polyhedral subcone (cf. [27]). When $\dim X = 2$, it is in fact continuous on the whole of $\mathrm{Psef}(X)$, but this fails in general when $\dim X \ge 3$ (see respectively Proposition III.1.19 and Example IV.2.8 in [40]).

4.3. *The center of an extremal function*

The following fact plays a key role in what follows.

Theorem 31. *For any $\theta \in \mathrm{Psef}(X)$, the function $V_\theta \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is of divisorial type (see Definition 4). Further, its center $Z_X(V_\theta)$ coincides with the diminished base locus $\mathbb{B}_-(\theta)$ (see Section 1.1).*

The proof relies on the next result, which corresponds to [40, Corollary III.1.11] (see also [6, Theorem 3.12] in the analytic context).

Lemma 32. *Pick $\theta \in \mathrm{Psef}(X)$, and assume $E_1, \dots, E_r \subset X$ are distinct prime divisors such that* $\mathrm{ord}_{E_i}(\theta) > 0$ *for all i. Then $[E_1], \dots, [E_r]$ are linearly independent in $\mathrm{N}^1(X)$. In particular, $r \le \rho(X) = \dim \mathrm{N}^1(X)$.*

Proof. We reproduce the simple argument of [8, Theorem 3.5 (v)] for the convenience of the reader. By (11), after adding to θ a small enough ample class we assume that θ is big. Suppose $\sum_i c_i[E_i] = 0$ with $c_i \in \mathbb{R}$, so that $G := \sum_i c_i E_i$ is numerically equivalent to 0, and choose $0 < \varepsilon \ll 1$ such that $\mathrm{ord}_{E_i}(\theta) + \varepsilon c_i > 0$ for all i. Pick any effective $\mathbb{R}$-divisor $D \equiv \theta$ and set $D' := D + \varepsilon G$. Then D' is effective, since

$$\mathrm{ord}_{E_i}(D') = \mathrm{ord}_{E_i}(D) + \varepsilon c_i \ge \mathrm{ord}_{E_i}(\theta) + \varepsilon c_i > 0$$

for all i. Since $G \equiv 0$, we also have $D' \equiv \theta$, and (12) thus yields for each i

$$\mathrm{ord}_{E_i}(\theta) \le \mathrm{ord}_{E_i}(D') = \mathrm{ord}_{E_i}(D) + \varepsilon c_i.$$

Taking the infimum over D we get $\mathrm{ord}_{E_i}(\theta) \le \mathrm{ord}_{E_i}(\theta) + \varepsilon c_i$ (see Proposition 28 (ii)), i.e. $c_i \ge 0$ for all i. Thus $G \ge 0$, and hence $G = 0$, since $G \equiv 0$. This proves $c_i = 0$ for all i which shows, as desired, that the $[E_i]$ are linearly independent. □

Proof of Theorem 31. By (10), the first assertion means that there are only finitely many prime divisors $E \subset X$ such that $\mathrm{ord}_E(\theta) > 0$, and is thus a direct consequence of Lemma 32. Pick $v \in X^{\mathrm{div}}$. The second point is equivalent to $v(\theta) > 0 \Leftrightarrow c_X(v) \in \mathbb{B}_-(\theta)$. When θ is big, this is the content of [22, Theorem B]. In the general case, pick $\omega \in \mathrm{Amp}(X)$. Then $v(\theta) > 0$ iff $v(\theta + \varepsilon\omega) > 0$ for $0 < \varepsilon \ll 1$, by (11), while $c_X(v) \in \mathbb{B}_-(\theta)$ iff $c_X(v) \in \mathbb{B}_-(\theta + \varepsilon\omega)$ for $0 < \varepsilon \ll 1$, by (1). The result follows. □

For later use, we also note:

Lemma 33. *For any polyhedral subcone $C \subset \mathrm{Psef}(X)$, we have:*

(i) *$\theta \mapsto v(\theta)$ is continuous on C for all $v \in X^{\mathrm{div}}$;*

(ii) *the set of prime divisors $E \subset X$ such that $\mathrm{ord}_E(\theta) > 0$ for some $\theta \in C$ is finite.*

Proof. As mentioned in Remark 30, any convex, lsc function on a polyhedral cone is continuous (see [27]), and (i) follows. To see (ii), pick a finite set of generators (θ_i) of C. Each $\theta \in C$ can be written as $\theta = \sum_i t_i\theta_i$ with $t_i \ge 0$. By convexity and homogeneity of minimal vanishing orders, this implies $\mathrm{ord}_E(\theta) \le \sum_i t_i \,\mathrm{ord}_E(\theta_i)$, so that $\mathrm{ord}_E(\theta) > 0$ implies $\mathrm{ord}_E(\theta_i) > 0$ for some i. The result now follows from Lemma 32. □

5. Zariski decompositions

Next we study the close relationship between the extremal function in Section 4, and the various versions of the Zariski decomposition of a psef numerical class.

5.1. *The b-divisorial Zariski decomposition*

Pick $\theta \in \mathrm{N}^1(X)$ a psef class. By Theorem 31, the function $X^{\mathrm{div}} \ni v \mapsto v(\theta) = -V_\theta(v)$ is of divisorial type. We denote by

$$\mathrm{N}(\theta) \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$$

the corresponding effective b-divisor, which thus satisfies

$$\psi_{\mathrm{N}(\theta)}(v) = v(\mathrm{N}(\theta)) = v(\theta) = -V_\theta(v)$$

for all $v \in X^{\mathrm{div}}$.

Theorem 34. *For any $\theta \in \mathrm{Psef}(X)$, the b-divisor class*

$$\mathrm{P}(\theta) := \overline{\theta} - [\mathrm{N}(\theta)] \in \mathrm{N}^1_{\mathrm{b}}(X)$$

is nef, and $\mathrm{N}(\theta)$ is the smallest effective b-divisor with this property. Moreover,

$$\mathrm{N}(\theta) \ge \overline{\mathrm{N}(\theta)_Y} \tag{13}$$

for all birational models $Y \to X$.

We call $\overline{\theta} = \mathrm{P}(\theta) + [\mathrm{N}(\theta)]$ the *b-divisorial Zariski decomposition* of θ. At least when θ is big, this construction is basically equivalent to [33, Theorem D], and to the case $p = 1$ of [9, §2.2].

Note that the b-divisorial Zariski decomposition is birationally invariant:

Lemma 35. *For any $\theta \in \mathrm{Psef}(X)$ and any birational model $\pi\colon Y \to X$, we have*

$$\mathrm{N}(\pi^\star\theta) = \mathrm{N}(\theta) \quad \text{and} \quad \mathrm{P}(\pi^\star\theta) = \mathrm{P}(\theta)$$

in $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}} = \mathrm{Z}^1_{\mathrm{b}}(Y)_{\mathbb{R}}$ and $\mathrm{N}^1_{\mathrm{b}}(X)_{\mathbb{R}} = \mathrm{N}^1_{\mathrm{b}}(Y)_{\mathbb{R}}$, respectively.

Proof. Since $\mathrm{PSH}(\pi^\star\theta) = \pi^\star\,\mathrm{PSH}(\theta)$, see (2), we have $V_{\pi^\star\theta} = \pi^\star V_\theta$, and the result follows. □

Proof of Theorem 34. Since $\psi_{-\mathrm{N}(\theta)} = V_\theta$ is θ-psh, Proposition 15 shows that $\overline{\theta} - [\mathrm{N}(\theta)]$ is nef, which yields the last point, by the Negativity Lemma (see Lemma 11). Conversely, if $E \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is effective with $\overline{\theta} - [E]$ nef, then $-\psi_E \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$, again by Proposition 15. Thus $-\psi_E \le V_\theta = -\psi_{\mathrm{N}(\theta)}$, and hence $E \ge \mathrm{N}(\theta)$. □

As a consequence of Proposition 28, we get

Corollary 36. *The map $\mathrm{Psef}(X) \ni \theta \mapsto \mathrm{N}(\theta) \in \mathrm{Z}^1_{\mathrm{b}}(X)$ is homogeneous, lsc, and convex.*

5.2. *The divisorial Zariski decomposition*

For any $\theta \in \mathrm{Psef}(X)$, we denote by $\mathrm{N}_X(\theta) := \mathrm{N}(\theta)_X$ the incarnation of $\mathrm{N}(\theta) \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ on X, which thus satisfies

$$\mathrm{N}_X(\theta) = \sum_{E \subset X} \mathrm{ord}_E(\theta) E \tag{14}$$

with E ranging over all prime divisors of X, and $\mathrm{ord}_E(\theta) = 0$ for all but finitely many E.

For any effective $\mathbb{R}$-divisor D on X with numerical class $[D] \in \mathrm{Psef}(X)$, (12) yields

$$\mathrm{N}_X(D) := \mathrm{N}_X([D]) \le D. \tag{15}$$

More generally, the following variational characterization holds.

Theorem 37. *For any $\theta \in \mathrm{Psef}(X)$, the class*

$$\mathrm{P}_X(\theta) := \theta - [\mathrm{N}_X(\theta)] \in \mathrm{N}^1(X)$$

is movable, and $\mathrm{N}_X(\theta)$ is the smallest effective $\mathbb{R}$-divisor on X with this property.

Following [6], we call the decomposition

$$\theta = \mathrm{P}_X(\theta) + [\mathrm{N}_X(\theta)]$$

the *divisorial Zariski decomposition* of θ. It coincides with the *σ-decomposition* of [40].

Proof of Theorem 37. By definition, $\mathrm{P}_X(\theta)$ is the incarnation on X of $\overline{\theta} - [\mathrm{N}(\theta)]$. By Theorem 34, the latter class is nef, and $\mathrm{P}_X(\theta)$ is thus movable, by Lemma 10.

To prove the converse, assume first that θ is movable. We then need to show $\mathrm{N}_X(\theta) = 0$, i.e. $\mathrm{ord}_E(\theta) = 0$ for each $E \subset X$ prime (see (14)). By (12), this is clear if $\theta = c_1(L)$ for a big line bundle L with base locus of codimension at least 2. Since the movable cone $\mathrm{Mov}(X)$ is generated by the classes of such line bundles, the continuity of $\theta \mapsto \mathrm{ord}_E(\theta)$ on the big cone yields the result when θ is further big, and the case of an arbitrary movable class follows by (11).

Finally, consider any $\theta \in \mathrm{Psef}(X)$ and any effective $\mathbb{R}$-divisor D on X such that $\theta - [D]$ is movable. For any $E \subset X$ prime we then have $\mathrm{ord}_E(\theta - [D]) = 0$ by the previous step, and $\mathrm{ord}_E([D]) \le \mathrm{ord}_E(D)$ by (15)). Thus

$$\mathrm{ord}_E(\theta) \le \mathrm{ord}_E(\theta - [D]) + \mathrm{ord}_E(D) = \mathrm{ord}_E(D).$$

This shows $\mathrm{N}_X(\theta) \le D$, which concludes the proof. □

Remark 38. Theorem 37 implies the following converse of Lemma 10: a class $\theta \in \mathrm{N}^1(X)$ is movable iff $\theta = \alpha_X$ for a nef b-divisor class $\alpha \in \mathrm{Nef_b}(X)$.

Corollary 39. *Pick $\theta \in \mathrm{Psef}(X)$ and a prime divisor $E \subset X$. Then $(\theta - \mathrm{ord}_E(\theta)E)|_E \in \mathrm{N}^1(E)$ is pseudoeffective.*

Proof. We have $\theta - \mathrm{ord}_E(\theta)[E] = \mathrm{P}_X(\theta) + \sum_{F \neq E} \mathrm{ord}_F(\theta)[F]$, where F ranges over all prime divisors of X distinct from E. Since $\mathrm{P}_X(\theta)$ is movable, $\mathrm{P}_X(\theta)|_E$ is psef. On the other hand, $[F]|_E$ is psef for any $F \neq E$, and the result follows. □

Lemma 40. *For any $\theta \in \mathrm{Psef}(X)$ and any birational model $\pi \colon Y \to X$, the incarnation of $\mathrm{N}(\theta)$ on Y coincides with $\mathrm{N}_Y(\pi^\star\theta)$. Further, the following are equivalent:*

(i) *the b-divisor $\mathrm{N}(\theta)$ is $\mathbb{R}$-Cartier, and determined on Y;*
(ii) *$\mathrm{P}_Y(\pi^\star\theta)$ is nef.*

Proof. The first point follows from Lemma 35. If (i) holds then the nef b-divisor class $\overline{\theta} - \mathrm{N}(\theta)$ is $\mathbb{R}$-Cartier and determined on Y. Thus $(\overline{\theta} - \mathrm{N}(\theta))_Y = \pi^\star\theta - \mathrm{N}_Y(\pi^\star\theta) = \mathrm{P}_Y(\pi^\star\theta)$ is nef, and hence (i) ⇒ (ii).

Conversely, assume (ii). Then $\overline{\mathrm{N}(\theta)_Y} = \overline{\mathrm{N}_Y(\pi^\star\theta)}$ is an effective b-divisor, and the b-divisor class $\overline{\theta} - [\overline{\mathrm{N}(\theta)_Y}] = \overline{\mathrm{P}_Y(\pi^\star\theta)}$ is nef. By Theorem 34 this implies $\mathrm{N}(\theta) \le \overline{\mathrm{N}(\theta)_Y}$, while $\mathrm{N}(\theta) \ge \overline{\mathrm{N}(\theta)_Y}$ always holds (see (13)). This proves (ii) ⇒ (i). □

Since any movable class on a surface is nef, we get:

Corollary 41. *If $\dim X = 2$ then $\mathrm{N}(\theta) = \overline{\mathrm{N}_X(\theta)}$ for all $\theta \in \mathrm{Psef}(X)$.*

In contrast, see [40, Theorem IV.2.10] for an example of a big line bundle L on a 4-fold X such that the b-divisor $\mathrm{N}(L)$ is not $\mathbb{R}$-Cartier, i.e. $\mathrm{P}_Y(\pi^\star L)$ is not nef for any model $\pi \colon Y \to X$.

5.3. *Zariski exceptional divisors and faces*

This section revisits [6, §3.3].

Definition 42. *We say that:*

(i) *an effective $\mathbb{R}$-divisor D on X is* Zariski exceptional *if $\mathrm{N}_X(D) = D$, or equivalently, $\mathrm{P}_X([D]) = 0$;*
(ii) *a finite family (E_i) of prime divisors $E_i \subset X$ is* Zariski exceptional *if every effective $\mathbb{R}$-divisor supported in the E_i's is Zariski exceptional.*

We also define a Zariski exceptional face F *of* $\mathrm{Psef}(X)$ *as an extremal subcone such that* $\mathrm{P}_X|_F \equiv 0$.

Here a closed subcone $C \subset \mathrm{Psef}(X)$ is extremal iff $\alpha, \beta \in \mathrm{Psef}(X)$, $\alpha + \beta \in C$ implies $\alpha, \beta \in C$.
We first note:

Lemma 43. *An effective $\mathbb{R}$-divisor D on X is Zariski exceptional iff $\mathrm{N}(D) = \overline{D}$.*

Proof. Assume $\mathrm{N}_X(D) = D$. Then $\mathrm{N}(D) \le \overline{D}$, by Theorem 34, and $\mathrm{N}(D) \ge \overline{\mathrm{N}_X(D)} = \overline{D}$ (see (13)). The result follows. □

The above notions are related as follows:

Theorem 44. *The following properties hold:*

(i) *if $E \subset X$ is a prime divisor, then E is either movable (in which case $E|_E$ is psef), or it is Zariski exceptional;*
(ii) *the set of Zariski exceptional families of prime divisors on X is at most countable;*
(iii) *for any $\theta \in \mathrm{Psef}(X)$, the irreducible components of $\mathrm{N}_X(\theta)$ form a Zariski exceptional family; in particular, $\mathrm{N}_X(\theta)$ is Zariski exceptional;*
(iv) *each Zariski exceptional family (E_i) is linearly independent in $\mathrm{N}^1(X)$, and generates a Zariski exceptional face $F := \sum_i \mathbb{R}_{\ge 0}[E_i]$ of $\mathrm{Psef}(X)$;*
(v) *conversely, each Zariski exceptional face F of $\mathrm{Psef}(X)$ arises as in* (iv).

Proof. Assume $E \subset X$ is a prime divisor. Then $\mathrm{N}_X(E) \le E$ (see (15)), and hence $\mathrm{N}_X(E) = cE$ with $c \in [0,1]$. If $c = 1$, then E is Zariski exceptional. Otherwise,

$$E = (1-c)^{-1}(E - \mathrm{N}_X(E)) \equiv (1-c)^{-1}\,\mathrm{P}_X(E)$$

is movable (and $c = 0$). This proves (i).

To see (ii), note that a Zariski exceptional prime divisor satisfies $E = \mathrm{N}_X(E)$, and hence is uniquely determined by its numerical class $[E] \in \mathrm{N}^1(X)_{\mathbb{Q}}$. As a consequence, the set of Zariski exceptional primes is at most countable, and hence so is the set of Zariski exceptional families.

Pick $\theta \in \mathrm{Psef}(X)$. We first claim that $D := \mathrm{N}_X(\theta)$ is Zariski exceptional. Since $\mathrm{P}_X(\theta) = \theta - [D]$ and $\mathrm{P}_X(D) = [D - \mathrm{N}_X(D)]$ are both movable, $\theta - [\mathrm{N}_X(D)]$ is movable as well. Theorem 37 thus yields $\mathrm{N}_X(D) \ge \mathrm{N}_X(\theta) = D$, which proves the claim in view of (15). Denote by $D = \sum_{i=1}^r c_i E_i$ the irreducible decomposition of D, and set $f_i(x) := \mathrm{ord}_{E_i}(\sum_j x_j E_j)$ for $1 \le i \le r$. This defines a convex function $f_i : \mathbb{R}^r_{\ge 0} \to \mathbb{R}_{\ge 0}$ which satisfies $f_i(x) \le x_i$ for all x, by (15). Since equality holds at the interior point $x = c \in \mathbb{R}^r_{>0}$, we necessarily have $f_i(x) = x_i$ for all $x \in \mathbb{R}^r_{\ge 0}$, which proves (iii).

Next pick a Zariski exceptional family (E_i). By Lemma 32, the $[E_i]$ are linearly independent in $\mathrm{N}^1(X)$. By definition, we have $\mathrm{P}_X \equiv 0$ on $F := \sum_i \mathbb{R}_{\ge 0}[E_i]$. To see that F is an extremal face of $\mathrm{Psef}(X)$, pick $D := \sum_i c_i E_i$ with $c_i \ge 0$, and assume $[D] = \alpha + \beta$ with $\alpha, \beta \in \mathrm{Psef}(X)$. We need to show that both α and β lie in F. By Definition 42 we have $D = \mathrm{N}_X(D) \le \mathrm{N}_X(\alpha) + \mathrm{N}_X(\beta)$, and hence

$$[\mathrm{N}_X(\alpha)] + [\mathrm{N}_X(\beta)] \le \mathrm{P}_X(\alpha) + \mathrm{P}_X(\beta) + [\mathrm{N}_X(\alpha)] + [\mathrm{N}_X(\beta)] = \alpha + \beta = [D] \le [\mathrm{N}_X(\alpha)] + \mathrm{N}_X(\beta)], \tag{16}$$

with respect to the psef order on $\mathrm{N}^1(X)$. Since $\mathrm{Psef}(X)$ is strict, we infer $\mathrm{P}_X(\alpha) = \mathrm{P}_X(\beta) = 0$ and $[D] = [\mathrm{N}_X(\alpha)] + [\mathrm{N}_X(\beta)]$. Since $\mathrm{N}_X(\alpha) + \mathrm{N}_X(\beta) - D$ is effective, it follows that $\mathrm{N}_X(\alpha) + \mathrm{N}_X(\beta) = D$. This implies that $\mathrm{N}_X(\alpha)$ and $\mathrm{N}_X(\beta)$ are supported in the E_i's, which proves, as desired, that $\alpha = [\mathrm{N}_X(\alpha)]$ and $\beta = [\mathrm{N}_X(\beta)]$ both lie in F. Thus (iv) holds.

Conversely, assume that $F \subset \mathrm{Psef}(X)$ is a Zariski exceptional face, and pick a class θ in its relative interior $\mathring{F}$. By (iii), the components (E_i) of $\mathrm{N}_X(\theta)$ form a Zariski exceptional family, which thus generates a Zariski exceptional face $F' := \sum_i \mathbb{R}_{\ge 0}[E_i]$. Since F and F' are both extremal faces containing θ in their relative interior, we conclude $F = F'$, which proves (v). □

As a result, Zariski exceptional families are in 1–1 correspondence with Zariski exceptional faces, which are rational simplicial cones generated by Zariski exceptional primes.

For surfaces, the notions above admit the following interpretation: see e.g. Theorems 5.4 and 4.8 in [6]:

Theorem 45. *Assume* $\dim X = 2$. *Then:*

(i) *a finite family* (E_i) *of prime divisors on* X *is Zariski exceptional iff the intersection matrix* $(E_i \cdot E_j)$ *is negative definite;*

(ii) *for any* $\theta \in \mathrm{Psef}(X)$, $\theta = \mathrm{P}_X(\theta) + [\mathrm{N}_X(\theta)]$ *coincides with the classical Zariski decomposition, i.e.* $\mathrm{P}_X(\theta)$ *is nef,* $\mathrm{N}_X(\theta)$ *is Zariski exceptional, and* $\mathrm{P}_X(\theta) \cdot \mathrm{N}_X(\theta) = 0$.

5.4. *Piecewise linear Zariski decompositions*

We introduce the following terminology:

Definition 46. *Given any convex subcone* $C \subset \mathrm{Psef}(X)$, *we say that* the Zariski decomposition is piecewise linear *(*PL *for short) on* C *if the map* $\mathrm{N}\colon C \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ *extends to a PL map* $\mathrm{N}^1(X) \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$, *i.e. a map that is linear on each cone of some finite fan decomposition of* $\mathrm{N}^1(X)$. *If the fan and the linear maps on its cones can further be chosen rational, then we say that* the Zariski decomposition is $\mathbb{Q}$-PL *on* C.

Lemma 47. *Let* $C \subset \mathrm{Psef}(X)$ *be a convex cone, and assume that* C *is written as the union of finitely many convex subcones* C_i. *Then the Zariski decomposition is PL (resp.* $\mathbb{Q}$*-PL) on* C *iff it is PL (resp.* $\mathbb{Q}$*-PL) on each* C_i.

Proof. The "only if" part is clear. Conversely, assume the Zariski decomposition is PL (resp. $\mathbb{Q}$-PL) on each C_i. After further subdividing each C_i according to a fan decomposition of $\mathrm{N}^1(X)$, we may assume that there exists a linear (resp. rational linear) map $L_i\colon \mathrm{N}^1(X) \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ that coincides with N on C_i. If C_i has nonempty interior in C, then $L_i|_{\mathrm{Vect}\,C}$ is uniquely determined as the derivative of N at any interior point of C_i, and we have $\mathrm{N} \ge L_i$ on C by convexity of N, see Corollary 36. Set $F := \max_i L_i$, where the maximum is over all C_i with nonempty interior in C. Then $F\colon \mathrm{N}^1(X) \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is PL (resp. $\mathbb{Q}$-PL), $\mathrm{N} \ge F$ on C, and equality holds outside the union A of all C_i with empty interior in C. Since A has zero measure, its complement is dense in C. Since $\mathrm{N} - F$ is lsc, see Corollary 36, we infer $\mathrm{N} \le F$ on C, which proves the "if" part. □

As a consequence of [22, Theorem 4.1] and its proof (especially Proposition 4.7) we have:

Example 48. If X is a Mori dream space (e.g. of log Fano type), then:

- for each $\theta \in \mathrm{Psef}(X)$, the b-divisor $\mathrm{N}(\theta)$ is $\mathbb{R}$-Cartier;
- $\mathrm{Psef}(X)$ is a rational polyhedral cone;
- the Zariski decomposition is $\mathbb{Q}$-PL on $\mathrm{Psef}(X)$.

The next result is closely related to the theory of Zariski chambers studied in [2].

Proposition 49. *If* $\dim X = 2$, *then the Zariski decomposition is* $\mathbb{Q}$*-PL on any convex cone* $C \subset \mathrm{Psef}(X)$ *with the property that the set of prime divisors* $E \subset X$ *with* $\mathrm{ord}_E(\theta) > 0$ *for some* $\theta \in C$ *is finite.*

By Lemma 33 (ii), the finiteness condition on C is satisfied as soon as C is polyhedral.

Proof. For each Zariski exceptional face F of $\mathrm{Psef}(X)$ with relative interior $\mathring{F}$, set $Z_F := \mathrm{N}_X^{-1}(\mathring{F})$. Thus $\theta \in \mathrm{Psef}(X)$ lies in Z_F iff the irreducible decomposition of $\mathrm{N}_X(\alpha)$ are precisely the generators of F. By Theorem 45 (ii), Z_F is a convex subcone of $\mathrm{Psef}(X)$ (whose intersection with $\mathrm{Big}(X)$ is a Zariski chamber in the sense of [2]); further, $\mathrm{N}_X|_{Z_F} : Z_F \to \mathring{F}$ is the restriction of the orthogonal projection onto $\mathrm{Vect}\,F$, which is a rational linear map. By Corollary 41, the Zariski decomposition is thus $\mathbb{Q}$-PL on Z_F. Finally, the finiteness assumption guarantees that C meets only finitely many Z_F's, and the result is thus a consequence of Lemma 47. □

We conclude this section with a higher-dimensional situation in which Zariski decompositions can be analyzed. Assuming again that $\dim X$ is arbitrary, consider next a 2-dimensional cone $C \subset \mathrm{N}^1(X)$ generated by two classes $\theta, \alpha \in \mathrm{N}^1(X)$ such that $\theta \in \mathrm{Nef}(X)$ and $\alpha \notin \mathrm{Psef}(X)$. Set

$$C_{\mathrm{nef}} := C \cap \mathrm{Nef}(X) \subset C_{\mathrm{psef}} := C \cap \mathrm{Psef}(X) \subset C,$$

and introduce the thresholds

$$\lambda_{\mathrm{nef}} := \sup\{\lambda \geq 0 \mid \theta + \lambda\alpha \in \mathrm{Nef}(X)\}, \quad \lambda_{\mathrm{psef}} := \sup\{\lambda \geq 0 \mid \theta + \lambda\alpha \in \mathrm{Psef}(X)\},$$

so that C_{nef} (resp. C_{psef}) is generated by θ and $\theta_{\mathrm{nef}} := \theta + \lambda_{\mathrm{nef}}\alpha$ (resp. $\theta_{\mathrm{psef}} := \theta + \lambda_{\mathrm{psef}}\alpha$).

The next result is basically contained in [41, §6.5].

Proposition 50. *With the above notation, suppose that C contains the class of a prime divisor $S \subset X$ such that* $\mathrm{Nef}(S) = \mathrm{Psef}(S)$ *and $S|_S$ is not nef. Then:*

(i) $\theta_{\mathrm{psef}} = t[S]$ *with* $t > 0$*;*

(ii) $\lambda_{\mathrm{nef}} = \lambda^S_{\mathrm{nef}} := \sup\{\lambda \geq 0 \mid (\theta + \lambda\alpha)|_S \in \mathrm{Nef}(S)\}$*;*

(iii) *the Zariski decomposition is PL on C_{psef}, with*

$$\mathrm{N} \equiv 0 \text{ on } C_{\mathrm{nef}}, \quad \mathrm{N}(a\theta_{\mathrm{nef}} + b[S]) = b\bar{S} \text{ for all } a, b \geq 0.$$

Proof. The assumptions imply that $S|_S$ is not psef. By Theorem 44 (i), S is thus Zariski exceptional, and $[S]$ generates an extremal ray of $\mathrm{Psef}(X)$. This ray is also extremal in C_{psef}, which proves (i).

Next, note that $\lambda_{\mathrm{nef}} \leq \lambda^S_{\mathrm{nef}} \leq \lambda_{\mathrm{psef}}$, by (i). Pick a curve $\gamma \subset X$. We need to show $(\theta + \lambda^S_{\mathrm{nef}}\alpha) \cdot \gamma \geq 0$. This is clear if $\gamma \subset S$ (since $(\theta + \lambda^S_{\mathrm{nef}}\alpha)|_S$ is nef), or if $\alpha \cdot \gamma \geq 0$ (since $\theta \cdot \gamma \geq 0$ and $\lambda^S_{\mathrm{nef}} \geq 0$). Otherwise, we have $S \cdot \gamma \geq 0$ and $\alpha \cdot \gamma \leq 0$, and we get again $(\theta + \lambda^S_{\mathrm{nef}}\alpha) \cdot \gamma \geq 0$ since

$$\theta + \lambda^S_{\mathrm{nef}}\alpha \equiv \theta_{\mathrm{psef}} + (\lambda^S_{\mathrm{nef}} - \lambda_{\mathrm{psef}})\alpha = t[S] + (\lambda^S_{\mathrm{nef}} - \lambda_{\mathrm{psef}})\alpha$$

with $\lambda^S_{\mathrm{nef}} - \lambda_{\mathrm{psef}} \leq 0$. This proves (ii).

For (iii), note that $\mathrm{N} \equiv 0$ on $\mathrm{Nef}(X) \supset C_{\mathrm{nef}}$ (see Theorem 34). Further, $\mathrm{N}([S]) = \bar{S}$ (see Lemma 43), and hence $\mathrm{N}(a\theta_{\mathrm{nef}} + b[S]) \leq b\bar{S}$ for $a, b \geq 0$. In particular, $c := \mathrm{ord}_S(a\theta_{\mathrm{nef}} + b[S]) \leq b$. On the other hand, (13) yields

$$\mathrm{N}(a\theta_{\mathrm{nef}} + b[S]) \geq \overline{\mathrm{N}(a\theta_{\mathrm{nef}} + b[S])} \geq c\bar{S},$$

and it thus remains to see $c = b$. By Corollary 39, $((a\theta_{\mathrm{nef}} + b[S]) - c[S])|_S$ lies in $\mathrm{Psef}(S) = \mathrm{Nef}(S)$. By (ii), we infer $a\theta_{\mathrm{nef}} + (b - c)[S] \in C_{\mathrm{nef}}$, and hence $b - c = 0$, since $C_{\mathrm{nef}} = \mathbb{R}_{\geq 0}\theta + \mathbb{R}_{\geq 0}\theta_{\mathrm{nef}}$ intersects $\mathbb{R}_{\geq 0}\theta_{\mathrm{nef}} + \mathbb{R}_{\geq 0}[S]$ only along $\mathbb{R}_{\geq 0}\theta_{\mathrm{nef}}$. □

6. Green's functions and Zariski decompositions

In this section we fix an ample class $\omega \in \mathrm{Amp}(X)$.

6.1. *Green's functions and equilibrium measures*

A subset $\Sigma \subset X^{\mathrm{an}}$ is *pluripolar* if $\Sigma \subset \{\varphi = -\infty\}$ for some $\varphi \in \mathrm{PSH}(\omega)$. By [13, Theorem 4.5], Σ is nonpluripolar iff

$$\mathrm{T}(\Sigma) := \sup_{\varphi \in \mathrm{PSH}(\omega)} (\sup \varphi - \sup_{\Sigma} \varphi) \in [0, +\infty]$$

is finite. The invariant $\mathrm{T}(\Sigma)$, which plays an important role in [5, 14], is modeled on the Alexander–Taylor capacity (which corresponds to $e^{-\mathrm{T}(\Sigma)}$) in complex analysis.

Definition 51. *For any subset $\Sigma \subset X^{\mathrm{an}}$ we set*

$$\varphi_\Sigma = \varphi_{\omega,\Sigma} := \sup\{\varphi \in \mathrm{PSH}(\omega) \mid \varphi|_\Sigma \leq 0\}. \tag{17}$$

Note that $\varphi_\Sigma(v_{\mathrm{triv}}) = \sup \varphi_\Sigma = \mathrm{T}(\Sigma)$, and hence

$$\varphi_\Sigma \in \mathrm{PL}(X) \Longrightarrow \mathrm{T}(\Sigma) \in \mathbb{Q}. \tag{18}$$

Theorem 52. *For any compact subset $\Sigma \subset X^{\mathrm{an}}$, the following holds:*

(i) $\varphi_\Sigma = \sup\{\varphi \in \mathrm{CPSH}(\omega) \mid \varphi|_\Sigma \le 0\}$*; in particular, φ_Σ is lsc;*

(ii) *if Σ is pluripolar then $\varphi_\Sigma^\star \equiv +\infty$;*

(iii) *if Σ is nonpluripolar, then $\varphi_\Sigma^\star$ is ω-psh and nonnegative; further, $\mu_\Sigma := \mathrm{MA}(\varphi_\Sigma^\star)$ is supported in Σ, $\int \varphi_\Sigma^\star \mu_\Sigma = 0$, and μ_Σ is characterized as the unique minimizer of the energy $\|\mu\|$ over all Radon probability measures μ with support in Σ.*

Since the energy of a Radon probability measure μ only appears in this statement, we simply recall here that it is defined as

$$\|\mu\| = \sup_{\varphi \in \mathscr{E}^1(\omega)} \left(\mathrm{E}(\varphi) - \int \varphi\, \mu \right) \in [0, +\infty], \tag{19}$$

and refer to [13, §9.1] for more details.

Definition 53. *Assuming Σ is nonpluripolar, we call μ_Σ its* equilibrium measure*, and $\varphi_\Sigma^\star$ its* Green's function.

The latter is characterized as the normalized potential of μ_Σ (in the terminology of [15, §1.6]), i.e. the unique $\varphi \in \mathscr{E}^1(\omega)$ such that $\mathrm{MA}(\varphi) = \mu_\Sigma$ and $\int \varphi\, \mu_\Sigma = 0$.

Proof of Theorem 52. Denote by φ'_Σ the right-hand side in (i), which obviously satisfies $\varphi'_\Sigma \le \varphi_\Sigma$. Pick $\varphi \in \mathrm{PSH}(\omega)$ with $\varphi|_\Sigma \le 0$, and write φ as the limit of a decreasing net (φ_i) in $\mathrm{CPSH}(\omega)$. For any $\varepsilon > 0$, a Dini type argument shows that $\varphi_i < \varepsilon$ on Σ for i large enough. Thus $\varphi_i \le \varphi'_\Sigma + \varepsilon$, and hence $\varphi \le \varphi'_\Sigma + \varepsilon$. This shows $\varphi_\Sigma \le \varphi'_\Sigma$, which proves (i).

Next, (ii) and the first half of (iii) follow from [13, Lemma 13.15]. Since the negligible set $\{\varphi_\Sigma < \varphi_\Sigma^\star\}$ is pluripolar (see [13, Theorem 13.17]), it has zero measure for any measure μ of finite energy [13, Lemma 9.2]. If μ has support in Σ, this yields $\int \varphi_\Sigma^\star \mu = \int \varphi_\Sigma \mu = 0$. By (19) we infer $\|\mu\| \ge \mathrm{E}(\varphi_\Sigma^\star) = \|\mu_\Sigma\|$. This proves that μ_Σ minimizes the energy, while uniqueness follows from the strict convexity of the energy [13, Proposition 10.10]. ☐

Further mimicking classical terminology in the complex analytic setting, we introduce:

Definition 54. *We say that a compact subset $\Sigma \subset X^{\mathrm{an}}$ is* regular *if $\varphi_\Sigma \in \mathrm{CPSH}(\omega)$.*

In particular, Σ is then nonpluripolar (see Theorem 52).

Lemma 55. *For any compact subset $\Sigma \subset X^{\mathrm{an}}$, the following hold:*

(i) *Σ is regular iff $\varphi_\Sigma^\star \le 0$ on Σ;*

(ii) *the regularity of Σ is independent of $\omega \in \mathrm{Amp}(X)$;*

(iii) *if $\Sigma \subset X^{\mathrm{lin}}$ then Σ is regular.*

Proof. If Σ is regular, then $\varphi_\Sigma^\star = \varphi_\Sigma$ vanishes on Σ. Conversely, assume $\varphi_\Sigma^\star \le 0$ on Σ. By (ii) and (iii) of Theorem 52, Σ is necessarily nonpluripolar, and $\varphi_\Sigma^\star$ is ω-psh. It is thus a competitor in (17), which implies that $\varphi_\Sigma = \varphi_\Sigma^\star$ is ω-psh, and also continuous by Theorem 52 (i).

Assume Σ is regular for ω, and pick $\omega' \in \mathrm{Amp}(X)$. Then $t\omega - \omega'$ is nef for $t \gg 1$, and hence $\mathrm{PSH}(\omega') \subset t\,\mathrm{PSH}(\omega)$. This implies $\varphi_{\omega',\Sigma} \le t\varphi_{\omega,\Sigma}$, and hence $\varphi_{\omega',\Sigma}^\star \le t\varphi_{\omega,\Sigma}$. In particular, $\varphi_{\omega',\Sigma}^\star|_\Sigma \le 0$, which proves that Σ is regular for ω', by (i).

Finally, assume $\Sigma \subset X^{\mathrm{lin}}$. Since $\{\varphi_\Sigma < \varphi_\Sigma^\star\}$ is pluripolar (see [13, Theorem 13.17]), it is disjoint from X^{lin}. As a result, $\varphi_\Sigma^\star \in \mathrm{PSH}(\omega)$ vanishes on Σ, and it again follows from (i) that Σ is regular. ☐

6.2. *The Green's function of a real divisorial set*

In what follows, we consider a *real divisorial set*, by which we mean a finite set $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$ of real divisorial valuations. By Lemma 55 (iii), $\Sigma \subset X^{\mathrm{lin}}$ is regular, i.e. $\varphi_\Sigma \in \mathrm{CPSH}(\omega)$. When $\Sigma = \{v\}$ for a single $v \in X^{\mathrm{div}}_{\mathbb{R}}$, we simply write $\varphi_v := \varphi_\Sigma$.

Example 56. Assume $\omega = c_1(L)$ with $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ ample and $v \in X^{\mathrm{div}}$. Then v is *dreamy* (with respect to L) in the sense of K.Fujita iff $\varphi_v \in \mathcal{H}(L)$; see [14, §1.7, Appendix A].

If $v_{\mathrm{triv}} \in \Sigma$, then $\varphi_\Sigma \equiv 0$, and we henceforth assume $v_{\mathrm{triv}} \notin \Sigma$. Pick a smooth birational model $\pi\colon Y \to X$ which extracts each $v \in \Sigma$, i.e. $v = t_v \operatorname{ord}_{E_v}$ for a prime divisor $E_v \subset Y$ and $t_v \in \mathbb{R}_{>0}$. We then introduce the effective $\mathbb{R}$-divisor on Y

$$D := \sum_\alpha t_\alpha^{-1} E_\alpha,$$

whose set of Rees valuations Γ_D coincides with Σ (see Definition 16).

Theorem 57. *With the above notation, the following holds:*

(i) $\sup \varphi_\Sigma = \mathrm{T}(\Sigma)$ *coincides with the pseudoeffective threshold*

$$\lambda_{\mathrm{psef}} := \max\left\{\lambda \geq 0 \mid \pi^\star\omega - \lambda D \in \mathrm{Psef}(Y)\right\};$$

(ii) $\varphi_\Sigma \in \mathrm{CPSH}(\omega)$ *is of divisorial type, and the associated family of b-divisors $(B_\lambda)_{\lambda \leq \lambda_{\mathrm{psef}}}$ (see Theorem 18) is given by*

$$-B_\lambda = \begin{cases} \mathrm{N}(\pi^\star\omega - \lambda D) + \lambda \overline{D} & \text{for } \lambda \in [0, \lambda_{\mathrm{psef}}] \\ 0 & \text{for } \lambda \leq 0. \end{cases}$$

Proof. Pick $\lambda \in \mathbb{R}$. For any $\psi \in \mathrm{PSH}(\omega)$, we have $\psi + \lambda \leq \varphi_\Sigma \Leftrightarrow \psi|_\Sigma \leq -\lambda$, and hence

$$\widehat{\varphi}^\lambda_\Sigma = \sup\{\psi \in \mathrm{PSH}_{\mathrm{hom}}(\omega) \mid \psi|_\Sigma \leq -\lambda\}.$$

When $\lambda \leq 0$ this yields $\widehat{\varphi}^\lambda_\Sigma = 0$. Now assume $\lambda > 0$. Using Proposition 17 and $\mathrm{PSH}_{\mathrm{hom}}(\pi^\star\omega) = \pi^\star \mathrm{PSH}_{\mathrm{hom}}(\omega)$, we get

$$\pi^\star \widehat{\varphi}^\lambda_\Sigma = \sup\{\tau \in \mathrm{PSH}_{\mathrm{hom}}(\pi^\star\omega - \lambda D)\} - \lambda\psi_D = V_{\pi^\star\omega - \lambda D} - \lambda\psi_D. \tag{20}$$

Now the left-hand side is not identically $-\infty$ iff $\lambda \leq \sup\varphi$, while for the right-hand side this holds iff $\lambda \leq \lambda_{\mathrm{psef}}$, by Proposition 27. This proves (i), and also (ii), by Theorem 31. □

Corollary 58. *The center of φ_Σ satisfies*

$$Z_X(\varphi_\Sigma) = \pi\left(\mathbb{B}_-(\pi^\star\omega - \lambda_{\mathrm{psef}} D)\right) \cup Z_X(\Sigma).$$

In particular, $Z_X(\varphi_\Sigma)$ is Zariski dense in X iff $\mathbb{B}_-(\pi^\star\omega - \lambda_{\mathrm{psef}} D)$ is Zariski dense in Y.

Proof. By Lemma 22, we have

$$Z_X(\varphi_\Sigma) = Z_X(\widehat{\varphi}^{\max}_\Sigma) = \pi(Z_Y(\pi^* \widehat{\varphi}^{\max}_\Sigma)).$$

It follows from Theorem 57 and its proof that

$$\pi^\star \widehat{\varphi}^{\max}_\Sigma = V_{\pi^\star\omega - \lambda_{\mathrm{psef}} D} - \lambda_{\mathrm{psef}}\psi_D.$$

Now $Z_Y(V_{\pi^\star\omega - \lambda_{\mathrm{psef}} D}) = \mathbb{B}_-(\pi^\star\omega - \lambda_{\mathrm{psef}} D)$ by Theorem 31, whereas we see from Example 21 that $Z_Y(-\lambda_{\mathrm{psef}}\psi_D) = Z_Y(\Sigma)$, so we conclude using Lemma 25. □

6.3. *Dimension one and two*

In this section we consider the case $\dim X \le 2$.

Proposition 59. *If* $\dim X = 1$*, then for any real divisorial set* $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$*, we have* $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$*. If* ω *is rational and* $\Sigma \subset X^{\mathrm{div}}$*, then we further have* $\varphi_\Sigma \in \mathrm{PL}^+(X)$*.*

Proof. We may assume $v_{\mathrm{triv}} \notin \Sigma$, or else $\varphi_\Sigma \equiv 0$. Thus assume $\Sigma = \{v_i\}_{i\in I}$, where $v_i = t_i \operatorname{ord}_{p_i}$, $t_i \in \mathbb{R}_{>0}$, and $p_i \in X$ is a closed point. We may assume $p_i \neq p_j$ for $i \neq j$, or else $\varphi_\Sigma = \varphi_{\Sigma'}$ for $\Sigma' = \{v_i\}_{i\in I'}$, where $I' \subset I$ is defined by $i \in I'$ iff for all $j \neq i$, either $p_j \neq p_i$ or $t_j > t_i$. Under these assumptions,

$$\varphi_\Sigma = A \max\left\{1 + \sum_i t_i^{-1} \log|\mathfrak{m}_{p_i}|, 0\right\},$$

where $A > 0$ satisfies $A \sum_i t_i^{-1} = \deg \omega$, see [13, Example 3.19]. Thus $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$. Further, if $\Sigma \subset X^{\mathrm{div}}$, then $t_i \in \mathbb{Q}_{>0}$ for all i, so if ω is rational, then $A \in \mathbb{Q}_{>0}$, and hence $\varphi_\Sigma \in \mathrm{PL}^+(X)$. □

Theorem 60. *If* $\dim X = 2$*, then for any real divisorial set* $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$*, we have* $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$*. If* ω *is rational and* $\Sigma \subset X^{\mathrm{div}}$*, then we further have*

$$\varphi_\Sigma \in \mathrm{PL}(X) \Longleftrightarrow \varphi_\Sigma \in \mathrm{PL}^+(X) \Longleftrightarrow \mathrm{T}(\Sigma) \in \mathbb{Q}. \tag{21}$$

We will see in Example 63 that $\mathrm{T}(\Sigma)$ can be irrational.

Lemma 61. *Assume* $\dim X \le 2$*, and pick* $B \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$*. Then* B *is relatively nef iff it is relatively semiample.*

Proof. Assume B is relative nef, and pick a determination $\pi\colon Y \to X$ of B. The relatively nef cone of $\mathrm{N}^1(Y/X)$ is dual to the cone generated by the (finite) set of π-exceptional prime divisors, and is thus a rational polyhedral cone. As a consequence, we can write $B_Y = \sum_i t_i D_i$ with $t_i > 0$ and $D_i \in \mathrm{Div}(Y)_{\mathbb{Q}}$ relatively nef. By [38, Theorem 12.1 (ii)], each D_i is relatively semiample, and the result follows. □

Proof of Theorem 60. Use the notation of Theorem 57. By Proposition 49, the Zariski decomposition is $\mathbb{Q}$-PL on the cone

$$C = (\mathbb{R}_+\pi^\star\omega + \mathbb{R}_+[-D]) \cap \mathrm{Psef}(Y) = \mathbb{R}_+\pi^\star\omega + \mathbb{R}_+(\pi^\star\omega - \lambda_{\mathrm{psef}}[D]).$$

We can thus find $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_N = \lambda_{\mathrm{psef}}$ such that

$$\lambda \longmapsto B_\lambda = -(\mathrm{N}(\pi^\star\omega - \lambda[D]) + \lambda\overline{D})$$

is affine linear on $[\lambda_i, \lambda_{i+1}]$ for $1 \le i < N$. Setting $B_i := B_{\lambda_i}$, it follows that

$$\varphi_\Sigma = \sup_{\lambda\in[0,\lambda_{\mathrm{psef}}]} \{\psi_{B_\lambda} + \lambda\} = \max_{1\le i\le N} \{\psi_{B_i} + \lambda_i\}.$$

Since $\overline{\omega} + [B_i]$ is nef, the antieffective divisor B_i is relatively nef, and hence relatively semiample (see Lemma 61). By Proposition 7, we infer $\psi_{B_i} \in \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$, and hence $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$.

Now assume ω and $\mathrm{T}(\Sigma) = \lambda_{\mathrm{psef}}$ are both rational, and that $\Sigma \subset X^{\mathrm{div}}$. Then D is rational as well, and C is thus a rational polyhedral cone. Since the Zariski decomposition on C is the restriction of a $\mathbb{Q}$-PL map on $\mathrm{N}^1(Y)$, this implies that the λ_i above can be chosen rational. Using again that the Zariski decomposition is $\mathbb{Q}$-PL on C, we infer that B_i is a $\mathbb{Q}$-divisor, hence $\psi_{B_i} \in \mathrm{PL}^+_{\mathrm{hom}}(X)$, which shows $\varphi_\Sigma \in \mathrm{PL}^+(X)$. The rest follows from (18). □

7. Examples of Green's functions

We now exhibit examples of Green's functions with various types of behavior. These examples serve as the underpinnings of Theorems A and B of the introduction.

7.1. *Divisors on abelian varieties*

As a direct application of Theorem 57, we show:

Proposition 62. *Assume* $\mathrm{Nef}(X) = \mathrm{Psef}(X)$*. Consider a real divisorial set* $\Sigma = \{v_\alpha\} \subset X^{\mathrm{div}}_{\mathbb{R}}$ *with* $v_\alpha = t_\alpha \operatorname{ord}_{E_\alpha}$ *for* $E_\alpha \subset X$ *prime and* $t_\alpha > 0$*, and set* $D := \sum_\alpha t_\alpha^{-1} E_\alpha$*. Then*

$$\mathrm{T}(\Sigma) = \lambda_{\mathrm{psef}} = \sup\{\lambda \ge 0 \mid \omega - \lambda D \in \mathrm{Psef}(X)\}$$

and

$$\varphi_\Sigma = \mathrm{T}(\Sigma)\max\{0, 1 - \psi_D\}.$$

In particular, $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$*. If we further assume* $\Sigma \subset X^{\mathrm{div}}$*, then*

$$\varphi_\Sigma \in \mathrm{PL}(X) \Longleftrightarrow \varphi_\Sigma \in \mathrm{PL}^+(X) \Longleftrightarrow \mathrm{T}(\Sigma) \in \mathbb{Q}. \tag{22}$$

Proof. Using the notation of Theorem 57, we have $\mathrm{N}(\omega - \lambda D) = 0$ for $\lambda \le \lambda_{\mathrm{psef}} = \mathrm{T}(\Sigma)$. Thus $\widehat{\varphi}^\lambda_\Sigma = -\lambda\psi_D$, and hence

$$\varphi_\Sigma = \sup_{0 \le \lambda \le \lambda_{\mathrm{psef}}} \{\lambda - \lambda\psi_D\} = \lambda_{\mathrm{psef}} \max\{0, 1 - \psi_D\}.$$

Since $-\psi_D = \sum_\alpha t_\alpha^{-1} \log|\mathcal{O}_X(-E_\alpha)|$ lies in $\mathrm{PL}^+(X)_{\mathbb{R}}$, it follows that $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$. If $\Sigma \subset X^{\mathrm{div}}$, then D is a $\mathbb{Q}$-divisor, and hence $-\psi_D \in \mathrm{PL}^+_{\mathrm{hom}}(X)$. If we further assume $\mathrm{T}(\Sigma) \in \mathbb{Q}$, we get $\varphi_\Sigma \in \mathrm{PL}^+(X)$, and the remaining implication follows from (18). □

Example 63. Suppose X is an abelian surface, $\omega = c_1(L)$ with $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ ample, and $v = \operatorname{ord}_E$ with $E \subset X$ a prime divisor. Then $\mathrm{Nef}(X) = \mathrm{Psef}(X)$, and $\mathrm{T}(v) = \lambda_{\mathrm{psef}}$ is the smallest root of the quadratic equation $(L - \lambda E)^2 = 0$, see [34, Remark 1.5.6]. If X has Picard number $\rho(X) \ge 2$, then λ_{psef} is irrational for a typical choice of L and E, and hence $\varphi_v \notin \mathrm{PL}(X)$. (Compare [34, Example 2.3.8]). In particular, v is not dreamy (with respect to L) in the sense of Fujita, see Example 56.

7.2. *The Cutkosky example*

Building on a construction of Cutkosky [21] and Proposition 50 (itself based on [41, §6.5]), we provide an example of a divisorial valuation on $\mathbb{P}^3$ for which (21) fails. This relies on the following general result.

Proposition 64. *Consider a flag of smooth subvarieties* $Z \subset S \subset X$ *with* $\operatorname{codim} S = 1$, $\operatorname{codim} Z = 2$ *and ideals* $\mathfrak{b}_S \subset \mathfrak{b}_Z \subset \mathcal{O}_X$*, and assume that*

(i) $S \equiv \omega$*;*
(ii) $\mathrm{Nef}(S) = \mathrm{Psef}(S)$*;*
(iii) $\omega|_S - Z$ *is not nef on* S*, i.e.* $\lambda^S_{\mathrm{nef}} := \sup\{\lambda \ge 0 \mid \omega|_S - \lambda[Z] \in \mathrm{Nef}(S)\} < 1$.

The Green's function of $v := \operatorname{ord}_Z \in X^{\mathrm{div}}$ *is then given by*

$$\varphi_v = \max\{0, \lambda^S_{\mathrm{nef}}(\log|\mathfrak{b}_Z| + 1), \log|\mathfrak{b}_S| + 1\}.$$

In particular, $\mathrm{T}(v) = 1$*,* $\varphi_v \in \mathbb{R}\mathrm{PL}^+(X)$*, and*

$$\varphi_v \in \mathrm{PL}(X) \Longleftrightarrow \varphi_v \in \mathrm{PL}^+(X) \Longleftrightarrow \lambda^S_{\mathrm{nef}} \in \mathbb{Q}.$$

Proof. Let $\pi\colon Y \to X$ be the blowup along Z, with exceptional divisor E, and denote by $S' = \pi^\star S - E$ the strict transform of S. Since Z has codimension 1 on S, π maps S' isomorphically onto S, and takes $S'|_{S'} = \pi^\star S|_{S'} - E|_{S'}$ to $S|_S - Z \equiv \omega|_S - [Z]$. By (ii) and (iii), we thus have $\mathrm{Nef}(S') = \mathrm{Psef}(S')$, and $S'|_{S'}$ is not nef.

Consider the cone $C \subset \mathrm{N}^1(Y)$ generated by $\theta := \pi^\star\omega \in \mathrm{Nef}(Y)$ and $\alpha := -[E] \notin \mathrm{Psef}(Y)$. Since C contains the class of S', it follows from Proposition 50 that

$$1 = \lambda_{\mathrm{psef}} := \sup\{\lambda \geq 0 \mid \pi^\star\omega - \lambda[E] \in \mathrm{Psef}(Y)\}$$

and $\lambda \mapsto \mathrm{N}(\pi^\star\omega - \lambda E)$ vanishes on $[0, \lambda^S_{\mathrm{nef}}]$, and is affine linear on $[\lambda^S_{\mathrm{nef}}, 1]$, with value S' at $\lambda = 1$. By Theorem 57, the concave family $(B_\lambda)_{\lambda \leq 1}$ of b-divisors associated to φ_v is affine linear on $(-\infty, 0]$, $[0, \lambda^S_{\mathrm{nef}}]$ and $[\lambda^S_{\mathrm{nef}}, 1]$, with value

$$B_\lambda = 0, \quad \lambda^S_{\mathrm{nef}}\overline{E} \quad \text{and} \quad \overline{S' + E} = \overline{S}$$

at $\lambda = 0$, λ^S_{nef} and 1, respectively. By (6), the result follows, since $-\psi_{\overline{E}} = \log|\mathfrak{b}_Z|$ and $-\psi_{\overline{S}} = \log|\mathfrak{b}_S|$. □

Example 65. Assume $k = \mathbb{C}$, and set $(X, L) = (\mathbb{P}^3, \mathcal{O}(4))$. By [21], there exists a smooth quartic surface $S \subset X$ without (-2)-curves, and hence such that $\mathrm{Nef}(S) = \mathrm{Psef}(S)$, containing a smooth curve Z such that λ^S_{nef} is irrational and less than 1. By Proposition 64, we infer $\mathrm{T}(v) = 1$ and $\varphi_v \in \mathbb{R}\mathrm{PL}^+(X) \setminus \mathrm{PL}(X)$ (in contrast with (21)).

7.3. *The Lesieutre example*

Based on an example by Lesieutre [35], we now exhibit a Green's function that is not $\mathbb{R}$-PL. This forms the basis for Theorem B in the introduction.

Proposition 66. *Suppose that X admits a class $\theta \in \mathrm{Psef}(X)$ whose diminished base locus $\mathbb{B}_-(\theta)$ is Zariski dense. Then there exist $\omega \in \mathrm{Amp}(X)$ and $v \in X^{\mathrm{div}}$ such that $Z_X(\varphi_{\omega,v})$ is Zariski dense in X. In particular, $\varphi_{\omega,v} \notin \mathbb{R}\mathrm{PL}(X)$.*

Proof. Note first that θ cannot be big. Otherwise, there would exist an effective $\mathbb{R}$-divisor $D \equiv \theta$, and hence $\mathbb{B}_-(\theta)$ would be contained in $\operatorname{supp} D$. Pick an ample prime divisor E on X, choose $c \in \mathbb{Q}_{>0}$ large enough such that $\omega := \theta + c[E]$ is ample, and set $v := c^{-1}\operatorname{ord}_E \in X^{\mathrm{div}}$. Since ω is ample and $\omega - c[E] = \theta$ lies on the boundary of $\mathrm{Psef}(X)$, the threshold $\lambda_{\mathrm{psef}} = \sup\{\lambda \geq 0 \mid \omega - \lambda[E] \in \mathrm{Psef}(X)\}$ is equal to c. Thus $\mathbb{B}_-(\omega - \lambda_{\mathrm{psef}}[E])$ is Zariski dense, and hence so is $Z_X(\varphi_{\omega,v})$, by Corollary 58. The last point follows from Lemma 26. □

Example 67. By [35, Theorem 1.1], the assumptions in Proposition 66 are satisfied when $k = \mathbb{C}$ and X is the blowup of $\mathbb{P}^3$ at nine sufficiently general points.

If θ in Proposition 66 is rational, then the proof shows that ω can be taken rational as well, i.e. $\omega = c_1(L)$ for an ample $\mathbb{Q}$-line bundle. While no such rational example appears to be known at present, we can nevertheless exploit the structure of Lesieutre's example to get:

Proposition 68. *Set $(X, L) := (\mathbb{P}^3, \mathcal{O}(1))$. Then there exists a finite set $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$ such that $Z_X(\varphi_{L,\Sigma})$ is Zariski dense in X, and hence $\varphi_{L,\Sigma} \notin \mathbb{R}\mathrm{PL}(X)$.*

Proof. Let $\pi\colon Y \to X$ be the blowup at nine sufficiently general points, and denote by $\sum_{i=1}^9 E_i$ the exceptional divisor. By [35, Remark 4.5, Lemma 5.2], we can pick $D = \sum_i c_i E_i$ with $c_i \in \mathbb{R}_{>0}$ such that the diminished base locus of $\pi^\star L - D$ is Zariski dense. As above, this implies that this class lies on the boundary of the psef cone (it even generates an extremal ray, see [35, Lemma 5.1]), and the psef threshold

$$\lambda_{\mathrm{psef}} = \sup\{\lambda \geq 0 \mid \pi^\star L - \lambda D \in \mathrm{Psef}(Y)\}$$

is thus equal to 1. The result now follows from Corollary 58, with $\Sigma = \{c_i^{-1}\operatorname{ord}_{E_i}\}_{1 \leq i \leq 9}$. □

It is natural to ask:

Question 69. *Can an example as in Proposition 68 be found with $\Sigma \subset X^{\mathrm{div}}$?*

8. The non-trivially valued case

In this section, we work over the non-Archimedean field $K = k((\varpi))$ of formal Laurent series, with valuation ring $K^\circ := k[[\varpi]]$. We use [10] as our main reference.

Thus X now denotes a smooth projective variety of dimension n over K. (In Section 9, it will be obtained as the base change of a smooth projective k-variety). Working "additively", we view the elements of the analytification X^{an} as valuations $x\colon K(Y)^\times \to \mathbb{R}$ for subvarieties $Y \subset X$, restricting to the given valuation on K.

8.1. *Models*

We define a *model* of X to be a normal, flat, projective K°-scheme $\mathcal{X}$ together with the data of an isomorphism $\mathcal{X}_K \simeq X$. The *special fiber* of $\mathcal{X}$ is the projective k-scheme $\mathcal{X}_0 := \mathcal{X} \times_{\operatorname{Spec} K} \operatorname{Spec} k$. Each $x \in X^{\mathrm{an}}$ can be viewed as a semivaluation on $\mathcal{X}$, whose center is denoted by $\mathrm{red}_{\mathcal{X}}(x) \in \mathcal{X}_0$. This defines a *reduction map* $\mathrm{red}_{\mathcal{X}}\colon X^{\mathrm{an}} \to \mathcal{X}_0$, which is surjective and anticontinuous (i.e. the preimage of an open set is closed). For each $x \in X^{\mathrm{an}}$ we also set

$$Z_{\mathcal{X}}(x) := \overline{\{\mathrm{red}_{\mathcal{X}}(x)\}} \subset \mathcal{X}_0.$$

The preimage under $\mathrm{red}_{\mathcal{X}}$ of the set of generic points of $\mathcal{X}_0$ is finite. We denote it by $\Gamma_{\mathcal{X}} \subset X^{\mathrm{an}}$, and call its elements the *Shilov points* of $\mathcal{X}$. As $\mathcal{X}$ is normal, each irreducible component E of $\mathcal{X}_0$ defines a *divisorial valuation* $x_E \in X_K^{\mathrm{an}}$ given by

$$x_E := b_E^{-1} \operatorname{ord}_E, \; b_E := \operatorname{ord}_E(\varpi);$$

it is the unique preimage under $\mathrm{red}_{\mathcal{X}}$ of the generic point of E, and the Shilov points of $\mathcal{X}$ are exactly these valuations x_E.

One says that another model $\mathcal{X}'$ *dominates* $\mathcal{X}$ if the canonical birational map $\mathcal{X}' \dashrightarrow \mathcal{X}$ extends to a morphism (necessarily unique, by separatedness). In that case, $\mathrm{red}_{\mathcal{X}}$ is the composition of $\mathrm{red}_{\mathcal{X}'}$ with the induced projective morphism $\mathcal{X}'_0 \to \mathcal{X}_0$. The set of models forms a filtered poset with respect to domination. The set

$$X^{\mathrm{div}} = \bigcup_{\mathcal{X}} \Gamma_{\mathcal{X}}$$

of all divisorial valuations is a dense subset of X^{an}.

8.2. *Piecewise linear functions*

A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on a model $\mathcal{X}$ of X is *vertical* if it is supported in $\mathcal{X}_0$; it then defines a continuous function on X^{an} called a *model function.* The $\mathbb{Q}$-vector space $\mathrm{PL}(X)$ of such functions is stable under max, and dense in $\mathrm{C}^0(X^{\mathrm{an}})$.

Definition 70. *We define the space* $\mathbb{R}\mathrm{PL}(X)$ *of* real piecewise linear functions *on* X^{an} *(*$\mathbb{R}$-PL functions *for short) as the smallest* $\mathbb{R}$*-linear subspace of* $\mathrm{C}^0(X^{\mathrm{an}})$ *that is stable under max (and hence also min) and contains* $\mathrm{PL}(X)$.

Fix a model $\mathcal{X}$. An ideal $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}}$ is *vertical* if its zero locus $V(\mathfrak{a})$ is contained in $\mathcal{X}_0$. This defines a nonpositive function $\log|\mathfrak{a}| \in \mathrm{PL}(X)$, determined by minus the exceptional divisor of the blowup of $\mathcal{X}$ along $\mathfrak{a}$, and such that

$$\log|\mathfrak{a}|(x) < 0 \Longleftrightarrow Z_{\mathcal{X}}(x) \subset V(\mathfrak{a}). \tag{23}$$

Functions of the form $\log|\mathfrak{a}|$ for a vertical ideal $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}}$ span the $\mathbb{Q}$-vector space $\mathrm{PL}(X)$ (see [10, Proposition 2.2]). As in Section 1.3, it follows that any function in $\mathbb{R}\mathrm{PL}(X)$ can be written as a difference of finite maxima of $\mathbb{R}_+$-linear combinations of functions of the form $\log|\mathfrak{a}|$.

8.3. *Dual complexes and retractions*

We use [10, 39] as references.

An *snc model* is a regular model $\mathscr{X}$ such that the Cartier divisor $\mathscr{X}_0$ has simple normal crossing support. Denote by $\mathscr{X}_0 = \sum_{i\in I} b_i E_i$ its irreducible decomposition. A *stratum* of $\mathscr{X}_0$ is defined as a non-empty irreducible component of $E_J := \bigcap_{j\in J} E_j$ for some $J \subset I$. By resolution of singularities, the set of snc models is cofinal in the poset of all models.

The *dual complex* $\Delta_{\mathscr{X}}$ of an snc model $\mathscr{X}$ is defined as the dual intersection complex of $\mathscr{X}_0$. Its faces are in 1–1 correspondence with the strata of $\mathscr{X}_0$, and further come with a natural integral affine structure. In particular, the vertices of $\Delta_{\mathscr{X}}$ are in 1–1 correspondence with the E_i's, and admit a natural realization in X^{an} as the set $\Gamma_{\mathscr{X}}$ of Shilov points x_{E_i}.

This extends to a canonical embedding $\Delta_{\mathscr{X}} \hookrightarrow X^{\mathrm{an}}$ onto the set of monomial points with respect to $\sum_i E_i$. The reduction $\mathrm{red}_{\mathscr{X}}(x) \in \mathscr{X}_0$ of a point $x \in \Delta_{\mathscr{X}} \subset X^{\mathrm{an}}$ is the generic point of the stratum of $\mathscr{X}_0$ associated with the unique simplex of $\Delta_{\mathscr{X}}$ containing x in its relative interior. In particular, $Z_{\mathscr{X}}(x)$ is a stratum of $\mathscr{X}_0$. This embedding is further compatible with the PL structures, in the sense that the $\mathbb{Q}$-vector space $\mathrm{PL}(\Delta_{\mathscr{X}})$ of piecewise rational affine functions on $\Delta_{\mathscr{X}}$ is precisely the image of $\mathrm{PL}(X)$ under restriction.

If another snc model $\mathscr{X}'$ dominates $\mathscr{X}$, then $\Delta_{\mathscr{X}}$ is contained in $\Delta_{\mathscr{X}'}$, and $\mathrm{PL}(\Delta_{\mathscr{X}'})$ restricts to $\mathrm{PL}(\Delta_{\mathscr{X}})$. Furthermore, the set

$$X^{\mathrm{qm}} := \bigcup_{\mathscr{X}} \Delta_{\mathscr{X}} \subset X^{\mathrm{an}}$$

of *quasimonomial valuations* coincides with the set of Abhyankar points of X, see [10, Remark 3.8] and [29, Proposition 3.7], while the subset of rational points $\bigcup_{\mathscr{X}} \Delta_{\mathscr{X}}(\mathbb{Q})$ coincides with the set X^{div} of divisorial valuations. For later use, we also note:

Lemma 71. *If $\mathscr{X}$ is an snc model, then the image $\mathrm{red}_{\mathscr{X}'}(\Delta_{\mathscr{X}}) \subset \mathscr{X}'_0$ of the dual complex of $\mathscr{X}$ under the reduction map of any other model $\mathscr{X}'$ is finite.*

Proof. Pick an snc model $\mathscr{X}''$ that dominates both $\mathscr{X}$ and $\mathscr{X}'$. Then $\Delta_{\mathscr{X}}$ is contained in $\Delta_{\mathscr{X}''}$, and $\mathrm{red}_{\mathscr{X}'}(\Delta_{\mathscr{X}})$ is thus contained in the image of $\mathrm{red}_{\mathscr{X}''}(\Delta_{\mathscr{X}''})$ under the induced morphism $\mathscr{X}''_0 \to \mathscr{X}_0$. After replacing both $\mathscr{X}$ and $\mathscr{X}'$ with $\mathscr{X}''$, we may thus assume without loss that $\mathscr{X} = \mathscr{X}'$. For any $x \in \Delta_{\mathscr{X}}$, $\mathrm{red}_{\mathscr{X}}(x)$ is then the generic point of some stratum of $\mathscr{X}_0$, and $\mathrm{red}_{\mathscr{X}}(\Delta_{\mathscr{X}})$ is thus a finite set. $\square$

Dually, each snc model $\mathscr{X}$ comes with a canonical *retraction* $p_{\mathscr{X}}\colon X^{\mathrm{an}} \to \Delta_{\mathscr{X}}$ that takes $x \in X^{\mathrm{an}}$ to the unique monomial valuation $y = p_{\mathscr{X}}(x)$ such that

- $Z_{\mathscr{X}}(y)$ is the minimal stratum containing $Z_{\mathscr{X}}(x)$;
- x and y take the same values on the E_i's.

This induces a homeomorphism $X^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{\mathscr{X}} \Delta_{\mathscr{X}}$, which is compatible with the PL structures in the sense that

$$\mathrm{PL}(X) = \bigcup_{\mathscr{X}} p_{\mathscr{X}}^{\star}\,\mathrm{PL}(\Delta_{\mathscr{X}}). \tag{24}$$

This implies

$$\mathbb{R}\mathrm{PL}(X) = \bigcup_{\mathscr{X}} p_{\mathscr{X}}^{\star}\,\mathbb{R}\mathrm{PL}(\Delta_{\mathscr{X}}), \tag{25}$$

where $\mathbb{R}\mathrm{PL}(\Delta_{\mathscr{X}})$ is the space $\mathbb{R}$-PL functions on $\Delta_{\mathscr{X}}$, i.e. functions that are real affine linear on a sufficiently fine decomposition of each face into real simplices.

8.4. *Psh functions and Monge–Ampère measures*

We use [10, 11, 26] as references.

A *closed* $(1,1)$*-form* $\theta \in \mathcal{Z}^{1,1}(X)$ in the sense of [10, §4.2] is represented by a relative numerical equivalence class on some model $\mathcal{X}$, called a *determination* of θ. It induces a numerical class $[\theta] \in \mathrm{N}^1(X)$. We say that θ is *semipositive*, written $\theta \geq 0$, if θ is determined by a nef numerical class on some model. In that case, $[\theta]$ is nef as well.

To each tuple $\theta_1, \dots, \theta_n$ in $\mathcal{Z}^{1,1}(X)$ is associated a signed Radon measure $\theta_1 \wedge \cdots \wedge \theta_n$ on X^{an} of total mass $[\theta_1] \cdot \ldots \cdot [\theta_n]$, with finite support in X^{div}. More precisely, if all θ_i are determined by a normal model $\mathcal{X}$, then $\theta_1 \wedge \cdots \wedge \theta_n$ has support in $\Gamma_{\mathcal{X}}$ (see [11, §2.7]).

Each $\varphi \in \mathrm{PL}(X)$ is determined by a vertical $\mathbb{Q}$-Cartier divisor D on some model $\mathcal{X}$, whose numerical class defines a closed $(1,1)$-form $\mathrm{dd}^c\varphi \in \mathcal{Z}^{1,1}(X)$. We say that φ is θ*-psh* for a given $\theta \in \mathcal{Z}^{1,1}(X)$ if $\theta + \mathrm{dd}^c\varphi \geq 0$.

From now on, we fix a semipositive form $\omega \in \mathcal{Z}^{1,1}(X)$ such that $[\omega]$ is ample. A function $\varphi\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ is ω*-plurisubharmonic* (ω*-psh* for short) if $\varphi \not\equiv -\infty$ and φ can be written as the pointwise limit of a decreasing net of ω-psh PL functions. The space $\mathrm{PSH}(\omega)$ is closed under max and under decreasing limits.

By Dini's lemma, the space $\mathrm{CPSH}(\omega)$ of continuous ω-psh functions coincides with the closure in $\mathrm{C}^0(X)$ (with respect to uniform convergence) of the space of ω-psh PL functions.

Each $\varphi \in \mathrm{PSH}(\omega)$ satisfies the "maximum principle"

$$\sup_X \varphi = \max_{\Gamma_{\mathcal{X}}} \varphi \tag{26}$$

for any model $\mathcal{X}$ determining ω (see [26, Proposition 4.22]). For snc models, [10, §7.1] more precisely yields:

Lemma 72. *Pick* $\varphi \in \mathrm{PSH}(\omega)$ *and an snc model* $\mathcal{X}$ *on which* ω *is determined. Then:*

(i) *the restriction of* φ *to any face of* $\Delta_{\mathcal{X}}$ *is continuous and convex;*
(ii) *the net* $(\varphi \circ p_{\mathcal{X}})_{\mathcal{X}}$ *is decreasing and converges pointwise to* φ.

Remark 73. The definition of $\mathrm{PSH}(\omega)$ given here differs from the one in [10], but Theorem 8.7 in loc. cit. implies that the two definitions are equivalent.

To each continuous ω-psh function φ (or, more generally, any ω-psh function of finite energy) is associated its *Monge–Ampère measure* $\mathrm{MA}(\varphi) = \mathrm{MA}_\omega(\varphi)$, a Radon probability measure on X uniquely determined by the following properties:

- if φ is PL, then $\mathrm{MA}(\varphi) = V^{-1}(\omega + \mathrm{dd}^c\varphi)^n$ with $V := [\omega]^n$;
- $\varphi \mapsto \mathrm{MA}(\varphi)$ is continuous along decreasing nets.

By the main result of [11], any Radon probability measure μ with support in the dual complex $\Delta_{\mathcal{X}}$ of some snc model can be written as $\mu = \mathrm{MA}(\varphi)$ for some $\varphi \in \mathrm{CPSH}(\omega)$, unique up to an additive constant.

8.5. *Green's functions*

As in the trivially valued case, we can consider the Green's function associated to a nonpluripolar set $\Sigma \subset X^{\mathrm{an}}$. Here we will only consider the following case. Suppose $x \in X^{\mathrm{div}}$ is a divisorial point, and define

$$\varphi_x := \varphi_{\omega,x} := \sup\{\varphi \in \mathrm{PSH}(\omega) \mid \varphi(x) \leq 0\}.$$

It follows from [11, §8.4] that $\varphi_x \in \mathrm{CPSH}(\omega)$ satisfies $\mathrm{MA}(\varphi_x) = \delta_x$ and $\varphi_x(x) = 0$.

Proposition 74. *If* $\dim X = 1$ *and* $[\omega]$ *is a rational class, then* $\varphi_x \in \mathrm{PL}(X)$.

Proof. This follows from Proposition 3.3.7 in [42], and can also be deduced from properties of the intersection form on $\mathcal{X}_0$ for any snc model $\mathcal{X}$, as in [23, Theorem 7.17]. □

This proves part (i) of Theorem A in the introduction. We will prove (ii) in Section 9.5.

8.6. *Invariance under retraction*

It will be convenient to introduce the following terminology:

Definition 75. *We say that a function φ on X^{an} is* invariant under retraction *if $\varphi = \varphi \circ p_{\mathcal{X}}$ for some (and hence any sufficiently high) snc model $\mathcal{X}$ of X.*

Example 76. By (24) and (25), a function $\varphi \in \mathrm{C}^0(X^{\mathrm{an}})$ lies in $\mathrm{PL}(X)$ (resp. $\mathbb{R}\mathrm{PL}(X)$) iff φ is invariant under retraction and restricts to a $\mathbb{Q}$-PL (resp. $\mathbb{R}$-PL) function on the dual complex associated to any (equivalently, any sufficiently high) snc model.

Remark 77. The condition $\varphi = \varphi \circ p_{\mathcal{X}}$ in Definition 75 is stronger than the "comparison property" of [36, Definition 3.11], which merely requires $\varphi = \varphi \circ p_{\mathcal{X}}$ to hold on the preimage under $p_{\mathcal{X}}$ of the n-dimensional open faces of some dual complex $\Delta_{\mathcal{X}}$, i.e. the preimage of the 0-dimensional strata of $\mathcal{X}_0$ under the reduction map.

Proposition 78. *If $\varphi \in \mathrm{PSH}(\omega)$ is invariant under retraction, then $\varphi \in \mathrm{CPSH}(\omega)$, and $\mathrm{MA}(\varphi)$ is supported in some dual complex.*

The first point is a direct consequence of Lemma 72, while the second one is a special case of the following more precise result. Recall first that the *ω-psh envelope* of $f \in \mathrm{C}^0(X^{\mathrm{an}})$ is defined as

$$\mathrm{P}(f) = \mathrm{P}_\omega(f) := \sup\{\varphi \in \mathrm{PSH}(\omega) \mid \varphi \le f\}.$$

By [10], it lies in $\mathrm{CPSH}(\omega)$.

Theorem 79. *For any $\varphi \in \mathrm{CPSH}(\omega)$ and any snc model $\mathcal{X}$ on which ω is determined, the following properties are equivalent:*

(i) $\mathrm{MA}(\varphi)$ *is supported in* $\Delta_{\mathcal{X}}$*;*
(ii) $\varphi = \mathrm{P}(\varphi \circ p_{\mathcal{X}})$.

Proof. For any $\psi \in \mathrm{PSH}(\omega)$, we have $\psi \le \psi \circ p_{\mathcal{X}}$ (see Lemma 72 (ii)), and hence

$$\mathrm{P}(\varphi \circ p_{\mathcal{X}}) = \sup\left\{\psi \in \mathrm{PSH}(\omega) \mid \psi \le \varphi \text{ on } \Delta_{\mathcal{X}}\right\}. \tag{27}$$

Assume (i). By the domination principle (see [11, Lemma 8.4]), any $\psi \in \mathrm{PSH}(\omega)$ such that $\psi \le \varphi$ on $\operatorname{supp}\mathrm{MA}(\varphi) \subset \Delta_{\mathcal{X}}$ satisfies $\psi \le \varphi$ on X^{an}. In view of (27) this yields (ii). Conversely, assume (ii). For any finite set of rational points $\Sigma \subset \Delta_{\mathcal{X}}(\mathbb{Q}) \subset X^{\mathrm{div}}$, consider the envelope

$$\varphi_\Sigma := \sup\{\psi \in \mathrm{PSH}(\omega) \mid \psi \le \varphi \text{ on } \Sigma\}.$$

Then φ_Σ lies in $\mathrm{CPSH}(\omega)$, and $\mathrm{MA}(\varphi_\Sigma)$ is supported in Σ (see [11, Lemma 8.5]). The net (φ_Σ), indexed by the filtered poset of finite subsets $\Sigma \subset \Delta_{\mathcal{X}}(\mathbb{Q})$, is clearly decreasing, and bounded below by φ. Its limit $\psi := \lim_\Sigma \varphi_\Sigma$ is thus ω-psh, and we claim that it coincides with φ. Indeed, we have $\psi \le \varphi$ on $\bigcup_\Sigma \Sigma = \Delta_{\mathcal{X}}(\mathbb{Q})$, and hence on $\Delta_{\mathcal{X}}$, where both ψ and φ are continuous. By (27), this yields $\psi \le \mathrm{P}(\varphi \circ p_{\mathcal{X}}) = \varphi$. By continuity of the Monge–Ampère operator along decreasing nets, we infer $\mathrm{MA}(\varphi_\Sigma) \to \mathrm{MA}(\varphi)$ weakly on X, which yields (i) since each $\mathrm{MA}(\varphi_\Sigma)$ is supported in $\Delta_{\mathcal{X}}$. □

In view of Proposition 78 and Example 76, it is natural to conversely ask:

Question 80. *If the Monge–Ampère measure $\mathrm{MA}_\omega(\varphi)$ of $\varphi \in \mathrm{CPSH}(\omega)$ is supported in some dual complex, is φ invariant under retraction?*

This question appears as [25, Question 2], and is equivalent to asking whether $\varphi \circ p_{\mathscr{X}}$ is ω-psh for some high enough model $\mathscr{X}$, by Theorem 79. In Example 99 below (see also Theorem A) we show that the answer is negative. In this example, the support of $\mathrm{MA}_\omega(\varphi)$ is even a finite set. One can nevertheless ask:

Question 81. *Assume that $\varphi \in \mathrm{CPSH}(\omega)$ is such that the support of the Monge–Ampère measure $\mathrm{MA}_\omega(\varphi)$ is a finite set contained in some dual complex.*

(i) *is φ $\mathbb{R}$-PL on each dual complex?*
(ii) *if ω is rational, is φ $\mathbb{Q}$-PL on each dual complex?*

Example 99 below provides a negative answer to (ii). Indeed the function φ in this example is $\mathbb{R}$-PL but not $\mathbb{Q}$-PL, and by (24), (25), this implies that φ fails to be $\mathbb{Q}$-PL on some dual complex $\Delta_{\mathscr{X}}$. The answer to (i) is also likely negative in general, as suggested by Nakayama's counterexample to the existence of Zariski decompositions on certain toric bundles over an abelian suface [40, p. IV.2.10].

Question 82. *Suppose X is a toric variety, and let $\varphi \in \mathrm{CPSH}(\omega)$ be a torus invariant ω-psh function such that $\mathrm{MA}_\omega(\varphi)$ is supported on a compact subset of $N_{\mathbb{R}} \subset X^{\mathrm{an}}$. Is φ invariant under retraction?*

Question 83. *If $\varphi \in \mathrm{CPSH}(\omega)$ is invariant under retraction, is the same true for $\varphi|_{Z^{\mathrm{an}}}$, if $Z \subset X$ is a smooth subvariety?*

8.7. *The center of a plurisubharmonic function*

We end this section by a version of Theorem 24 in our present context. In analogy with (7), for any subset $S \subset X^{\mathrm{an}}$ and any model $\mathscr{X}$ we set

$$Z_{\mathscr{X}}(S) := \bigcup_{x \in S} Z_{\mathscr{X}}(x).$$

This is thus the smallest subset of $\mathscr{X}_0$ that is invariant under specialization and contains the image $\mathrm{red}_{\mathscr{X}}(S)$ of S under the reduction map $\mathrm{red}_{\mathscr{X}} \colon X^{\mathrm{an}} \to \mathscr{X}_0$. For any higher model $\mathscr{X}'$, the induced proper morphism $\mathscr{X}'_0 \to \mathscr{X}_0$ maps $Z_{\mathscr{X}'}(S)$ onto $Z_{\mathscr{X}}(S)$.

We say that $S \subset X^{\mathrm{an}}$ is *invariant under retraction* if $p_{\mathscr{X}}^{-1}(S) = S$ for some (and hence any sufficiently high) snc model $\mathscr{X}$.

Lemma 84. *If $S \subset X^{\mathrm{an}}$ is invariant under retraction, then $Z_{\mathscr{X}}(S)$ is Zariski closed for any model $\mathscr{X}$.*

Proof. Pick an snc model $\mathscr{X}'$ dominating $\mathscr{X}$ such that $S = p_{\mathscr{X}'}^{-1}(S)$. Since $Z_{\mathscr{X}}(S)$ is the image of $Z_{\mathscr{X}'}(S)$ under the proper morphism $\mathscr{X}'_0 \to \mathscr{X}_0$, we may replace $\mathscr{X}$ with $\mathscr{X}'$ and assume without loss that $\mathscr{X} = \mathscr{X}'$. The set $Z_{\mathscr{X}}(S)$ obviously contains $Z_{\mathscr{X}}(S \cap \Delta_{\mathscr{X}})$, which is Zariski closed since $Z_{\mathscr{X}}(y)$ is a stratum of $\mathscr{X}_0$ for any $y \in \Delta_{\mathscr{X}}$. Conversely, pick $x \in S$, and set $y := p_{\mathscr{X}}(x) \in \Delta_{\mathscr{X}}$. Then $y \in p_{\mathscr{X}}^{-1}(S) = S$, and $Z_{\mathscr{X}}(x) \subset Z_{\mathscr{X}}(y)$ since it follows from the definition of $p_{\mathscr{X}}$ that $\mathrm{red}_{\mathscr{X}}(x)$ is a specialization of $\mathrm{red}_{\mathscr{X}}(y)$. This shows, as desired, that $Z_{\mathscr{X}}(S) = Z_{\mathscr{X}}(S \cap \Delta_{\mathscr{X}})$ is Zariski closed. □

Definition 85. *Given $\varphi \in \mathrm{PSH}(\omega)$ and a model $\mathscr{X}$, we define the* center of φ on $\mathscr{X}$ *as*

$$Z_{\mathscr{X}}(\varphi) := Z_{\mathscr{X}}(\{\varphi < \sup \varphi\}) = \bigcup \{Z_{\mathscr{X}}(x) \mid x \in X, \varphi(x) < \sup \varphi\}.$$

Example 86. If $\varphi = \log|\mathfrak{a}|$ for a vertical ideal $\mathfrak{a} \subset \mathscr{O}_{\mathscr{X}}$, then $Z_{\mathscr{X}}(\varphi) = V(\mathfrak{a})$.

Theorem 87. *For any $\varphi \in \mathrm{PSH}(\omega)$ and any model $\mathscr{X}$, the following holds:*

(i) *$Z_{\mathscr{X}}(\varphi)$ is an at most countable union of subvarieties of $\mathscr{X}_0$;*
(ii) *if φ is invariant under retraction, then $Z_{\mathscr{X}}(\varphi)$ is Zariski closed;*
(iii) *$Z_{\mathscr{X}}(\varphi) = \mathrm{red}_{\mathscr{X}}(\{\varphi < \sup \varphi\})$;*
(iv) *$Z_{\mathscr{X}}(\varphi)$ is a strict subset of $\mathscr{X}_0$ as soon as $\mathscr{X}$ determines ω.*

Question 88. *Is it true that* $\{\varphi < \sup\varphi\} = \mathrm{red}_{\mathcal{X}}^{-1}(Z_{\mathcal{X}}(\varphi))$ *as in Theorem 24?*

Proof. By [11, Proposition 4.7], φ can be written as the pointwise limit of a decreasing sequence $(\varphi_m)_{m\in\mathbb{N}}$ of ω-psh PL functions. Since each φ_m is in particular invariant under retraction (see Example 76), Lemma 84 implies that $Z_{\mathcal{X}}\{(\varphi_m < \sup\varphi\})$ is Zariski closed for each m. On the other hand, since $\varphi_m \searrow \varphi$ pointwise on X, we have $\{\varphi < \sup\varphi\} = \bigcup_m\{\varphi_m < \sup\varphi\}$, and hence $Z_{\mathcal{X}}(\varphi) = \bigcup_m Z_{\mathcal{X}}(\{\varphi_m < \sup\varphi\})$. This proves (i), while (ii) is a direct consequence of Lemma 84.

Pick $x \in X^{\mathrm{an}}$ such that $\varphi(x) < \sup\varphi$. To prove (iii), we need to show that any $\xi \in Z_{\mathcal{X}}(x)$ lies in $\mathrm{red}_{\mathcal{X}}(\{\varphi < \sup\varphi\})$. By Lemma 72, we can find a high enough snc model $\mathcal{X}'$ such that $x' := p_{\mathcal{X}'}(x)$ satisfies $\varphi(x') < \sup\varphi$. By properness of $\mathcal{X}'_0 \to \mathcal{X}_0$, $Z_{\mathcal{X}}(x)$ is the image of $Z_{\mathcal{X}'}(x)$, which is itself contained in $Z_{\mathcal{X}'}(x')$. After replacing $\mathcal{X}$ with $\mathcal{X}'$ and x with x', we may thus assume without loss that $\mathcal{X}$ is snc and x lies in $\Delta_{\mathcal{X}}$. Pick $y \in X^{\mathrm{an}}$ with $\mathrm{red}_{\mathcal{X}}(y) = \xi$ (which exists by surjectivity of the reduction map, see [24, Lemma 4.12]). Set $z := p_{\mathcal{X}}(y)$, and denote by σ the unique face of $\Delta_{\mathcal{X}}$ that contains z in its relative interior, the corresponding stratum of $\mathcal{X}_0$ being the smallest one containing ξ. Since the latter point lies on the stratum $Z_{\mathcal{X}}(x)$, it follows that σ contains x (possibly on its boundary). Since φ is convex and continuous on σ (see Lemma 72), it can only achieve its supremum at the interior point z if it is constant on σ. As $x \in \sigma$ satisfies $\varphi(x) < \sup\varphi$, it follows that $\varphi(z) < \sup\varphi$ as well. Since $z = p_{\mathcal{X}}(y)$, this implies $\varphi(y) \le \varphi(z) < \sup\varphi$ (again by Lemma 72). Thus $\xi = \mathrm{red}_{\mathcal{X}}(y) \in \mathrm{red}_{\mathcal{X}}(\{\varphi < \sup\varphi\})$, which proves (iii).

Finally, assume that $\mathcal{X}$ determines ω. By (26), we can find an irreducible component E of $\mathcal{X}_0$ whose corresponding Shilov point $x_E \in \Gamma_{\mathcal{X}}$ satisfies $\varphi(x_E) = \sup\varphi$. Since x_E is the only point of X^{an} whose reduction on $\mathcal{X}_0$ is the generic point of E, it follows that the latter does not belong to $Z_{\mathcal{X}}(\varphi)$, which is thus a strict subset of $\mathcal{X}_0$. $\square$

9. The isotrivial case

We now consider the *isotrivial* case, in which the variety over $K = k((\varpi))$ is the base change X_K of a smooth projective variety X over the (trivially valued) field k.

9.1. *Ground field extension*

We have a natural projection

$$\pi\colon X_K^{\mathrm{an}} \longrightarrow X^{\mathrm{an}},$$

while Gauss extension provides a continuous section

$$\sigma\colon X^{\mathrm{an}} \hookrightarrow X_K^{\mathrm{an}}$$

onto the set of $k^\times$-invariant points (see [12, Proposition 1.6]). By [12, Corollary 1.5], we further have:

Lemma 89. *If $v \in X^{\mathrm{an}}$ is divisorial (resp real divisorial) then $\sigma(v) \in X_K^{\mathrm{an}}$ is divisorial (resp. quasimonomial).*

The base change of X to the valuation ring $K^\circ := k[[\varpi]]$ defines the *trivial model*

$$\mathcal{X}_{\mathrm{triv}} := X_{K^\circ}$$

of X_K, whose special fiber $\mathcal{X}_{\mathrm{triv},0}$ will be identified with X. More generally, each *test configuration* $\mathcal{X} \to \mathbb{A}^1 = \operatorname{Spec} k[\varpi]$ for X induces via base change under $k[\varpi] \to k[[\varpi]] = K^\circ$ a $k^\times$-invariant model of X_K, that shares the same vertical ideals and vertical divisors as $\mathcal{X}$, and will simply be denoted by $\mathcal{X}$, for simplicity.

9.2. *Psh functions*

For any $\theta \in \mathrm{N}^1(X)$, we denote by $\pi^\star\theta \in \mathcal{Z}^{1,1}(X_K)$ the induced closed $(1,1)$-form, determined by the relative numerical class induced by θ on the trivial model. If $\omega \in \mathrm{Amp}(X)$, then $[\pi^\star\omega] \in \mathrm{N}^1(X_K)$ coincides with the base change of ω, and hence is ample.

Theorem 90. *Pick $\omega \in \mathrm{Amp}(X)$ and $\varphi \in \mathrm{PSH}(\omega)$. Then:*

(i) *$\pi^\star\varphi \in \mathrm{PSH}(\pi^\star\omega)$;*
(ii) *if φ further lies in $\mathrm{CPSH}(\omega)$, then $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi) = \sigma_\star \mathrm{MA}_\omega(\varphi)$.*

Lemma 91. *For any $\varphi \in \mathrm{PL}(X)$ and $\theta \in \mathrm{N}^1(X)$, the following holds:*

(i) *$\pi^\star\varphi \in \mathrm{PL}(X_K)$;*
(ii) *$(\pi^\star\theta + \mathrm{dd}^c\pi^\star\varphi)^n = \sigma_\star(\theta + \mathrm{dd}^c\varphi)^n$;*
(iii) *φ is θ-psh iff $\pi^\star\varphi$ is $\pi^\star\theta$-psh.*

Proof. The function φ is determined by a vertical $\mathbb{Q}$-Cartier divisor D on a test configuration $\mathcal{X}$, that may be taken to dominate the trivial one (see [13, Theorem 2.7]). The induced vertical divisor on the induced model of X_K then determines $\pi^\star\varphi$. This proves (i), and also (ii), by comparing [11, (2.2)] and [13, (3.6)]. Finally, denote by $\theta_{\mathcal{X}}$ the pullback of θ to $\mathrm{N}^1(\mathcal{X}/\mathbb{A}^1)$. Then φ is θ-psh iff $(\theta_{\mathcal{X}} + [D])|_{\mathcal{X}_0}$ is nef, which is also equivalent to $\pi^\star\varphi$ being $\pi^\star\theta$-psh. This proves (iii). □

Proof of Theorem 90. Write φ as the limit on X^{an} of a decreasing net of ω-psh PL functions φ_i. By Lemma 91, $\pi^\star\varphi_i$ is PL and $\pi^\star\omega$-psh. Since it decreases pointwise on X_K^{an} to $\pi^\star\varphi$, the latter is $\pi^\star\omega$-psh, which proves (i). For each i, Lemma 91 (ii) further implies $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi_i) = \sigma_\star \mathrm{MA}_\omega(\varphi_i)$. If φ is continuous, then $\mathrm{MA}_\omega(\varphi)$ and $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi)$ are both defined, and are the limits of $\mathrm{MA}_\omega(\varphi_i)$ and $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi_i)$, respectively. This proves (ii). □

9.3. *PL structures*

As a direct consequence of Lemma 91, the projection $\pi\colon X_K^{\mathrm{an}} \to X^{\mathrm{an}}$ is compatible with the PL structures:

Corollary 92. *We have $\pi^\star \mathrm{PL}(X) \subset \mathrm{PL}(X_K)$ and $\pi^\star \mathbb{R}\mathrm{PL}(X) \subset \mathbb{R}\mathrm{PL}(X_K)$.*

As we next show, this is also the case for Gauss extension.

Theorem 93. *We have $\sigma^\star \mathrm{PL}(X_K) = \mathrm{PL}(X)$ and $\sigma^\star \mathbb{R}\mathrm{PL}(X_K) = \mathbb{R}\mathrm{PL}(X)$.*

Any vertical ideal $\mathfrak{a}$ on $\mathcal{X}_{\mathrm{triv}}$, being trivial outside the central fiber, can be viewed as a vertical ideal on $X \times \mathbb{A}^1$, and $\tilde{\mathfrak{a}} := \mathbb{G}_{\mathrm{m}} \cdot \mathfrak{a}$ is then the smallest flag ideal containing $\mathfrak{a}$.

Lemma 94. *With the above notation we have $\sigma^\star \log|\mathfrak{a}| = \varphi_{\tilde{\mathfrak{a}}}$.*

Proof. Pick an ample line bundle L on X, and denote by $\mathcal{L}_{\mathrm{triv}}$ the trivial model of L_K, i.e. the pullback of L to the trivial model $\mathcal{X}_{\mathrm{triv}} = X_{K^\circ}$. After replacing L with a large enough multiple, we may assume $\mathcal{L}_{\mathrm{triv}} \otimes \mathfrak{a}$ is generated by finitely many sections $s_i \in \mathrm{H}^0(\mathcal{X}_{\mathrm{triv}}, \mathcal{L}_{\mathrm{triv}})$. Then $\log|\mathfrak{a}| = \max_i \log|s_i|$, where $|s_i|$ denotes the pointwise length of s_i in the model metric induced by $\mathcal{L}_{\mathrm{triv}}$. For each i write $s_i = \sum_{\lambda\in\mathbb{Z}} s_{i,\lambda}\varpi^\lambda$ where $s_{i,\lambda} \in \mathrm{H}^0(X, L)$, and denote by $\mathfrak{b}_\lambda \subset \mathcal{O}_X$ the ideal locally generated by $(s_{i,\lambda})_i$. Then $\tilde{\mathfrak{a}} = \sum_{\lambda\in\mathbb{Z}} \mathfrak{b}_\lambda\varpi^\lambda$. By definition of Gauss extension, we have for any $v \in X^{\mathrm{an}}$

$$\log|s_i|(\sigma(v)) = \max_{\lambda\in\mathbb{Z}}\{\log|s_{i,\lambda}| + \lambda\}.$$

Thus $\sigma^\star \log|\mathfrak{a}| = \max_{\lambda\in\mathbb{Z}}\{\psi_\lambda - \lambda\}$ with $\psi_\lambda := \max_i \log|s_{i,\lambda}| = \log|\mathfrak{b}_\lambda|$, and hence $\sigma^\star \log|\mathfrak{a}| = \max_\lambda\{\log|\mathfrak{b}_\lambda| - \lambda\} = \varphi_{\tilde{\mathfrak{a}}}$. □

Proof of Theorem 93. By Corollary 92 we have $\pi^\star \mathrm{PL}(X) \subset \mathrm{PL}(X_K)$. Since $\mathrm{PL}(X_K)$ is generated by functions of the form $\log|\mathfrak{a}|$ for a vertical ideal $\mathfrak{a} \subset \mathcal{O}_{\mathscr{X}_{\mathrm{triv}}}$, Lemma 94 yields $\sigma^\star \mathrm{PL}(X_K) \subset \mathrm{PL}(X)$, and hence also $\sigma^\star \mathbb{R}\mathrm{PL}(X_K) \subset \mathbb{R}\mathrm{PL}(X)$. This completes the proof, since $\sigma^\star\pi^\star = \mathrm{id}$. □

9.4. *Centers*

Next we study the relationships between the two center maps $Z_X\colon X^{\mathrm{an}} \to X$ and $Z_{\mathscr{X}_{\mathrm{triv}}}\colon X_K^{\mathrm{an}} \to \mathscr{X}_{\mathrm{triv},0} = X$.

Lemma 95. *For all $x \in X_K^{\mathrm{an}}$ and $v \in X^{\mathrm{an}}$ we have*

$$Z_{\mathscr{X}_{\mathrm{triv}}}(x) \subset Z_X(\pi(x)), \quad Z_X(v) = Z_{\mathscr{X}_{\mathrm{triv}}}(\sigma(v)).$$

Proof. Denote by $\mathfrak{b} \subset \mathcal{O}_X$ the ideal of the subvariety $Z_X(\pi(x))$. Then $\mathfrak{a} := \mathfrak{b} + (\varpi)$ is a vertical ideal on $\mathscr{X}_{\mathrm{triv}}$ such that $V(\mathfrak{a}) = V(\mathfrak{b}) = Z_X(\pi(x))$ under the identification $\mathscr{X}_{\mathrm{triv},0} = X$. Further,

$$\log|\mathfrak{a}|(x) = \max\{\log|\mathfrak{b}|(\pi(x)), -1\} < 0,$$

and hence $Z_{\mathscr{X}_{\mathrm{triv}}}(x) \subset V(\mathfrak{a}) = Z_X(\pi(x))$, see (23).

Applying this to $x = \sigma(v)$ yields $Z_{\mathscr{X}_{\mathrm{triv}}}(\sigma(v)) \subset Z_X(v)$. To prove the converse inclusion, denote by $\mathfrak{a} \subset \mathcal{O}_{\mathscr{X}_{\mathrm{triv}}}$ the ideal of $Z_{\mathscr{X}_{\mathrm{triv}}}(\sigma(v))$. Since $\sigma(v)$ is $k^\times$-invariant, $\mathfrak{a} = \sum_{\lambda\in\mathbb{Z}} \mathfrak{a}_\lambda \varpi^{-\lambda}$ is (induced by) a flag ideal. Further, $\varphi_{\mathfrak{a}}(v) = \log|\mathfrak{a}|(\sigma(v)) < 0$, and hence $Z_X(v) \subset Z_X(\varphi_{\mathfrak{a}})$. By Example 14 we have $Z_X(\varphi_{\mathfrak{a}}) = V(\mathfrak{a}_0)$. The latter is also equal to the zero locus of $\mathfrak{a}_0 + (\varpi)$ on $\mathscr{X}_{\mathrm{triv}}$, which is contained in $V(\mathfrak{a}) = Z_{\mathscr{X}_{\mathrm{triv}}}(\sigma(v))$ since $\mathfrak{a} \subset \mathfrak{a}_0 + (\varpi)$. Thus $Z_X(v) \subset Z_{\mathscr{X}_{\mathrm{triv}}}(\sigma(v))$, which concludes the proof. □

As a consequence we get:

Proposition 96. *If $\omega \in \mathrm{Amp}(X)$ and $\varphi \in \mathrm{PSH}(\omega)$, then $Z_{\mathscr{X}_{\mathrm{triv}}}(\pi^\star\varphi) = Z_X(\varphi)$.*

Proof. Pick $v \in X^{\mathrm{an}}$ such that $\varphi(v) < \sup\varphi$, and set $x := \sigma(v)$. Then $\pi^\star\varphi(x) = \varphi(v)$ and $\sup\pi^\star\varphi = \sup\varphi$, so x lies in $\{\pi^\star\varphi < \sup\pi^\star\varphi\}$, and hence $Z_X(v) = Z_{\mathscr{X}_{\mathrm{triv}}}(x) \subset Z_{\mathscr{X}_{\mathrm{triv}}}(\pi^\star\varphi)$ by Lemma 95. This implies $Z_X(\varphi) \subset Z_{\mathscr{X}_{\mathrm{triv}}}(\pi^\star\varphi)$. Conversely, assume $x \in X_K^{\mathrm{an}}$ satisfies $\pi^\star\varphi(x) < \sup\pi^\star\varphi$. Then $v := \pi(x)$ lies in $\{\varphi < \sup\varphi\}$, and hence $Z_X(v) \subset Z_X(\varphi)$. In view of Lemma 95, this implies $Z_{\mathscr{X}_{\mathrm{triv}}}(x) \subset Z_X(\varphi)$, and hence $Z_{\mathscr{X}_{\mathrm{triv}}}(\pi^\star\varphi) \subset Z_X(\varphi)$. □

Combining Proposition 96 and Theorem 87, we obtain

Corollary 97. *Let $\varphi \in \mathrm{PSH}(\omega)$, where $\omega \in \mathrm{Amp}(X)$, and suppose that $\pi^\star\varphi \in \mathrm{PSH}(\pi^\star\omega)$ is invariant under retraction. Then $Z_X(\varphi) \subset X$ is a Zariski closed proper subset of X.*

9.5. *Examples*

We are now ready to prove Theorems A and B in the introduction, and also provide additional examples. As in the previous section, X denotes a smooth projective variety over k. Pick a class $\omega \in \mathrm{Amp}(X)$, a $k^\times$-invariant divisorial point $x \in X_K^{\mathrm{div}}$, and denote as in Section 8.5 by $\varphi_x \in \mathrm{CPSH}(\pi^\star\omega)$ the Green's function associated to x; this is the unique solution to the Monge–Ampère equation

$$\mathrm{MA}_{\pi^\star\omega}(\varphi_x) = \delta_x \quad \text{and} \quad \varphi_x(x) = 0.$$

By Lemma 89, we have $x = \sigma(v)$ with $v := \pi(x) \in X^{\mathrm{div}}$. If $\varphi_v \in \mathrm{CPSH}(\omega)$ denotes the Green's function of $\{v\}$, see Section 6.1, then we have

$$\varphi_x = \pi^\star\varphi_v.$$

Indeed, $\pi^\star\varphi_v(x) = \varphi_v(v) = 0$, and by Theorem 90, we have $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi_v) = \sigma_\star\delta_v = \delta_x$.

Our goal is to investigate the regularity of φ_x.

Corollary 98. *If* $\dim X = 1$, *then* $\varphi_x \in \mathrm{PL}(X_K)$. *If* $\dim X = 2$, *then* $\varphi_x \in \mathbb{R}\mathrm{PL}(X_K)$.

Proof. The first statement follows from Proposition 74. Now suppose $\dim X = 2$. By Theorem 60, $\varphi_v \in \mathbb{R}\mathrm{PL}(X)$, so that $\varphi_x \in \mathbb{R}\mathrm{PL}(X_K)$, see Corollary 92. ☐

However, even when ω is rational, φ_x is in general not $\mathbb{Q}$-PL:

Example 99. Example 63 gives an example of an abelian surface X, a rational class $\omega \in \mathrm{Amp}(X)$, and a divisorial valuation $v \in X^{\mathrm{div}}$ such that $\varphi_v \in \mathbb{R}\mathrm{PL}(X) \setminus \mathrm{PL}(X)$. If $x = \sigma(v)$, then $\varphi_x = \pi^\star \varphi_v \in \mathbb{R}\mathrm{PL}(X_K) \setminus \mathrm{PL}(X_K)$, by Theorem 93.

Example 100. Similarly, Example 65 gives an example of a divisorial valuation $v \in \mathbb{P}^{3,\mathrm{div}}$ such that if we set $\omega = c_1(\mathcal{O}(4))$, then $\varphi_v := \varphi_{\omega,v} \in \mathbb{R}\mathrm{PL}(X) \setminus \mathrm{PL}(X)$. If $x = \sigma(v)$, then $\varphi_x = \pi^\star \varphi_v \in \mathbb{R}\mathrm{PL}(X_K) \setminus \mathrm{PL}(X_K)$, by Theorem 93.

Examples 99 and 100 establish Theorem A (ii). They also provide a negative answer to Question 81 (ii). Indeed, a function $\varphi \in \mathrm{C}^0(X_K^{\mathrm{an}})$ lies in $\mathbb{R}\mathrm{PL}(X_K)$ (resp. $\mathrm{PL}(X_K)$) iff φ is invariant under retraction and restricts to an $\mathbb{R}$-PL (resp. $\mathbb{Q}$-PL) function on each dual complex, see Example 76.

As the next example shows, if $\dim X = 3$, then φ_x need not be $\mathbb{R}$-PL. In fact, it may not even be invariant under retraction.

Example 101. Example 67 shows that we may have $\dim X = 3$ and $Z_X(\varphi_v)$ Zariski dense in X, and it follows from Corollary 97 that φ_x cannot be invariant under retraction.

It could, however, a priori be the case that the restriction φ_x to any dual complex is $\mathbb{R}$-PL, see Question 81 (i).

In Example 101, based on Lesieutre's work, the class ω is irrational. We do not know of an example for which the class ω is rational. However, the following example provides a proof of Theorem B in the introduction.

Example 102. Set $X = \mathbb{P}^3_k$ and $\omega := c_1(\mathcal{O}(1)) \in \mathrm{N}^1(X)$. By Proposition 68, there exists $\psi \in \mathrm{CPSH}(\omega)$ such that $\mathrm{MA}_\omega(\psi)$ is supported in a finite subset $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$, and $Z_X(\psi)$ is Zariski dense in X. Theorem 90 then shows that $\varphi := \pi^\star \psi$ lies in $\mathrm{CPSH}(\pi^\star \omega)$, $\mathrm{MA}_{\pi^\star\omega}(\varphi) = \sigma_\star \mathrm{MA}_\omega(\psi)$ has finite support in some dual complex (see Lemma 89), while Corollary 97 shows that φ cannot be invariant under retraction.

References

[1] R. Andreasson and J. Hultgren, "Monge–Ampère equations and tropical affine structures on reflexive polytopes", 2023. https://arxiv.org/abs/2303.05276.

[2] T. Bauer, A. Küronya and T. Szemberg, "Zariski chambers, volumes, and stable base loci", *J. Reine Angew. Math.* **576** (2004), pp. 209–233.

[3] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, American Mathematical Society, 1990, pp. ix+169.

[4] R. J. Berman, S. Boucksom, V. Guedj and A. Zeriahi, "A variational approach to complex Monge–Ampère equations", *Publ. Math., Inst. Hautes Étud. Sci.* **117** (2013), pp. 179–245.

[5] H. Blum and M. Jonsson, "Thresholds, valuations, and K-stability", *Adv. Math.* **365** (2020), article no. 107062 (57 pages).

[6] S. Boucksom, "Divisorial Zariski decompositions on compact complex manifolds", *Ann. Sci. Éc. Norm. Supér.* **37** (2004), pp. 45–76.

[7] S. Boucksom, T. De Fernex and C. Favre, "The volume of an isolated singularity", *Duke Math. J.* **161** (2012), no. 8, pp. 1455–1521.

[8] S. Boucksom, J.-P. Demailly, M. Păun and T. Peternell, "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension", *J. Algebr. Geom.* **22** (2013), no. 2, pp. 201–248.

[9] S. Boucksom, C. Favre and M. Jonsson, "Differentiability of volumes of divisors and a problem of Teissier", *J. Algebr. Geom.* **18** (2009), no. 2, pp. 279–308.

[10] S. Boucksom, C. Favre and M. Jonsson, "Singular semipositive metrics in non-Archimedean geometry", *J. Algebr. Geom.* **25** (2015), no. 1, pp. 77–139.

[11] S. Boucksom, C. Favre and M. Jonsson, "Solution to a non-Archimedean Monge-Ampère equation", *J. Am. Math. Soc.* **28** (2015), no. 3, pp. 617–667.

[12] S. Boucksom and M. Jonsson, "Singular semipositive metrics on line bundles on varieties over trivially valued fields", 2018. https://arxiv.org/abs/1801.08229v1.

[13] S. Boucksom and M. Jonsson, "Global pluripotential theory over a trivially valued field", *Ann. Fac. Sci. Toulouse, Math.* **31** (2022), no. 3, pp. 647–836.

[14] S. Boucksom and M. Jonsson, "A non-Archimedean approach to K-stability, I: Metric geometry of spaces of test configurations and valuations", 2023. https://arxiv.org/abs/2107.11221v3, to appear in *Ann. Inst. Fourier.*

[15] S. Boucksom and M. Jonsson, "A non-Archimedean approach to K-stability, II: Divisorial stability and openness", *J. Reine Angew. Math.* **805** (2023), pp. 1–53.

[16] S. Boucksom and M. Jonsson, "Addendum to "Global pluripotential theory over a trivially valued field"", 2023. https://arxiv.org/abs/2206.07183, to appear in *Ann. Fac. Sci. Toulouse Math.*

[17] S. Boucksom, A. Küronya, C. Maclean and T. Szemberg, "Vanishing sequences and Okounkov bodies", *Math. Ann.* **361** (2015), no. 3-4, pp. 811–834.

[18] J. I. Burgos Gil, W. Gubler, P. Jell, K. Künnemann and F. Martin, "Differentiability of non-archimedean volumes and non-archimedean Monge–Ampère equations", *Algebr. Geom.* **7** (2020), no. 2, pp. 113–152.

[19] A. Chambert-Loir, "Mesures et équidistribution sur les espaces de Berkovich", *J. Reine Angew. Math.* **595** (2006), pp. 215–235.

[20] A. Chambert-Loir and A. Ducros, "Formes différentielles réelles et courants sur les espaces de Berkovich", 2012. https://arxiv.org/abs/1204.6277.

[21] S. D. Cutkosky, "Irrational asymptotic behaviour of Castelnuovo–Mumford regularity", *J. Reine Angew. Math.* **522** (2000), pp. 93–103.

[22] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, "Asymptotic invariants of base loci", *Ann. Inst. Fourier* **56** (2006), no. 6, pp. 1701–1734.

[23] W. Gubler, "Local and canonical heights of subvarieties", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **2** (2003), no. 4, pp. 711–760.

[24] W. Gubler, "A guide to tropicalizations", in *Algebraic and combinatorial aspects of tropical geometry*, American Mathematical Society, 2013, pp. 125–189.

[25] W. Gubler, P. Jell, K. Künnemann and F. Martin, "Continuity of plurisubharmonic envelopes in non-archimedean geometry and test ideals", *Ann. Inst. Fourier* **69** (2019), no. 5, pp. 2331–2376.

[26] W. Gubler and F. Martin, "On Zhang's semipositive metrics", *Doc. Math.* **24** (2019), pp. 331–372.

[27] R. Howe, "Automatic continuity of concave functions", *Proc. Am. Math. Soc.* **103** (1988), no. 4, pp. 1196–1200.

[28] J. Hultgren, M. Jonsson, E. Mazzon and N. McCleerey, "Tropical and non-Archimedean Monge–Ampère equations for a class of Calabi–Yau hypersurfaces", *Adv. Math.* **439** (2024), article no. 109494 (42 pages).

[29] M. Jonsson and M. Mustaţă, “Valuations and asymptotic invariants for sequences of ideals”, *Ann. Inst. Fourier* **62** (2012), no. 6, pp. 2145–2209.

[30] S. Kołodziej, “The complex Monge–Ampère equation”, *Acta Math.* **180** (1998), no. 1, pp. 69–117.

[31] M. Kontsevich and Y. Soibelman, “Affine structures and non-Archimedean analytic spaces”, in *The unity of mathematics*, Birkhäuser, 2006, pp. 321–385.

[32] M. Kontsevich and Y. Tschinkel, “Non-Archimedean Kähler geometry”. unpublished.

[33] A. Küronya and C. Maclean, “Zariski decomposition of b-divisors”, *Math. Z.* **273** (2013), no. 1-2, pp. 427–436.

[34] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: Line bundles and linear series*, Springer, 2014, pp. 1729–1741.

[35] J. Lesieutre, “The diminished base locus is not always closed”, *Compos. Math.* **150** (2014), no. 10, pp. 1729–1741.

[36] Y. Li, “Metric SYZ conjecture and non-archimedean geometry”, *Duke Math. J.* **172** (2023), no. 17, pp. 3227–3255.

[37] Y. Li, “Metric SYZ conjecture for certain toric Fano hypersurfaces”, *Camb. J. Math.* **12** (2024), no. 1, pp. 223–252.

[38] J. Lipman, “Rational singularities with applications to algebraic surfaces and unique factorization”, *Publ. Math., Inst. Hautes Étud. Sci.* **36** (1969), pp. 195–279.

[39] M. Mustaţă and J. Nicaise, “Weight functions on non-archimedean analytic spaces and the Kontsevich–Soibelman skeleton”, *Algebr. Geom.* **2** (2015), no. 3, pp. 365–404.

[40] N. Nakayama, *Zariski-decomposition and abundance*, Mathematical Society of Japan, 2004.

[41] M. Perera, *La décomposition de Zariski : une approche valuative*, PhD thesis, Sorbonne Université, 2021.

[42] A. Thuillier, *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov*, PhD thesis, Université de Rennes I, 2005. available at `http://tel.archives-ouvertes.fr/docs/00/04/87/50/PDF/tel-00010990.pdf`.

[43] S.-T. Yau, “On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation”, *Commun. Pure Appl. Math.* **31** (1978), pp. 339–411.

[44] X. Yuan and S.-W. Zhang, “The arithmetic Hodge Theorem for adelic line bundles”, *Math. Ann.* **367** (2017), no. 3-4, pp. 1123–1171.