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Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Miyaoka–Yau inequalities and the topological characterization of certain klt varieties

Inégalités de Miyaoka–Yau et caractérisation topologique de certaines variétés klt

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Abstract. Ball quotients, hyperelliptic varieties, and projective spaces are characterized by their Chern classes, as the varieties where the Miyaoka–Yau inequality becomes an equality. Ball quotients, Abelian varieties, and projective spaces are also characterized topologically: if a complex, projective manifold X is homeomorphic to a variety of this type, then X is itself of this type. In this paper, similar results are established for projective varieties with klt singularities that are homeomorphic to singular ball quotients, quotients of Abelian varieties, or projective spaces.

Résumé. Les quotients de boules, les variétés hyperelliptiques et les espaces projectifs sont caractérisés par leurs classes de Chern, comme les variétés pour lesquelles l'inégalité de Miyaoka–Yau devient une égalité. Les quotients de boules, les variétés abéliennes et les espaces projectifs sont aussi caractérisés topologiquement : si une variété projective complexe X est homéomorphe à une variété de ce type, alors X est elle-même de ce type. Dans cet article, des résultats similaires sont établis pour les variétés projectives avec des singularités klt qui sont homéomorphes à des quotients de boules singulières, à des quotients de variétés abéliennes, ou à des espaces projectifs.

Keywords. Miyaoka–Yau inequality, klt singularities, uniformisation, homeomorphisms.

Mots-clés. Inégalité de Miyaoka–Yau, singularités klt, uniformisation, homéomorphismes.

2020 Mathematics Subject Classification. 32Q30, 32Q26, 14E20, 14E30.

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1. Introduction

1.1. *The Miyaoka–Yau inequality for projective manifolds*

Let X be an n-dimensional complex-projective manifold and let D be any divisor on X. Recall that X is said to "satisfy the Miyaoka–Yau inequality for D" if the following Chern class inequality holds,

$$\big(2(n+1)\cdot c_2(X) - n\cdot c_1(X)^2\big)\cdot [D]^{n-2} \ge 0.$$

It is a classic fact that n-dimensional projective manifolds X whose canonical bundles are ample or trivial satisfy Miyaoka–Yau inequalities. In case of equality, the universal covers are of particularly simple form.

Theorem 1 (Ball quotients and hyperelliptic varieties). *Let X be an n-dimensional complex projective manifold.*

- *If K_X is ample, then X satisfies the Miyaoka–Yau inequality for K_X. In case of equality, the universal cover of X is the unit the ball $\mathbb{B}^n$.*
- *If K_X is trivial and D is any ample divisor, then X satisfies the Miyaoka–Yau inequality for D. In case of equality, the universal cover of X is the affine space $\mathbb{C}^n$.*

We refer the reader to [24] for a full discussion and references to the original literature.

In the Fano case, where $-K_X$ is ample, the situation is more complicated, due to the fact that the tangent bundle $\mathscr{T}_X$ and the canonical extension $\mathscr{E}_X$ need not be semistable[1]. If $\mathscr{E}_X$ is semistable, then analogous results hold, see [21, Thm. 1.3], as well as further references given there.

Theorem 2 (Projective space). *Let X be an n-dimensional projective manifold. If $-K_X$ is ample and if the canonical extension is semistable with respect to $-K_X$, then X satisfies the Miyaoka–Yau inequality for $-K_X$. In case of equality, X is isomorphic to the projective space $\mathbb{P}^n$.*

In each of the three settings, the equality cases are characterized topologically: if M is any projective manifold homeomorphic to a ball quotient, a finite étale quotient of an Abelian variety or the projective space, then M itself is biholomorphic to a ball quotient, to a finite étale quotient of an Abelian variety, or to the projective space. For ball quotients, this is a theorem of Siu [40]. The torus case is due to Catanese [6], whereas the Fano case is due to Hirzebruch–Kodaira [27] and Yau [44].

1.2. *Spaces with MMP singularities*

In general, it is rarely the case that the canonical bundle of a projective variety has a definite "sign". Minimal model theory offers a solution to this problem, at the expense of introducing singularities. It is therefore natural to extend our study from projective manifolds to projective varieties with Kawamata log terminal (= klt) singularities. For klt varieties whose canonical sheaves are ample, trivial or negative, analogues of Theorems 1 and 2 have been found in the last few years. We refer the reader to [23, Thm. 1.5] for a characterization of singular ball quotients among projective varieties with klt singularities (see Definition 8 for the notion of singular ball quotients). Characterizations of torus quotients and quotients of the projective space can be found in [33], [20, Thm. 1.2] and [21, Thm. 1.3]. In each case, we find it striking that the Chern class equalities imply that the underlying space has no worse than quotient singularities.

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[1] Recall that the canonical extension $\mathscr{E}_X$ is defined as the middle term of the exact sequence $0 \to \mathscr{O}_X \to \mathscr{E}_X \to \mathscr{T}_X \to 0$ whose extension class equals $c_1(X) \in H^1\big(X, \Omega^1_X\big)$.

1.3. *Main results of this paper*

This paper asks whether the topological characterizations of ball quotients, Abelian varieties and the projective spaces have analogues in the klt settings. Section 2 establishes a topological characterization of singular ball quotients. The main result of this section, Theorem 10, can be seen as a direct analogue of Siu's rigidity theorems.

Theorem 3 (Rigidity in the klt setting, see Theorem 10). *Let X be a singular quotient of an irreducible bounded symmetric domain and let M be a normal projective variety that is homeomorphic to X. If $\dim X \geq 2$, then, M is biholomorphic or conjugate-biholomorphic to X.*

Using somewhat different methods, Section 3 generalizes Catanese's result to the klt setting.

Theorem 4 (Varieties homeomorphic to torus quotients, see Theorem 18). *Let M be a compact complex space with klt singularities. Assume that M is bimeromorphic to a Kähler manifold. If M is homeomorphic to a singular torus quotient, then M is a singular torus quotient.*

In both cases, we find that certain Chern classes equalities are invariant under homeomorphisms.

Varieties homeomorphic to projective spaces are harder to investigate. Section 4 gives a full topological characterization of $\mathbb{P}^3$, but cannot fully solve the characterization problem in higher dimensions.

Theorem 5 (Topological $\mathbb{P}^3$, see Theorem 40). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^3$. Then, $X \cong \mathbb{P}^3$.*

However, we present some partial results that severely restrict the geometry of potential exotic varieties homeomorphic to $\mathbb{P}^n$. These allow us to show the following.

Theorem 6 ($\mathbb{Q}$-Fanos in dimension 4 and 5, see Theorem 41). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$ with $n = 4$ or $n = 5$. Then, $X \cong \mathbb{P}^n$, unless K_X is ample.*

Dedication

We dedicate this paper to the memory of Jean-Pierre Demailly. His passing is a tremendous loss to the mathematical community and to all who knew him.

Greb

When I was a PhD student, Jean-Pierre's book "Complex Analytic and Differential Geometry" was a revelation for me, as it connected the classical concepts of Complex Analysis with those of modern Complex Differential Geometry and Algebraic Geometry. This greatly shaped my mathematical interests and still influences me today. When I later got to know him during several "Komplexe Analysis" Oberwolfach meetings, I was deeply impressed by his vast knowledge of the field that he shared generously and in his kind and gentle manner, especially with younger people.

Kebekus

I first met Jean-Pierre in the late 90s, when he graciously invited me to Grenoble for my first extended research stay abroad. From the moment I arrived, I was struck by his relaxed and positive air, and by his can-do attitude towards the hardest problems. Over the years, I tried and tested his legendary patience, when he generously shared his vast knowledge with newcomers to the field, myself included. Jean-Pierre's unparalleled clarity made even the most challenging mathematical concepts accessible, and I cherished our discussions on a wide range of topics, from free software to the intricacies of French labour laws[2].

Peternell

Since the late 1980s I had an invaluable close scientific and personal contact with Jean-Pierre, with various mutual joint visits in Bayreuth and Grenoble. I will always commemorate Jean-Pierre's scientific wisdom and his great personality.

Acknowledgements

We thank Markus Banagl, Sebastian Goette, Wolfgang Lück, Jörg Schürmann and Michael Weiss for providing detailed guidance regarding Pontrjagin classes of topological manifolds. Igor Belegradek kindly answered our questions on MathOverflow. We also thank the referee, who suggested, among other improvements, to generalize the results of Section 3 to the Kähler case.

After finishing the paper we were informed by Haidong Liu that, using the recent preprint "Kawamata–Miyaoka type inequality for canonical $\mathbb{Q}$-Fano varieties" [31], instead of Ou's result cited in Proposition 4.19, Theorem 4.21 can be shown to hold also in dimensions 6 and 7.

2. Mostow Rigidity for singular quotients of symmetric domains

Consider a compact Kähler manifold X whose universal cover is a bounded symmetric domain. Siu has shown in [40, Thm. 4] and [41, Main Theorem] that any compact Kähler manifold M which is homotopy equivalent to X is biholomorphic or conjugate-biholomorphic[3] to X. We show an analogous result for homeomorphisms between *singular* varieties M and X. The following notion will be used.

Definition 7 (Quasi-étale cover). *A finite, surjective morphism between normal, irreducible complex spaces is called* quasi-étale cover *if it is unbranched in codimension one.*

Definition 8 (Singular quotient of bounded symmetric domtain). *Let Ω be an irreducible bounded symmetric domain. A normal projective variety X is called a* singular quotient of Ω *if there exists a quasi-étale cover $\widehat{X} \to X$, where $\widehat{X}$ is a smooth variety whose universal cover is Ω.*

Remark 9 (Singular quotients are quotients). Let X be a singular quotient of an irreducible bounded symmetric domain Ω. Passing to a suitable Galois closure, one finds a quasi-étale *Galois* cover $\widehat{X} \to X$, where $\widehat{X}$ is a smooth variety whose universal cover is Ω. In particular, it follows that X is a quotient variety and that it has quotient singularities. Moreover, it can be shown as in [22, §9] that X is actually a quotient of Ω by the fundamental group of X_{reg}, which acts properly discontinously on Ω. In addition, the action is free in codimension one.

Theorem 10 (Mostow rigidity in the klt setting). *Let X be a singular quotient of an irreducible bounded symmetric domain and let M be a normal projective variety that is homeomorphic to X. If* $\dim X \geq 2$, *then, M is biholomorphic or conjugate-biholomorphic to X.*

Remark 11 (Varieties conjugate-biholomorphic to ball quotients). We are particularly interested in the case where the bounded symmetric domain of Theorem 10 is the unit ball. For this, observe that the set of (singular) ball quotients is invariant under conjugation. It follows that if the variety M of Theorem 10 is biholomorphic or conjugate-biholomorphic to a (singular) ball quotient X, then M is itself a (singular) ball quotient.

[2] Solidarity strike = no food on campus because train drivers demand better working conditions

[3] See also [7, §7] and [1, Chapt. 5 and 6] as general references for the main ideas behind Siu's results and for related topics.

Before proving Theorem 10 in Sections 2.1–2.3 below, we note a first application: the Miyaoka–Yau Equality is a topological property. The symbols $\widehat{c}_\bullet(X)$ in Corollary 12 are the $\mathbb{Q}$-Chern classes of the klt space X, as defined and discussed for instance [22, §3.7].

Corollary 12 (Topological invariance of the Miyaoka–Yau equality). *Let X be a projective klt variety with K_X ample. Assume that the Miyaoka–Yau equality holds:*

$$\big(2(n+1)\cdot \widehat{c}_2(\mathscr{T}_X) - n\cdot \widehat{c}_1(\mathscr{T}_X)^2\big)\cdot [K_X]^{n-2} = 0.$$

Let M be a normal projective variety homeomorphic to X. Then M is klt, K_M is ample and

$$\big(2(n+1)\cdot \widehat{c}_2(\mathscr{T}_M) - n\cdot \widehat{c}_1(\mathscr{T}_M)^2\big)\cdot [K_M]^{n-2} = 0.$$

Proof. Since the Miyaoka–Yau Equality holds on X, there is a quasi-étale cover $\widetilde{X} \to X$ such that the universal cover of $\widetilde{X}$ is the ball, [22]. By Theorem 10, there is a quasi-étale cover $\widetilde{M} \to M$ such that $\widetilde{M} \cong \widetilde{X}$ biholomorphically or conjugate-biholomorphically. Hence, the universal cover of $\widetilde{M}$ is the ball. It follows that M is klt, K_M is ample, and that the Miyaoka–Yau Equality holds on M. □

2.1. *Preparation for the proof of Theorem 10*

The following lemma of independent interest might be well-known. We include a full proof for lack of a good reference.

Lemma 13. *Let X be a normal complex space. Then, the set $X_{\mathrm{sing,top}} \subset X$ of topological singularities is a complex-analytic set.*

Proof. Recall from [16, Thm. on p. 43] that X admits a Whitney stratification where all strata are locally closed complex-analytic submanifolds of X. Recall from [32, Chapt. IV.8] that the closures of the strata are complex-analytic subsets of X. Since Whitney stratifications are locally topologically trivial along the strata[4], it follows that $X_{\mathrm{sing,top}}$ is locally the union of finitely many strata. The additional observation that the set of topologically smooth points, $X \setminus X_{\mathrm{sing,top}}$, is open in the Euclidean topology implies that $X_{\mathrm{sing,top}}$ is locally the union of the closures of finitely many strata, hence analytic. □

2.2. *Proof of Theorem 10 if X is smooth*

We maintain the notation of Theorem 10 in this section and assume additionally that X is smooth. To begin, fix a homeomorphism $f: M \to X$ and choose a resolution of singularities, say $\pi: \widetilde{M} \to M$. The composed map $g = f \circ \pi$ is continuous and induces an isomorphism

$$g_*: H_{2n}\big(\widetilde{M}, \mathbb{Z}\big) \to H_{2n}\big(X, \mathbb{Z}\big). \tag{1}$$

Hence, by Siu's general rigidity result [40, Thm. 6] in combination with the curvature computations for the classical, respectively exceptional Hermitian symmetric domains done in [40, 41], the continuous map g is homotopic to a holomorphic or conjugate-holomorphic map $\widetilde{g}: \widetilde{M} \to X$. Replacing the complex structure on X by the conjugate complex structure, if necessary, we may assume without loss of generality that $\widetilde{g}$ is holomorphic and hence in particular algebraic. The isomorphism (1) maps the fundamental class of $\widetilde{M}$ to the fundamental class of X, and $\widetilde{g}$ is hence birational.

We claim that the bimeromorphic morphism $\widetilde{g}$ factors via π. To begin, observe that since g contracts the fibres of π and since $\widetilde{g}$ is homotopic to g, the map $\widetilde{g}$ contracts the fibres of π as well.

[4] See [16, Part I, §1.4] for a detailed discussion.

In fact, given any curve $\widetilde{C} \subset \widetilde{M}$ with $\pi(\widetilde{C})$ a point, consider its fundamental class $[\widetilde{C}] \in H_2(\widetilde{M}, \mathbb{R})$. By assumption, we find that

$$\widetilde{g}_*\big([\widetilde{C}]\big) = g_*\big([\widetilde{C}]\big) = 0 \in H_2\big(X, \mathbb{R}\big).$$

Given that X is projective, this is only possible if $\widetilde{g}(\widetilde{C})$ is a point. Since M is normal and since $\widetilde{g}$ contracts the (connected) fibres of the resolution map π, we obtain the desired factorisation of $\widetilde{g}$, as follows

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\pi"'] \arrow[rr, bend left, "\widetilde{g}"] & M \arrow[r, "\exists! \widetilde{f}"'] & X. \end{tikzcd}$$

We claim that the birational map $\widetilde{f}$ is biholomorphic.[5] By Zariski's Main Theorem, [25, V Thm. 5.2], it suffices to verify that it does not contract any curve $C \subset M$. Aiming for a contradiction, assume that there exists a curve $\widetilde{C} \subset \widetilde{M}$ whose image $C := \pi(\widetilde{C})$ is a curve in M, while $\widetilde{g}(\widetilde{C}) = \widetilde{f}(C) = (*)$ is a point in X. Let $d > 0$ be the degree of the restricted map $\pi|_{\widetilde{C}} : \widetilde{C} \to C$. Then, on the one hand,

$$f_*\big(d \cdot [C]\big) = f_*\big(\pi_*[\widetilde{C}]\big) = g_*[\widetilde{C}] = \widetilde{g}_*[\widetilde{C}] = 0 \in H_2\big(X, \mathbb{R}\big).$$

On the other hand, projectivity of M implies that $d \cdot [C]$ is a non-trivial element of $H_2\big(M, \mathbb{R}\big)$, which therefore must be mapped to a non-trivial element of $H_2\big(X, \mathbb{R}\big)$, since f is assumed to be a homeomorphism. This finishes the proof of Theorem 10 in the case where X is smooth.

2.3. *Proof of Theorem 10 in general*

Maintain the setting of Theorem 10.

Step 1: Setup

By assumption, there exists a bounded symmetric domain Ω and a quasi-étale cover $\tau_X : \widehat{X} \to X$ such that the universal cover of $\widehat{X}$ is Ω. Choose a homeomorphism $f : M \to X$ and let $\widehat{M} := \widehat{X} \times_X M$ be the topological fibre product. The situation is summarized in the following commutative diagram,

$$\begin{tikzcd} \widehat{M} \arrow[r, two heads, "\tau_M"] \arrow[d, "\simeq"'] & M \arrow[d, "\simeq"', "f"] \\ \widehat{X} \arrow[r, two heads, "\tau_X,\ \text{quasi-étale}"'] & X, \end{tikzcd} \tag{2}$$

in which the vertical maps are homeomorphisms and the horizontal maps are surjective with finite fibres.

Step 2: A complex structure on $\widehat{M}$

The spaces M, $\widehat{X}$ and X all carry complex structures. We aim to equip $\widehat{M}$ with a structure so that all horizontal arrows in (2) become holomorphic.

Claim 14. There exists a normal complex structure on $\widehat{M}$ that makes τ_M a finite, holomorphic, and quasi-étale cover.

Proof of Claim 14. Let X_0 be the smooth locus of X, set $M_0 := f^{-1}(X_0)$ and $\widehat{M}_0 := \tau_M^{-1}(M_0)$. The map $\tau_M|_{\widehat{M}_0}$ being a local homeomorphism, there is a uniquely determined complex structure on $\widehat{M}_0$ such that $\tau_M|_{\widehat{M}_0} : \widehat{M}_0 \to M_0$ is a finite holomorphic cover. Since X has quotient singularities,

[5] Cf. [7, Rem. 86(2)]

the topological and holomorphic singularities agree, $X_{\mathrm{sing,top}} = X_{\mathrm{sing}}$. Hence, f being a homeomorphism, we note that

$$M_{\mathrm{sing,top}} = f^{-1}\left(X_{\mathrm{sing}}\right) \quad \text{and} \quad M \setminus M_0 = M_{\mathrm{sing,top}}.$$

We have seen in Lemma 13 that $M_{\mathrm{sing,top}}$ is an analytic set. Therefore, by [9, Thm. 3.4] and [42, Satz 1], the complex structure on $\widehat{M}_0$ uniquely extends to a normal complex structure on the topological manifold $\widehat{M}$, making τ_M holomorphic and finite. The branch locus of τ_M has the same topological dimension as the branch locus of τ_X, so that τ_M is quasi-étale, as claimed. □

Note that as a finite cover of the projective variety M, the normal complex space $\widehat{M}$ is again projective.

Step 3: $\widehat{M}$ as a quotient of Ω

The homeomorphic varieties $\widehat{X}$ and $\widehat{M}$ reproduce the assumptions of Theorem 10. The partial results of Section 2.2 therefore apply to show that the complex spaces $\widehat{M}$ and $\widehat{X}$ are biholomorphic or conjugate-biholomorphic. Replacing the complex structures on M and $\widehat{M}$ by their conjugates, if necessary, we assume without loss of generality for the remainder of this proof that $\widehat{M}$ and $\widehat{X}$ are biholomorphic. This has two consequences.

(1) The projective variety $\widehat{M}$ is smooth. The universal cover of $\widehat{M}$ is biholomorphic to Ω.
(2) Its quotient M is a singular quotient of Ω and has only quotient singularities.

Recalling that quotient singularities are not topologically smooth, Item (2) implies that the homeomorphism $f : M \to X$ restricts to a homeomorphism between the smooth loci, X_{reg} and M_{reg}. The situation is summarized in the following commutative diagram,

$$\begin{tikzcd}
\Omega \arrow[r, two heads, "u_X\text{, univ. cover}"] \arrow[d, "\simeq"'] & \widehat{M} \arrow[r, two heads, "\tau_M\text{, quasi-étale}"] \arrow[d, "\simeq"'] & M \arrow[d, "\simeq"', "f"] & M_{\mathrm{reg}} \arrow[l, hook', "\text{inclusion}"'] \arrow[d, "\simeq"', "f|_{X_{\mathrm{reg}}}"] \\
\Omega \arrow[r, two heads, "u_M\text{, univ. cover}"'] & \widehat{X} \arrow[r, two heads, "\tau_X\text{, quasi-étale}"'] & X & X_{\mathrm{reg}}, \arrow[l, hook', "\text{inclusion}"]
\end{tikzcd}$$

where all horizontal maps are holomorphic, and all vertical maps are homeomorphic.

The description of M as a singular quotient of Ω can be made precise. The argument in [22, §9.1] shows that the fundamental group $\pi_1(M_{\mathrm{reg}})$ acts properly discontinously on Ω with quotient M. In particular, we have an injective homomorphism from $\pi_1(M_{\mathrm{reg}})$ into the holomorphic automorphism group $\mathrm{Aut}(\Omega)$ of Ω, with image a discrete cocompact subgroup $\Gamma_M \subseteq \mathrm{Aut}(\Omega)$.

The same reasoning also applies to X and presents X as a quotient $X = \Omega/\pi_1(X_{\mathrm{reg}})$, where $\pi_1(X_{\mathrm{reg}})$ again acts via an injective homomorphism $\pi_1(X_{\mathrm{reg}}) \hookrightarrow \mathrm{Aut}(\Omega)$, with image a cocompact, discrete subgroup Γ_X of $\mathrm{Aut}(\Omega)$.

As we have seen above, f induces a homeomorphism from M_{reg} to X_{red}, from which we obtain an abstract group isomorphism $\theta : \Gamma_M \to \Gamma_X$.

Step 4: End of proof

In the remainder of the proof we will show that not only $\widehat{M}$ and $\widehat{X}$ are (conjugate-)-biholomorphic, but that this actually holds for M and X. This will be a consequence of Mostow's rigidity theorem for lattices in connected semisimple real Lie groups. As the groups appearing in our situation are not necessarily connected, we have to do some work to reduce to the connected case[6].

Given that Ω is an irreducible Hermitian symmetric domain of dimension greater than one, the identity component $\mathrm{Aut}^\circ(\Omega) \subseteq \mathrm{Aut}(\Omega)$ coincides with the identity component $I^\circ(\Omega)$ of the

[6]Alternatively, one could trace the finite group actions through the proof of the results used in Section 2.2.

isometry group $I(\Omega)$ of the Riemannian symmetric space Ω, [26, VIII.Lem. 4.3][7], which is a non-compact simple Lie group without non-trivial proper compact normal subgroups and with trivial centre, [10, Prop. 2.1.1 and bottom of p. 379]. We also note that a Bergman-metric argument shows that $\mathrm{Aut}(\Omega)$ is contained in $I(\Omega)$. Furthermore, both Lie groups have only finitely many connected components.

Claim 15. There exists an isometry $F \in I(\Omega)$ such that

$$F \circ \gamma = \theta(\gamma) \circ F, \quad \text{for every } \gamma \in \Gamma_M. \tag{3}$$

Proof of Claim 15. If the rank of Ω is equal to one, then $\Omega \cong \mathbb{B}_n$, the unit ball in $\mathbb{B}^n$, see [26, §X.6.3/4]. Consequently, the group $\mathrm{Aut}(\Omega)$ is connected, and we may apply [34, Thm. A′ on p. 4] to obtain an automorphism of real Lie groups, $\Theta : \mathrm{Aut}(\Omega) \to \mathrm{Aut}(\Omega)$ such that $\Theta|_{\Gamma_M} = \theta$. The desired isometry is then produced by an application of [10, Prop. 3.9.11].

We consider the case $\mathrm{rank}(\Omega) \geq 2$ for the remainder of the present proof, where the automorphism group may be non-connected. To deal with this slight difficulty, we proceed as in [10, p. 379]: as $\mathrm{Aut}(\Omega)$ has finitely many connected components, we may assume that the subgroups $\Gamma_{\widehat{M}} \subseteq \Gamma_M$ and $\Gamma_{\widehat{X}} \subseteq \Gamma_X$ corresponding to the deck transformation groups of u_M and u_X, respectively, are contained in the identity component $I^\circ(\Omega) = \mathrm{Aut}^\circ(\Omega)$. Again, apply [34, Thm. A′ on p. 4] to obtain an automorphism of real Lie groups $\Theta : I^\circ(\Omega) \to I^\circ(\Omega)$ such that $\Theta|_{\Gamma_{\widehat{M}}} = \theta|_{\Gamma_{\widehat{M}}}$ and then [10, Prop. 3.9.11] to obtain an isometry $F \in I(\Omega)$ such that

$$F \circ g = \Theta(g) \circ F, \quad \text{for every } g \in I^\circ(\Omega).$$

This in particular yields (3) for all γ contained in the finite index subgroup $\Gamma_{\widehat{M}}$ of Γ_M. This is not yet enough.

However, noticing that for any finite index subgroup $\Gamma'_M < \Gamma_M$, every Γ'_M-periodic vector in the sense of [10, Def. 4.5.13] by definition is also Γ_M-periodic, we see with the argument given in [10, p. 379], which uses essentially the same notation as we have introduced here, that the set of Γ_M-periodic vectors is dense in the unit sphere bundle $S\Omega$ of Ω. The subsequent argument in [10, bottom of p. 379 and upper part of p. 380] then applies verbatim to yield the desired relation (3) for all $\gamma \in \Gamma_M$; this is [10, equation (5) on p. 380]. □

Now, since the Hermitian symmetric domain Ω is assumed to be irreducible, the Γ-equivariant isometry $F \in I(\Omega)$ is either holomorphic or conjugate-holomorphic, as follows for example from [5] together with [26, VIII.Prop. 4.2]. By the universal property of the quotient map π with respect to Γ-invariant holomorphic maps, F hence descends to a holomorphic or conjugate-holomorphic isomorphism from M to X. This completes the proof of Theorem 10.

3. Topological characterization of torus quotients

In line with the results of Section 2, we show that a Kähler space with klt singularities is a singular torus quotient if and only if it is homeomorphic to a singular torus quotient. In the smooth case, this was shown by Catanese [6], but see also [3, Thm. 2.2]. The following notion is a direct analogue of Definition 8 above.

Definition 16 (Singular torus quotient). *A normal complex space X is called a* singular torus quotient *if there exists a quasi-étale cover $\widehat{X} \to X$, where $\widehat{X}$ is a compact complex torus.*

Remark 17 (Singular torus quotients are quotients). Let X be a singular torus quotient. Passing to a suitable Galois closure, one finds a quasi-étale *Galois* cover $\widehat{X} \to X$, where $\widehat{X}$ is a compact torus.

[7] As Ω is irreducible, the compatible Riemannian metric on Ω is unique up to a positive real multiple that does not change the isometry group.

Theorem 18 (Varieties homeomorphic to torus quotients). *Let M be a compact complex space with klt singularities. Assume that M is bimeromorphic to a Kähler manifold. If M is homeomorphic to a singular torus quotient, then M is a singular torus quotient.*

Theorem 18 will be shown in Sections 3.1–3.2 below. In analogy to Corollary 12 above, we note that vanishing of $\mathbb{Q}$-Chern classes is a topological property among compact Kähler spaces with klt singularities.

Corollary 19 (Topological invariance of vanishing Chern classes). *Let X be a compact Kähler space with klt singularities. Assume that the canonical class vanishes numerically, $K_X \equiv 0$, and that the second $\mathbb{Q}$-Chern class of $\mathscr{T}_X$ satisfies*

$$\widehat{c}_2(\mathscr{T}_X) \cdot \alpha_1 \dots \alpha_{\dim X - 2} = 0,$$

for every $(\dim X - 2)$-tuple of Kähler classes on X. If M is any compact Kähler space with klt singularities that is homeomorphic to X, then $K_M \equiv 0$, and the second $\mathbb{Q}$-Chern class of $\mathscr{T}_M$ satisfies

$$\widehat{c}_2(\mathscr{T}_M) \cdot \beta_1 \dots \beta_{\dim M - 2} = 0,$$

for every $(\dim X - 2)$-tuple of Kähler classes on M.

Proof. The characterization of singular torus quotients in terms of Chern classes by Claudon, Graf and Guenancia, [8, Cor. 1.7], guarantees that X is a torus quotient[8]. By Theorem 18, then so is M. □

3.1. *Proof of Theorem 18 if M is homeomorphic to a torus*

As before, we prove Theorem 18 first in case where the (potentially singular) space M is homeomorphic to a torus. Recalling that klt singularities are rational, see [30, Thm. 5.22] for the algebraic case and [11, Thm. 3.12] (together with the vanishing theorems proven in [12]) for the analytic case, we show the following, slightly stronger statement.

Proposition 20. *Let M be a compact complex space with rational singularities. Assume that M is bimeromorphic to a Kähler manifold. If M is homotopy equivalent to a compact torus, then M is a compact torus.*

Proof. We follow the arguments of Catanese, [6, Thm. 4.8], and choose a resolution of singularities, $\pi : \widetilde{M} \to M$, which owing to the assumptions on M we may assume to be a compact Kähler manifold. Using the assumption that M has rational singularities together with the push-forward of the exponential sequence, we observe that the pull-back map $H^1(M, \mathbb{Z}) \to H^1(\widetilde{M}, \mathbb{Z})$ is an isomorphism. In particular, first Betti numbers of M and $\widetilde{M}$ agree. As a next step, consider the Albanese map of $\widetilde{M}$, observing that $\widetilde{M}$ is bimeromorphic to a Kähler manifold since M is. Again using that M has rational singularities, recall from [38, Prop. 2.3] that the Albanese factors via M,

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\mathrm{alb}"] \arrow[d, twoheadrightarrow, "{\pi,\ \mathrm{resolution}}"'] & \mathrm{Alb}. \\ M \arrow[ur, bend right, "\alpha"'] & \end{tikzcd}$$

Since the pull-back morphisms

$$\begin{aligned} \mathrm{alb}^* = \pi^* \circ \alpha^* &: H^1(\mathrm{Alb}, \mathbb{Z}) \to H^1(\widetilde{M}, \mathbb{Z}) \\ \pi^* \quad &: H^1(M, \mathbb{Z}) \to H^1(\widetilde{M}, \mathbb{Z}) \end{aligned}$$

[8] See [33, Thm. 1.2] for the projective case and see [19, Thm. 1.17] for the case where X is projective and smooth in codimension two.

are both isomorphic, we find that $\alpha^* : H^1(\mathrm{Alb}, \mathbb{Z}) \to H^1(M, \mathbb{Z})$ must likewise be an isomorphism. There is more that we can say. Since the topological cohomology ring of a torus is an exterior algebra,

$$H^*(\mathrm{Alb}, \mathbb{Z}) = \wedge^* H^1(\mathrm{Alb}, \mathbb{Z}) \quad \text{and} \quad H^*(M, \mathbb{Z}) = \wedge^* H^1(M, \mathbb{Z}),$$

we find that all pull-back morphisms are isomorphisms,

$$\alpha^* : H^q(\mathrm{Alb}, \mathbb{Z}) \xrightarrow{\cong} H^q(M, \mathbb{Z}), \quad \text{for every } 0 \le q \le 2 \cdot \dim M.$$

Applying this to $q = 2 \cdot \dim M$, we see α is surjective of degree one, hence birational. Again, more is true: if α failed to be isomorphic, Zariski's Main Theorem would guarantee that α contracts a positive-dimensional subvariety, so $b_2(M) > b_2(\mathrm{Alb})$. But we have seen above that equality holds and hence reached a contradiction. □

3.2. *Proof of Theorem 18 in general*

By assumption, there exists a homeomorphism $f : M \to X$, where X is a singular torus quotient. Choose a quasi-étale cover $\tau_X : \widehat{X} \to X$, where $\widehat{X}$ is a complex torus, and proceed as in the proof of Theorem 10, in order to construct a diagram of continuous mappings between normal complex spaces,

$$\begin{tikzcd} \widehat{M} \arrow[r, two heads, "\tau_M\text{, quasi-étale}"] \arrow[d, "\cong"'] & M \arrow[d, "\cong"', "f"] \\ \widehat{X} \arrow[r, two heads, "\tau_X\text{, quasi-étale}"'] & X, \end{tikzcd}$$

where

- the vertical maps are homeomorphisms, and
- the horizontal maps are holomorphic, surjective, and finite.

Since M is bimeromorphic to a Kähler manifold, so is $\widehat{M}$. Recalling from [30, Prop. 5.20] that also $\widehat{M}$ has no worse than klt singularities, Proposition 20 will then guarantee that $\widehat{M}$ is a complex torus, as claimed.

4. Rigidity results for projective spaces

Recall the classical theorem of Hirzebruch–Kodaira, which asserts that the projective space carries a unique structure as a Kähler manifold.

Theorem 21 (Rigidity of the projective space, [27, p. 367]). *Let X be a compact Kähler manifold. If X is homeomorphic to $\mathbb{P}^n$, then X is biholomorphic to $\mathbb{P}^n$.*

Remark 22. Strictly speaking, Hirzebruch–Kodaira proved a somewhat weaker result: X is biholomorphic to $\mathbb{P}^n$ if either n is odd, or if n is even and $c_1(X) \neq -(n+1)\cdot g$, where g is a generator of $H^2(X, \mathbb{Z})$ and the fundamental class of a Kähler metric on X. The second case was later ruled out by Yau's solution to the Calabi conjecture, which implies that then the universal cover of X is the ball, contradicting $\pi_1(X) = 0$.

Since the topological invariance of the Pontrjagin classes, [35], was not known at that time, Hirzebruch–Kodaira also had to assume that X is diffeomorphic to $\mathbb{P}^n$ rather than merely homeomorphic.

We ask whether an analogue of Hirzebruch–Kodaira's theorem remains true in the context of minimal model theory.

Question 23. *Let X be a projective variety with klt singularities. Assume that X is homeomorphic to $\mathbb{P}^n$. Is X then biholomorphic to $\mathbb{P}^n$?*

4.1. *Varieties homeomorphic to projective space*

We do not have a full answer to Question 23. The following proposition will, however, restrict the geometry of potential varieties substantially. It will later be used to answer Question 23 in a number of special settings.

Proposition 24 (Varieties homeomorphic to $\mathbb{P}^n$). *Let X be a projective klt variety. If X is homeomorphic to $\mathbb{P}^n$, then the following holds.*

(1) *We have $H^q\big(X, \mathscr{O}_X\big) = 0$ for every $1 \le q$.*
(2) *The Chern class map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism.*
(3) *The variety X is smooth in codimension two.*
(4) *The maps $r_q : H^q\big(X, \mathbb{Z}\big) \to H^q\big(X_{\mathrm{reg}}, \mathbb{Z}\big)$ are isomorphic, for every $0 \le q \le 4$. The same statement holds for $\mathbb{Z}_2$ coefficients.*
(5) *Every Weil divisor on X is Cartier, i.e., X is factorial. In particular, X is Gorenstein.*
(6) *The canonical divisor K_X is ample or anti-ample.*

Proof. We prove the items of Proposition 24 separately.

Item (1). This is a consequence of the rationality of the singularities of X and the isomorphisms $H^q\big(X, \mathbb{C}\big) \simeq H^q\big(\mathbb{P}^n, \mathbb{C}\big)$. In fact, since X has rational singularities, the morphisms

$$\varphi_q : H^q\big(X, \mathbb{C}\big) \to H^q\big(X, \mathscr{O}_X\big)$$

induced by the canonical inclusion $\mathbb{C} \to \mathscr{O}_X$, are surjective, [29, Thm. 12.3]. If q is odd, this already implies that $H^q(X, \mathscr{O}_X) = 0$. If q is even, it suffices to note that φ_q has a non-trivial kernel. For this, choose an ample line bundle $\mathscr{L} \in \mathrm{Pic}(X)$ and observe that

$$\varphi_q\big(c_1(\mathscr{L})^{q/2}\big) = 0 \in H^q\big(X, \mathscr{O}_X\big).$$

To prove the observation, pass to a desingularisation and use the Hodge decomposition there.

Item (2). The description of c_1 follows from (1) and the exponential sequence.

Item (3). Recall that klt varieties have quotient singularities in codimension two, [18, Prop. 9.3]. Smoothness follows because quotient singularities have non-trivial local fundamental groups and are hence not topologically smooth.

Item (4). We describe the relevant cohomology groups in terms of Borel-Moore homology, [4], and also refer to the reader to [15, §19.1] for a summary of the relevant facts (over $\mathbb{Z}$). The assumption that X is homeomorphic to an oriented, connected, real manifold implies that singular cohomology and Borel-Moore homology agree, [4, Thm. 7.6] and [15, p. 371]. The same holds for the non-compact manifold X_{reg}, i.e., for $R = \mathbb{Z}, \mathbb{Z}_2$ we have

$$H^q\big(X, R\big) = H^{BM}_{2\cdot n-q}\big(X, R\big) \quad \text{and} \quad H^q\big(X_{\mathrm{reg}}, R\big) = H^{BM}_{2\cdot n-q}\big(X_{\mathrm{reg}}, R\big), \quad \text{for every } q.$$

The isomorphisms identify the restriction maps r_q with the pull-back maps for Borel-Moore homology. These feature in the localization sequence for Borel-Moore homology, [4, Thm.3.8],

$$\dots \to H^{BM}_{2\cdot n-q}\big(X_{\mathrm{sing}}, R\big) \to H^{BM}_{2\cdot n-q}\big(X, R\big) \xrightarrow{r_q} H^{BM}_{2\cdot n-q}\big(X_{\mathrm{reg}}, R\big) \to H^{BM}_{2\cdot n-q-1}\big(X_{\mathrm{sing}}, R\big) \to \dots$$

Recalling from [15, Lem. 19.1.1] that $H^{BM}_i\big(X_{\mathrm{sing}}, \mathbb{Z}\big) = 0$ for every $i > 2 \cdot \dim_{\mathbb{C}} X_{\mathrm{sing}}$ and noticing that the inductive argument employed in the proof also works for $\mathbb{Z}_2$-coefficients, the claim of Item (4) thus follows from smoothness in codimension two, Item (3).

Item (5). Remaining in the analytic category, writing down the exponential sequences for X and X_{reg},

$$\begin{tikzcd} H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, "r_1"] & H^1(X,\mathscr{O}_X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r, "c_1"] \arrow[d, hook] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, "r_2"] & H^2(X,\mathscr{O}_X) \arrow[d] \\ H^1(X_{\mathrm{reg}},\mathbb{Z}) \arrow[r] & H^1(X_{\mathrm{reg}},\mathscr{O}_{X_{\mathrm{reg}}}) \arrow[r] & \mathrm{Pic}(X_{\mathrm{reg}}) \arrow[r, "c_1"'] & H^2(X_{\mathrm{reg}},\mathbb{Z}) \arrow[r] & H^2(X_{\mathrm{reg}},\mathscr{O}_{X_{\mathrm{reg}}}), \end{tikzcd}$$

and filling in what we already know, we find a commutative diagram with exact rows, as follows,

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r, hook, two heads, "c_1,\ \text{iso.}"] \arrow[d, hook] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, hook, two heads, "r_2,\ \text{iso.}"] & 0 \\ 0 \arrow[r] & H^1(X_{\mathrm{reg}},\mathscr{O}_{X_{\mathrm{reg}}}) \arrow[r] & \mathrm{Pic}(X_{\mathrm{reg}}) \arrow[r, two heads, "c_1"'] & H^2(X_{\mathrm{reg}},\mathbb{Z}) \arrow[r] & 0. \end{tikzcd}$$

The snake lemma now asserts that

$$H^1\big(X_{\mathrm{reg}},\mathscr{O}_{X_{\mathrm{reg}}}\big) \cong {}^{\mathrm{Pic}(X_{\mathrm{reg}})}\!/_{\mathrm{Pic}(X)}. \tag{4}$$

We claim that $H^1\big(X_{\mathrm{reg}},\mathscr{O}_{X_{\mathrm{reg}}}\big)$ vanishes. For this, recall that the singularities of X are rational, so every local ring $\mathscr{O}_{X,x}$ of the (holomorphic) structure sheaf has depth equal to n. Since the singular set of X has codimension at least 3 in X by Item (3), we may apply [39, §5, Korollar after Satz III] or alternatively [2, Chap. II, Cor. 3.9 and Thm. 3.6] to see that the restriction homomorphism

$$H^1\big(X,\mathscr{O}_X\big) \to H^1\big(X_{\mathrm{reg}},\mathscr{O}_{X_{\mathrm{reg}}}\big)$$

is bijective. However, the cohomology group on the left side was shown to vanish in Item (1) above.

In summary, we find that every invertible sheaf on X_{reg} extends to an invertible sheaf on X. If $D \in \mathrm{Div}(X)$ is any Weil divisor, the invertible sheaf $\mathscr{O}_{X_{\mathrm{reg}}}(D)$ will therefore extend to an invertible sheaf on X, which necessarily equals the (reflexive) Weil divisorial sheaf $\mathscr{O}_X(D)$. It follows that D is Cartier. This applies in particular to the canonical divisor, so X is $\mathbb{Q}$-Gorenstein of index one. Since X is Cohen–Macaulay, we conclude that X is Gorenstein.

Item (6). Given that $\mathrm{Pic}(X) = \mathbb{Z}$, every line bundle is ample, anti-ample, or trivial; we need to exclude the case that K_X is trivial. But if K_X were trivial, use that X is Gorenstein and apply Serre duality to find

$$h^n\big(X,\mathscr{O}_X\big) = h^0\big(X,\omega_X\big) = h^0\big(X,\mathscr{O}_X\big) = 1.$$

This contradicts Item (1) above. □

Notation 25 (Line bundles on varieties homeomorphic to $\mathbb{P}^n$). If X is a projective klt variety that is homeomorphic to $\mathbb{P}^n$, Item (2) shows the existence of a unique ample line bundle that generates $\mathrm{Pic}(X) \cong \mathbb{Z}$. We refer to this line bundle as $\mathscr{O}_X(1)$. Item (5) equips us with a unique number $r \in \mathbb{N}$ and such that $\omega_X \cong \mathscr{O}_X(r)$. Item (6) guarantees that $r \neq 0$.

Remark 26 (Pull-back of line bundles). The cohomology rings of X and $\mathbb{P}^n$ are isomorphic. If $\phi : X \to \mathbb{P}^n$ is any homeomorphism, then $\phi^* c_1\big(\mathscr{O}_{\mathbb{P}^n}(1)\big) = c_1\big(\mathscr{O}_X(\pm 1)\big)$. The cup products $c_1\big(\mathscr{O}_X(1)\big)^q$ generate the groups $H^{2q}\big(X,\mathbb{Z}\big) \cong \mathbb{Z}$.

4.2. *Characteristic classes*

We have seen in Proposition 24 that X is smooth away from a closed set of codimension ≥ 3. This allows defining a number of characteristic classes.

Notation 27 (Chern classes on varieties homeomorphic to $\mathbb{P}^n$). If X is a projective klt variety that is homeomorphic to $\mathbb{P}^n$, Item (4) allows defining first and second Chern classes, as well as a first Pontrjagin class and a second Stiefel–Whitney class

$$\begin{aligned} c_1(X) &= r_2^{-1} c_1(X_{\mathrm{reg}}) \in H^2\big(X, \mathbb{Z}\big) \\ c_2(X) &= r_4^{-1} c_2(X_{\mathrm{reg}}) \in H^4\big(X, \mathbb{Z}\big) \\ p_1(X) &= r_4^{-1} p_1(X_{\mathrm{reg}}) \in H^4\big(X, \mathbb{Z}\big) \\ w_2(X) &= r_2^{-1} w_2(X_{\mathrm{reg}}) \in H^2\big(X, \mathbb{Z}_2\big). \end{aligned}$$

Remark 28 (Pontrjagin and Chern classes). If X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$, the restriction maps $r_\bullet : H^\bullet\big(X, \mathbb{Z}\big) \to H^\bullet\big(X_{\mathrm{reg}}, \mathbb{Z}\big)$ commute with the cup products on X and X_{reg}, which implies in particular that

$$p_1(X) = r_4^{-1} p_1(X_{\mathrm{reg}}) = r_4^{-1}\Big(c_1(X_{\mathrm{reg}})^2 - 2\cdot c_2(X_{\mathrm{reg}})\Big) = c_1(X)^2 - 2\cdot c_2(X) \in H^4\big(X, \mathbb{Z}\big).$$

Remark 29 (Stiefel–Whitney class and first Chern class). By definition and the well-known relation in the smooth case, we have

$$w_2(X) = c_1(X) \bmod 2.$$

Novikov's result on the topological invariance of Pontrjagin classes extends to the generalized Pontrjagin class defined in Notation 27.

Proposition 30 (Topological invariance of Pontrjagin classes). *Let X be a projective klt variety. If $\phi : X \to \mathbb{P}^n$ is any homeomorphism, then $\phi^* p_1(\mathbb{P}^n) = p_1(X)$ in $H^4\big(X, \mathbb{Z}\big)$.*

Proof. Consider the open set $\mathbb{P}^n_{\mathrm{reg}} := \phi(X_{\mathrm{reg}})$ and the restricted homeomorphism $\phi_{\mathrm{reg}} : X_{\mathrm{reg}} \to \mathbb{P}^n_{\mathrm{reg}}$. Recalling from Item (4) of Propositions 24 that the restriction maps

$$r_4 : H^4\big(X, \mathbb{Z}\big) \to H^4\big(X_{\mathrm{reg}}, \mathbb{Z}\big) \quad \text{and} \quad r_4 : H^4\big(\mathbb{P}^n, \mathbb{Z}\big) \to H^4\big(\mathbb{P}^n_{\mathrm{reg}}, \mathbb{Z}\big)$$

are isomorphic, it suffices to show that the restricted classes in rational cohomology agree. More precisely,

$$\begin{aligned} & \phi^* p_1(\mathbb{P}^n) = p_1(X) && \text{in } H^4\big(X, \mathbb{Z}\big) \\ \iff\ & r_4\phi^* p_1(\mathbb{P}^n) = r_4 p_1(X) && \text{in } H^4\big(X_{\mathrm{reg}}, \mathbb{Z}\big)\text{, since } r_4\text{'s are iso.} \\ \iff\ & \phi^*_{\mathrm{reg}} p_1(\mathbb{P}^n_{\mathrm{reg}}) = p_1(X_{\mathrm{reg}}) && \text{in } H^4\big(X_{\mathrm{reg}}, \mathbb{Z}\big)\text{, definition, functoriality} \\ \iff\ & \phi^*_{\mathrm{reg}} p_1(\mathbb{P}^n_{\mathrm{reg}}) = p_1(X_{\mathrm{reg}}) && \text{in } H^4\big(X_{\mathrm{reg}}, \mathbb{Q}\big)\text{, since } H^4\big(X_{\mathrm{reg}}, \mathbb{Z}\big) = \mathbb{Z} \end{aligned}$$

The last equation is Novikov's famous topological invariance of Pontrjagin classes, [35][9]. □

Corollary 31 (Relation between Chern classes on varieties homeomorphic to $\mathbb{P}^n$). *If X is a projective klt variety that is homeomorphic to $\mathbb{P}^n$, then*

$$2\cdot c_2(X) = \big[r^2 - (n+1)\big]\cdot c_1\big(\mathscr{O}_X(1)\big)^2 \quad \textit{in } H^4\big(X, \mathbb{Z}\big).$$

Proof. Choose a homeomorphism $\phi : X \to \mathbb{P}^n$, in order to compare the Pontrjagin class of $\mathbb{P}^n$ with that of X.

$$\begin{aligned} & p_1(\mathbb{P}^n) = (n+1)\cdot c_1\big(\mathscr{O}_{\mathbb{P}^n}(1)\big)^2 && \text{in } H^4\big(\mathbb{P}^n, \mathbb{Z}\big) \\ \iff\ & \phi^* p_1(\mathbb{P}^n) = (n+1)\cdot \phi^* c_1\big(\mathscr{O}_{\mathbb{P}^n}(1)\big)^2 && \text{in } H^4\big(X, \mathbb{Z}\big) \\ \iff\ & p_1(X) = (n+1)\cdot c_1\big(\mathscr{O}_X(\pm 1)\big)^2 && \text{Prop. 30 and Rem. 26} \\ \iff\ & c_1\big(\mathscr{O}_X(r)\big)^2 - 2\cdot c_2(X) = (n+1)\cdot c_1\big(\mathscr{O}_X(1)\big)^2 && \text{Rem. 28} \end{aligned}$$

The claim thus follows. □

[9] See [17, Thm. 0] for the precise result used here and see [37, Appendix] for a history of the result. Igor Belegradek explains on MathOverflow (https://mathoverflow.net/q/442025) why compactness assumptions are not required.

Corollary 31 allows reformulating the $\mathbb{Q}$-Miyaoka–Yau inequality and $\mathbb{Q}$-Bogomolov–Gieseker inequality as inequalities between the index r and the dimension n. The first remark will be relevant for varieties of general type, whereas the second one will be used for Fano varieties.

Remark 32 (Reformulation of the $\mathbb{Q}$-Miyaoka–Yau inequality). Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. Since X is smooth in codimension two, the Miyaoka–Yau inequality for $\mathbb{Q}$-Chern classes,

$$\big(2(n+1)\cdot \widehat{c}_2(X) - n\cdot \widehat{c}_1(X)^2\big)\cdot [H]^{n-2} \geq 0, \quad \text{for one ample } H,$$

is equivalent to the assertion that there exists a non-negative constant $c \in \mathbb{R}^{\geq 0}$ such that

$$\begin{aligned} & \big(2(n+1)\cdot c_2(X) - n\cdot c_1(X)^2\big) \geq c\cdot c_1\big(\mathscr{O}_X(1)\big)^2 \quad \text{in } H^4\big(X, \mathbb{Z}\big) \\ \iff \quad & \big((n+1)(r^2-(n+1)) - n\cdot r^2\big)\cdot c_1\big(\mathscr{O}_X(1)\big)^2 \geq c\cdot c_1\big(\mathscr{O}_X(1)\big)^2 \quad \text{Cor. 31} \\ \iff \quad & \big(r^2-(n+1)^2\big)\cdot c_1\big(\mathscr{O}_X(1)\big)^2 \geq c\cdot c_1\big(\mathscr{O}_X(1)\big)^2 \\ \iff \quad & |r| \geq n+1. \end{aligned}$$

The Miyaoka–Yau inequality is an equality if and only if $|r| = n+1$.

Remark 33 (Reformulation of the $\mathbb{Q}$-Bogomolov–Gieseker inequality). Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. Since X is smooth in codimension two, the Bogomolov–Gieseker inequality for $\mathbb{Q}$-Chern classes,

$$\big(2n\cdot \widehat{c}_2(X) - (n-1)\cdot \widehat{c}_1(X)^2\big)\cdot [H]^{n-2} \geq 0, \quad \text{for one (equiv. every) ample } H,$$

is equivalent to the assertion that $|r| > n$.

We will also need the topological invariance of the second Stiefel–Whitney class w_2.

Proposition 34 (Topological invariance of the second Stiefel–Whitney class). *Let X be a projective klt variety. If $\phi : X \to \mathbb{P}^n$ is any homeomorphism, then $\phi^* w_2(\mathbb{P}^n) = w_2(X)$ in $H^2\big(X, \mathbb{Z}/2\mathbb{Z}\big)$.*

Proof. We can argue as in the proof of Proposition 30, replacing Novikov's Theorem by the corresponding invariance result for Stiefel–Whitney classes due to Thom, [43, Thm. III.8]. □

Corollary 35 (Parity of the first Chern class of varieties homeomorphic to $\mathbb{P}^n$). *If X is a projective klt variety that is homeomorphic to $\mathbb{P}^n$, then $r-(n+1)$ is even.*

Proof. This follows from the topological invariance established just above together with Remark 29 and the relation $\varphi^*(c_1(\mathscr{O}_{\mathbb{P}^n}(1))) = c_1(\mathscr{O}_X(\pm 1))$. □

4.3. *Partial answers to Question 23*

We conclude the present Section 4 with three partial answers to Question 23: for threefolds, we answer Question 23 in the affirmative. In dimension four and five, we give an affirmative answer for Fano manifolds. In higher dimensions, we can at least describe and restrict the geometry of potential exotic klt varieties homeomorphic to $\mathbb{P}^n$.

Proposition 36 (Topological $\mathbb{P}^n$ with ample canonical bundle). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. If K_X is ample, then $r > n+1$.*

Proof. Recall from [22, Thm. 1.1] that X satisfies the $\mathbb{Q}$-Miyaoka–Yau inequality. We have seen in Remark 32 that this implies $r = |r| \geq n+1$, with $r = n+1$ if and only if equality holds in $\mathbb{Q}$-Miyaoka–Yau inequality. In the latter case, recall from [22, Thm. 1.2] that X has no worse than quotient singularities. Since quotient singularities are not topologically smooth, it turns out that X cannot have any singularities at all. By Yau's theorem (or again by [22, Thm. 1.2]), X must then be a smooth ball quotient, contradicting $\pi_1(X) = \pi_1(\mathbb{P}_n) = \{1\}$. □

Proposition 37 (Topological $\mathbb{P}^n$ with ample anti-canonical bundle). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. If $-K_X$ is ample, then either $X \cong \mathbb{P}^n$ or $\mathscr{T}_X$ is unstable.*

Remark 38. Recall from [28, Cor. 32] that Fano varieties with unstable tangent bundles admit natural sequences of rationally connected foliations. These might be used to study their geometry further. If in the situation of Proposition 37 we additionally assume that the index is one, i.e., that $r = -1$, then $\Omega_X^{[1]}$ is always semistable: if $\mathscr{S} \subsetneq \Omega_X^{[1]}$ was destabilizing, then $\det \mathscr{S} \subseteq \Omega_X^{[\operatorname{rank} \mathscr{S}]}$ is either trivial (hence violating the non-existence of reflexive forms, [45, Thm. 1] and [18, Thm. 5.1]) or ample (hence violating the Bogomolov–Sommese vanishing theorem for klt varieties, [18, Thm. 7.2]).

Proof of Proposition 37. If $\mathscr{T}_X$ is semistable, then the $\mathbb{Q}$-Bogomolov–Gieseker inequality holds, and we have seen in Remark 33 that $-r = |r| > n$. Fujita's singular version of the Kobayashi–Ochiai theorem, [13, Thm. 1], will then apply to show that $X \cong \mathbb{P}^n$. □

While the Bogomolov–Gieseker inequality does not necessarily hold on a Fano variety with unstable tangent sheaf, we still get some restriction on the index from the following result.

Proposition 39. *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. If $-K_X$ is ample, then $r^2 \geq n+1$. In particular, if $n \geq 4$, then $r \geq 3$.*

Proof. Since X is factorial by Proposition 24 (5) and non-singular in codimension two by Proposition 24 (3), we may apply [36, Cor. 1.5] to obtain the bound $c_2(X) \cdot c_1(\mathscr{O}_X(1))^{n-2} \geq 0$. Then, we conclude by Corollary 31. □

In dimension three we can now fully answer Question 23.

Theorem 40 (Topological $\mathbb{P}^3$). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^3$. Then, $X \cong \mathbb{P}^3$.*

Proof. Since X is a threefold with isolated, rational Gorenstein singularities, Riemann–Roch takes a particularly simple form:

$$1 \overset{\text{Prop. 24 (1)}}{=} \chi(\mathscr{O}_X) = \frac{1}{24} \cdot [-K_X] \cdot c_2(X).$$

With Corollary 31, this reads

$$-48 = r \cdot (r^2 - 4).$$

This equation has only one real solution: $r = -4$; in particular, $-K_X$ is ample. As before, Fujita's theorem [13, Thm. 1] applies to show that $X \cong \mathbb{P}^n$. □

Finally, in dimensions four and five we show the following.

Theorem 41 ($\mathbb{Q}$-Fano 4- and 5-folds homeomorphic to projective spaces). *Let X be a projective klt variety homeomorphic to $\mathbb{P}^n$, with $n = 4$ or 5. Assume that K_X is not ample. Then, $X \cong \mathbb{P}^n$.*

Proof. Recall that X is a Gorenstein Fano variety of index $i = -r$, with canonical singularities, smooth in codimension two. By [13, Thm. 1 and 2], we may assume that $i \leq \dim X - 1$. Further, from Proposition 39, we see that $i \geq 3$. These cases have to be excluded.

If $i = \dim X - 1$, then by [14], X is a hypersurface of weighted degree 6 embedded in the smooth part of the weighted projective space $\mathbb{P}(3,2,1^n)$. Smooth such hypersurfaces have semistable tangent bundle by [21, Prop. 6.15]; in particular, they satisfy the Bogomolov–Gieseker inequality. Since X is smooth in codimension two, the "principle of conservation of numbers", [15, Thm. 10.2], implies that X satisfies the Bogomolov–Gieseker inequality as well, which in turn contradicts Remark 33.

The remaining case, $n = 5$ and $i = 3$, is ruled out by Corollary 35, which implies that $i = -r$ has to be even. □

References

[1] J. Amorós, M. Burger, K. Corlette, D. Kotschick and D. Toledo, *Fundamental groups of compact Kähler manifolds*, American Mathematical Society, 1996.

[2] C. Bănică and O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*, Editura Academiei; John Wiley & Sons, 1976. Translated from the Romanian.

[3] O. Baues and V. Cortés, "Aspherical Kähler manifolds with solvable fundamental group", *Geom. Dedicata* **122** (2006), pp. 215–229.

[4] A. Borel and J. C. Moore, "Homology theory for locally compact spaces", *Mich. Math. J.* **7** (1960), pp. 137–159.

[5] R. Bryant, *Complex structures on Hermitian symmetric space*, 2020. MathOverflow, https://mathoverflow.net/q/363432.

[6] F. Catanese, "Deformation types of real and complex manifolds", in *Contemporary trends in algebraic geometry and algebraic topology (Tianjin, 2000)*, World Scientific, 2002, pp. 195–238.

[7] F. Catanese, "Topological methods in moduli theory", *Bull. Math. Sci.* **5** (2015), no. 3, pp. 287–449.

[8] B. Claudon, P. Graf and H. Guenancia, "Equality in the Miyaoka–Yau inequality and uniformization of non-positively curved klt pairs", 2023. https://arxiv.org/abs/2305.04074.

[9] G. Dethloff and H. Grauert, "Seminormal complex spaces", in *Several Complex Variables VII*, Springer, 1994, pp. 183–220.

[10] P. B. Eberlein, *Geometry of nonpositively curved manifolds*, University of Chicago Press, 1996.

[11] O. Fujino, "Minimal model program for projective morphisms between complex analytic spaces", 2022. https://arxiv.org/abs/2201.11315.

[12] O. Fujino, "Vanishing theorems for projective morphisms between complex analytic spaces", 2023. https://arxiv.org/abs/2205.14801.

[13] T. Fujita, "On polarized manifolds whose adjoint bundles are not semipositive", in *Algebraic Geometry (Sendai, 1985)*, North-Holland, 1985, pp. 167–178.

[14] T. Fujita, "On singular del Pezzo varieties", in *Algebraic Geometry (L'Aquila, 1988)*, Springer, 1988, pp. 117–128.

[15] W. Fulton, *Intersection Theory*, 2nd edition, Springer, 1998.

[16] M. Goresky and R. D. MacPherson, *Stratified Morse theory*, Springer, 1988.

[17] A. Gramain, "L'invariance topologique des classes de Pontrjagin rationnelles (d'après S. P. Novikov et L. C. Siebenmann)", in *Séminaire Bourbaki, Vol. 9*, Société Mathématique de France, 1995, pp. 391–406. Exp. No. 304.

[18] D. Greb, S. Kebekus, S. J. Kovács and T. Peternell, "Differential forms on log canonical spaces", *Publ. Math., Inst. Hautes Étud. Sci.* **114** (2011), no. 1, pp. 87–169. An extended version with additional graphics is available at http://arxiv.org/abs/1003.2913.

[19] D. Greb, S. Kebekus and T. Peternell, "Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties", *Duke Math. J.* **165** (2004), no. 10, pp. 1965–2004.

[20] D. Greb, S. Kebekus and T. Peternell, "Projectively flat klt varieties", *J. Éc. Polytech., Math.* **8** (2021), pp. 1005–1036.

[21] D. Greb, S. Kebekus and T. Peternell, "Projective flatness over klt spaces and uniformisation of varieties with nef anti-canonical divisor", *J. Algebr. Geom.* **31** (2022), pp. 467–496.

[22] D. Greb, S. Kebekus, T. Peternell and B. Taji, "The Miyaoka–Yau inequality and uniformisation of canonical models", *Ann. Sci. Éc. Norm. Supér.* **52** (2019), no. 6, pp. 1487–1535.

[23] D. Greb, S. Kebekus, T. Peternell and B. Taji, “Harmonic metrics on Higgs sheaves and uniformization of varieties of general type”, *Math. Ann.* **378** (2020), no. 3-4, pp. 1061–1094.

[24] D. Greb, S. Kebekus and B. Taji, “Uniformisation of higher-dimensional varieties”, in *Algebraic Geometry (Salt Lake City 2015)*, American Mathematical Society, 2015, pp. 277–308.

[25] R. Hartshorne, *Algebraic geometry*, Springer, 1977.

[26] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, American Mathematical Society, 1978.

[27] F. Hirzebruch and K. Kodaira, “On the complex projective spaces”, *J. Math. Pures Appl.* **36** (1957), pp. 201–216.

[28] S. Kebekus, L. Solá Conde and M. Toma, “Rationally connected foliations after Bogomolov and McQuillan”, *J. Algebr. Geom.* **16** (2007), no. 1, pp. 65–81.

[29] J. Kollár, *Shafarevich maps and automorphic forms*, Princeton University Press, 1995.

[30] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press, 1998. With the collaboration of C. H. Clemens and A. Corti, translated from the 1998 Japanese original.

[31] H. Liu and J. Liu, “Kawamata–Miyaoka type inequality for canonical $\mathbb{Q}$-Fano varieties”, 2023. https://arxiv.org/abs/2308.10440.

[32] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, 1991. Translated from the Polish by Maciej Klimek.

[33] S. Lu and B. Taji, “A characterization of finite quotients of abelian varieties”, *Int. Math. Res. Not.* **2018** (2018), no. 1, pp. 292–319.

[34] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, University of Tokyo Press, 1973.

[35] S. P. Novikov, “Topological invariance of rational classes of Pontrjagin”, *Dokl. Akad. Nauk SSSR* **163** (1965), pp. 298–300.

[36] W. Ou, “On generic nefness of tangent sheaves”, *Math. Z.* **304** (2023), no. 4, article no. 58 (23 pages).

[37] A. Ranicki and M. Weiss, “On the construction and topological invariance of the Pontryagin classes”, *Geom. Dedicata* **148** (2010), pp. 309–343.

[38] M. Reid, “Projective morphisms according to Kawamata”, 1983. Preprint, University of Warwick, available at http://www.maths.warwick.ac.uk/~miles/3folds.

[39] G. Scheja, “Riemannsche Hebbarkeitssätze für Cohomologieklassen”, *Math. Ann.* **144** (1961), pp. 345–360.

[40] Y.-T. Siu, “The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds”, *Ann. Math.* **112** (1980), no. 1, pp. 73–111.

[41] Y.-T. Siu, “Strong rigidity of compact quotients of exceptional bounded symmetric domains”, *Duke Math. J.* **48** (1981), no. 4, pp. 857–871.

[42] K. Stein, “Analytische Zerlegungen komplexer Räume”, *Math. Ann.* **132** (1956), pp. 63–93.

[43] R. Thom, “Espaces fibrés en sphères et carrés de Steenrod”, *Ann. Sci. Éc. Norm. Supér.* **69** (1952), pp. 109–182.

[44] S.-T. Yau, “On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I”, *Commun. Pure Appl. Math.* **31** (1978), no. 3, pp. 339–411.

[45] Q. Zhang, “Rational connectedness of log **Q**-Fano varieties”, *J. Reine Angew. Math.* **590** (2006), pp. 131–142.