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Frobenius integrability of certain p-forms on singular spaces

Intégrabilité au sens de Frobenius pour certaines p-formes sur des espaces singuliers

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In memory of Jean-Pierre Demailly, our great academic teacher

Abstract. Demailly proved that on a smooth compact Kähler manifold the distribution defined by a holomorphic p-form with values in an anti-pseudoeffective line bundle is always integrable. We generalise his result to compact Kähler spaces with klt singularities.

Résumé. Demailly a montré que la distribution définie par une p-forme holomorphe à valeurs dans un fibré en droites est toujours intégrable si la variété est kählerienne compacte et le dual du fibré en droites est pseudoeffectif. Nous généralisons son résultat à des espaces kähleriennes compactes à singularités klt.

Keywords. holomorphic p-forms, klt spaces, foliations.

Mots-clés. p-forme holomorphe, espace à singularités klt, feuilletages.

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1. Introduction

Let X be a compact Kähler manifold, and let $u \in H^0(X, \Omega_X^p)$ be a holomorphic p-form on X. As a consequence of the Kähler identity for the Laplacians $\Delta_d = 2\Delta_\partial$ one obtains that the holomorphic form is d-closed, i.e. $du = 0$. Twenty years ago Jean-Pierre Demailly used a very clever "integration by parts" to generalise this statement to forms with values in certain line bundles:

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Theorem 1 ([3, Main Thm.]). *Let X be a compact Kähler manifold. Let L be a pseudo-effective holomorphic line bundle on X. Let*

$$u \in H^0(X, \Omega_X^p \otimes L^\star)$$

be a non-zero holomorphic section, and let $S_u \subset T_X$ be the saturated subsheaf given by vector fields ξ such that the contraction $i_\xi u$ vanishes. Then S_u is integrable, i.e. it defines a (possibly singular) holomorphic foliation on X.

Moreover, let h be a possibly singular metric such that $i\Theta_h(L) \geqslant 0$ on X in the sense of currents. Then one has $D'_{h^\star} u = 0$ and (L,h) has flat curvature along the leaves. Here $D'_{h^\star}$ is the $(1,0)$-part of the Chern connection with respect to the dual metric $h^\star$ on $L^\star$.

Demailly's main motivation for this result was to prove that if a compact Kähler manifold admits a contact structure, then the canonical bundle K_X is never pseudoeffective [3, Cor. 2]. Moreover Theorem 1 has turned out to be a very efficient tool for the study of foliations with vanishing first Chern class [10, 14, 17]. In view of the increased interest in foliations on singular spaces (cf. e.g. [2, 5]) it seems worthwhile to look at Demailly's arguments in this setting. In this paper we extend his result to singular spaces with klt (resp. log-canonical) singularities (see Section 2 for the definitions), i.e. the most general classes of singularities appearing in the minimal model program. Our main result is:

Theorem 2. *Let Y be a normal compact Kähler space. Let $\mathscr{A}$ be a rank one reflexive sheaf such that the reflexive power $\mathscr{A}^{[m]}$ is locally free and pseudoeffective for some $m \in \mathbb{N}$. Let*

$$u \in H^0(Y, (\Omega_Y^p \otimes \mathscr{A}^\star)^{\star\star})$$

be a non-zero holomorphic section. Let $S_u \subset T_Y$ be the saturated subsheaf given by vector fields ξ such that the contraction $i_\xi u$ vanishes. Assume one of the following:

(1) *Y has klt singularities; or*
(2) *Y has log-canonical singularities and $p = 1$.*

Then S_u is integrable, i.e. it defines a (possibly singular) foliation on Y.

For applications in foliation theory it is interesting to verify if $\mathscr{A}$ has flat curvature along the leaves of S_u. Since $\mathscr{A}$ is not locally free the precise formulation would be a bit awkward, but flatness holds for the corresponding line bundle (L,h) on a resolution of singularities (see Propositions 8, 10 and Remark 6).

Our basic strategy is similar to the proof of Theorem 1, except that we have to carry out the computation on a resolution of singularities $\pi : X \to Y$. If $\mathscr{A}$ is not locally free this leads to some well-known difficulties, for example the saturation of $\pi^\star \mathscr{A}$ in Ω_X^p is not always pseudoeffective [9, 16]. Therefore we consider forms with logarithmic poles along the exceptional divisor E of the resolution π, in particular we obtain that the saturation in $\Omega_X^p(\log E)$ is pseudoeffective, cf. Corollary 13.

This leads us to the following problem:

Question 3. *Let (X, ω_X) be a compact Kähler manifold, and let $E = \sum E_i$ be a snc divisor. Let (L,h) be a holomorphic line bundle on X where h is a possibly singular metric such that $i\Theta_h(L) \geqslant 0$ on X in the sense of currents. Let $(L^\star, h^\star)$ be the dual metric.*

Let $u \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$. Can we prove that S_u is a holomorphic foliation and $D'_{h^\star} u = 0$ on $X \setminus E$?

If $p = 1$, the problem is totally solved in [19, Thm. 5][1]. It is still open when $p \geqslant 2$. We give a positive answer to this question when the metric h is smooth (Proposition 5). Our main technical result (Proposition 8) gives a positive answer making an assumption on the singularity of h along

[1] We thank Stéphane Druel and Daniel Greb for bringing this reference to our attention.

certain irreducible components E_i. This integrability condition can be verified for a resolution of singularities $X \to Y$ of a klt space, thereby establishing the first part of Theorem 2. When $p = 1$, by using the techniques in our article, we can also give an alternative proof of [19, Thm. 5], cf. Proposition 10. This implies the second part of Theorem 2.

Patrick Graf indicated an alternative path of proof for the second part of Theorem 2: by [7, Thm. 1.4][2] a holomorphic 1-form on the smooth locus of a log-canonical space extends to a resolution, even without admitting logarithmic poles. Therefore we can copy the proof of Theorem 2 and verify the technical condition of Proposition 8. Note that [7, Thm. 1.6] gives an example of a 2-form on a log-canonical 3-fold that does not extend to a resolution unless we admit logarithmic poles. Therefore this approach does not allow to generalise the second part of Theorem 2 to forms in $(\Omega_Y^p \otimes \mathscr{A}^\star)^{\star\star}$ with $p \geqslant 2$.

As a first application, we can consider singular contact spaces, cp. [1, 18]: a normal compact Kähler space of dimension $2n+1$ with log-canonical singularities has a contact structure if there exists a reflexive subsheaf $\mathscr{F} \subset T_X$ of rank $2n$ such that on the smooth locus $X_{\mathrm{nons}} \subset X$,

- the inclusion $\mathscr{F} \subset T_X$ is an injective morphism of vector bundles; and
- the map
$$\wedge^2 \mathscr{F} \longrightarrow T_X/\mathscr{F}$$
induced by the Lie bracket is surjective. In particular $\mathscr{F} \subset T_X$ is not integrable.

If we set $L := (T_X/\mathscr{F})^{**}$, we obtain as in the smooth case that $\omega_X \simeq L^{[-(n+1)]}$. In particular some reflexive power of L is locally free.

Corollary 4. *Let X be a normal compact Kähler space with log-canonical singularities which admits a contact structure. Then the canonical sheaf ω_X is not pseudoeffective.*

Indeed ω_X is pseudoeffective if and only if L^* is pseudoeffective. Yet then we can apply Theorem 2 to the section of $(\Omega_X \otimes L)^{**}$ defined by the inclusion $L^* \to \Omega_X$ and obtain that its kernel $\mathscr{F} \subset T_X$ is integrable, a contradiction.

Since it is not clear if a singular contact space admits a resolution by a contact manifold, the corollary does not reduce to Demailly's theorem [3, Cor. 2].

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2. Notation and terminology

For general definitions in complex and algebraic geometry we refer to [4, 11], for the terminology of singularities of the MMP we refer to [13]. Manifolds and normal complex spaces will always be supposed to be irreducible.

For the convenience of the reader, let us recall the definition of klt (resp. log-canonical) singularities (cf. [13, Def. 2.34] for more details): let Y be a normal complex space such that some reflexive power $\omega_Y^{[m]}$ of the canonical sheaf ω_Y is locally free. Let $\mu : X \to Y$ be a resolution of

[2] The statement is formulated for algebraic varieties, but in view of [12] should hold for analytic spaces.

singularities such that the exceptional locus is a simple normal crossings divisor. Then we can write

$$\omega_X^m \simeq \mu^* \omega_Y^{[m]} \otimes \mathscr{O}_Y\left(\sum b_i E_i\right)$$

where the $E_i \subset Y$ are μ-exceptional prime divisors. The space Y has klt (resp. log-canonical) singularities if $\frac{b_i}{m} > -1$ (resp. $\frac{b_i}{m} \geqslant -1$) for all i.

Given a normal complex space Y, we denote by $\Omega_Y^{[p]} := (\Omega_Y^p)^{\star\star}$ the sheaf of holomorphic reflexive p-forms. If Y has klt singularities we know by [12, Thm. 1.1] that this coincides with the sheaf of holomorphic p-forms that extend to a resolution of singularities $f : X \to Y$, i.e. we have $f_* \Omega_X^p \simeq \Omega_Y^{[p]}$.

For a reflexive sheaf $\mathscr{F}$ on Y, we denote by $\mathscr{F}^{[m]} := (\mathscr{F}^{\otimes m})^{\star\star}$ the m-th reflexive power. Given a surjective morphism $\varphi : X \to Y$ we denote by $\varphi^{[\star]}\mathscr{F}$ the reflexive pull-back $(\varphi^\star \mathscr{F})^{\star\star}$.

3. Twisted logarithmic forms

Proposition 5. *Let X be a compact Kähler manifold, and let $E = \sum E_i$ be a snc divisor. Let (L, h) be a holomorphic line bundle on X where h is a smooth metric such that $i\Theta_h(L) \geqslant 0$. Let $u \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$ and $(L^\star, h^\star)$ be the dual metric on (L, h). Then $D'_{h^\star} u = 0$ on X and $i\Theta_h(L) \wedge u \wedge \overline{u} = 0$.*

Proof. If L is a trivial line bundle, it is done by [15]. We generalize it to the twisted setting by the following argument.

Step 1. Since h is a smooth metric, we know that $D'_{h^\star} u \in C^\infty(X, \Omega_X^{p+1}(\log E) \otimes L^\star)$. We show in this step that $D'_{h^\star} u \in C^\infty(X, \Omega_X^{p+1} \otimes L^\star)$.

We consider the residue of u and $D'_{h^\star} u$ on E_i. First of all, by a direct calculation, we have

$$\operatorname{Res}_{E_i}(D'_{h^\star} u) = -D'_{h^\star} \operatorname{Res}_{E_i}(u) \qquad \text{on } E_i. \tag{1}$$

In fact, let Ω be a neighborhood of a generic point of E_i. We suppose that E_i is defined by $z_1 = 0$ and $h = e^{-\varphi}$ on Ω. Then we can write

$$u = \frac{\mathrm{d}z_1}{z_1} \wedge f + g$$

for two smooth forms f, g on Ω.

For the RHS of (1), since $\operatorname{Res}_{E_i}(u) = f$ and we obtain

$$-D'_{h^\star} \operatorname{Res}_{E_i}(u) = -(\partial f + \partial\varphi \wedge f)|_{E_i}.$$

For the LHS of (1), we have

$$\operatorname{Res}_{E_i}(D'_{h^\star} u) = \operatorname{Res}_{E_i}\left(D'_{h^\star}\left(\frac{\mathrm{d}z_1}{z_1} \wedge f\right)\right) = \operatorname{Res}_{E_i}\left(-\frac{\mathrm{d}z_1}{z_1} \wedge \partial f + \partial\varphi \wedge \frac{\mathrm{d}z_1}{z_1} \wedge f\right) = -(\partial f + \partial\varphi \wedge f)|_{E_i}.$$

Then we obtain (1).

Note that $\operatorname{Res}_{E_i}(u) \in H^0(E_i, \Omega_{E_i}^{p-1}(\log(E - E_i)) \otimes L^\star)$. By induction on dimension, we know that $\operatorname{Res}_{E_i}(u)$ is $D'_{h^\star}$-closed on E_i. Then (1) implies that $\operatorname{Res}_{E_i}(D'_{h^\star} u) = 0$. Therefore the form $D'_{h^\star} u$ is a smooth form on the total space X.

Step 2. Let $N \in \mathbb{N}^\star$ and let $\Xi_N(x)$ be a smooth function which equals to 1 on $[0, N]$, equals to 0 on $[N+1, \infty]$ and $0 \leqslant \Xi'_N(x) \leqslant 1$. Let s_E be the canonical section of E. We consider the integration

$$\int_X \Xi_N(\log(-\log|s_E|))\{D'_{h^\star} u, D'_{h^\star} u\} \wedge \omega_X^{n-p-1}. \tag{2}$$

Here $|s_E|$ denotes the norm of s_E with respect to a fixed smooth metric on E such that $|s_E| < 1$ everywhere.

By integration by parts, (2) equals to

$$= \int_X \{D'_{h^\star}(\Xi_N(\log(-\log|s_E|))u), D'_{h^\star}u\} \wedge \omega_X^{n-p-1} - \int_X \{\partial(\Xi_N(\log(-\log|s_E|))) \wedge u, D'_{h^\star}u\} \wedge \omega_X^{n-p-1}$$
$$= -\int_X (-1)^p \Xi_N(\log(-\log|s_E|))\{u, \bar\partial(D'_{h^\star}u)\} \wedge \omega_X^{n-p-1} - \int_X \{\partial(\Xi_N(\log(-\log|s_E|))) \wedge u, D'_{h^\star}u\} \wedge \omega_X^{n-p-1}$$
$$= -\int_X i\Theta_h(L)\Xi_N \cdot \{u,u\} \wedge \omega_X^{n-p-1} - \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log|s_E|}, D'_{h^\star}u \right\} \wedge \omega_X^{n-p-1}. \tag{3}$$

Since $i\Theta_h(L) \geqslant 0$, the first term of (3) is semi-negative. For the second term of (3), by Step 1, we know that $D'_{h^\star}u$ is smooth on X. Together with $\frac{\mathrm{d}s_{E_i}}{s_{E_i}\log|s_{E_i}|} \wedge \frac{\mathrm{d}s_{E_i}}{s_{E_i}} = 0$, we know that the second term of (3) is controlled by

$$\int_{N \leqslant \log(-\log|s_E|) \leqslant N+1} \frac{1}{\prod_i |s_{E_i}|} \omega_X^n,$$

which converges to zero when $N \to 0$.

As a consequence, when $N \to +\infty$, the upper limit of (3) will not be strictly positive. Since (2) is always positive, we obtain

$$\lim_{N\to+\infty} \int_X \Xi_N(\log(-\log|s_E|))\{D'_{h^\star}u, D'_{h^\star}u\} \wedge \omega_X^{n-p} = 0. \tag{4}$$

Therefore $D'_{h^\star}u = 0$ on X. □

Remark 6. For the convenience of the reader let us recall why $D'_{h^\star}u = 0$ implies that (L,h) has flat curvature along the generic leaf, following [3, Main thm]. Let $x \in X$ be a general point and fix a holomorphic base e_L of L near x. Then the metric h is written locally as $h = e^{-\varphi}$. In these local coordinates, $D'_{h^\star}u = 0$ means that $\partial\varphi \wedge u = -\partial u$. By taking the $\bar\partial$, we obtain $\mathrm{d}\mathrm{d}^c\varphi \wedge u = 0$. Now we suppose that the leaves of the foliation near the generic point x is given by

$$z_1 = c_1, z_2 = c_2, \ldots, z_r = c_r$$

where the c_i are constants. Then u depends only on $\mathrm{d}z_1, \ldots, \mathrm{d}z_r$ near x. Therefore the condition $\mathrm{d}\mathrm{d}^c\varphi \wedge u = 0$ implies that $\frac{\partial^2\varphi}{\partial z_j \partial \bar z_k} = 0$ for $j,k > r$. In other words, (L,h) is flat along the generic leaf.

Remark 7. By a standard argument, it is easy to generalize the above proposition to the case when the metric (L,h) is of analytic singularity. However, it is unclear whether we can generalize it to the case of arbitrary singularity cf. Question 3.

In the rest of the section, we will confirm Question 3 in two special cases.

Proposition 8. *Let (X, ω_X) be a compact Kähler manifold, and let $E = \sum_{i=1}^r E_i$ be a snc divisor. Let (L,h) be a holomorphic line bundle on X where h is a possibly singular metric such that $i\Theta_h(L) \geqslant 0$ on X in the sense of currents. Let $(L^\star, h^\star)$ be the dual metric. Let $u \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$. We assume that $\mathrm{Res}_{E_i}(u) \neq 0$ for every $1 \leqslant i \leqslant k$ and $\mathrm{Res}_{E_i}(u) = 0$ for every $k < i \leqslant r$.*

We write $h = e^{-\varphi} \cdot h_0$, where φ is a quasi-psh function on X and h_0 is a smooth metric on L. If the weight function φ satisfies:

$$\varphi \leqslant -2\sum_{i=1}^k \ln(-\ln|s_{E_i}|) + C, \tag{5}$$

where s_{E_i} is the canonical section of E_i, then $D'_{h^\star}u = 0$ and $i\Theta_h(L) \wedge u \wedge \bar u = 0$ on $X \setminus E$, where $D'_{h^\star}$ is the connection with respect to $h^\star$.

Remark 9. Note that if the Lelong number of φ along E_i is strictly positive for every $i \leqslant k$, then φ satisfies the condition (5).

Proof. The proof is divided into two steps.

Step 1. Let $N \in \mathbb{N}^\star$ and let $\Xi_N(x)$ be a smooth function which equals to 1 on $[0, N]$, equals to 0 on $[N+1, \infty]$ and $0 \leqslant \Xi'_N(x) \leqslant 1$. We consider the integration

$$\int_X \Xi_N^2(\log(\log(-\log|s_E|)))\{D'_{h^\star}u, D'_{h^\star}u\} \wedge \omega_X^{n-2}. \tag{6}$$

Since $D'_{h^\star}u$ is L^2 in the support of $\Xi_N(\log(\log(-\log|s_E|)))$, we can still do the integration by parts as in [3]. In particular, (6) equals to

$$= \int_X \{D'_{h^\star}(\Xi_N^2(\log(\log(-\log|s_E|)))u), D'_{h^\star}u\} \wedge \omega_X^{n-2} - \int_X \{\partial(\Xi_N^2(\log(\log(-\log|s_E|))) \wedge u, D'_{h^\star}u\} \wedge \omega_X^{n-2}$$
$$= -\int_X i\Theta_h(L)\Xi_N^2(\log(-\log|s_E|))\{u,u\} \wedge \omega_X^{n-2} - \int_X \left\{ \frac{2 \cdot \Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \Xi_N \cdot D'_{h^\star}u \right\} \wedge \omega_X^{n-2}. \tag{7}$$

Since $i\Theta_h(L) \geqslant 0$, the first term of (7) is semi-negative. For the second term of (7), by using Cauchy inequality, we get

$$\left| \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \Xi_N \cdot D'_{h^\star}u \right\} \wedge \omega_X^{n-2} \right|^2$$
$$\leqslant \int_X \Xi_N^2 \{D'_{h^\star}u, D'_{h^\star}u\} \wedge \omega_X^{n-2} \cdot \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|} \right\} \wedge \omega_X^{n-2}.$$

As a consequence, we obtain

$$\int_X \Xi_N^2 \cdot \{D'_{h^\star}u, D'_{h^\star}u\} \wedge \omega_X^{n-2} \leqslant \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|} \right\} \wedge \omega_X^{n-2} \tag{8}$$

Step 2. In this step, we would like to show the RHS of (8) tends to zero when $N \to +\infty$.

Since $\frac{ds_{E_i}}{s_{E_i}} \wedge \frac{ds_{E_i}}{s_{E_i}} = 0$, the assumption (5) implies that $\{\partial \log|s_E| \wedge u, \partial \log|s_E| \wedge u\} \wedge \omega_X^{n-2}$ is upper bounded by

$$C' \frac{\omega_X^n}{\prod_{i=1}^k |s_{E_i}|^2 \log^2|s_{E_i}|} \cdot \left(\sum_{i=k+1}^r \frac{1}{|s_{E_i}|^2} \right)$$

for some constant C'. Then the RHS of (8) is controlled by

$$C' \sum_{i=k+1}^r \int_X \frac{(\Xi'_N)^2 \omega_X^n}{\prod_{i=1}^k |s_{E_i}|^2 \log^2|s_{E_i}|} \cdot \frac{1}{|s_{E_i}|^2 \log^2|s_{E_i}|}. \tag{9}$$

which converges to zero when $N \to 0$. As a consequence, the RHS of (8) tends to zero when $N \to +\infty$. Therefore $D'_{h^\star}u = 0$ on $X \setminus E$. $\square$

By using the argument in Proposition 8, we can give an alternative proof of [19, Thm. 5]:

Proposition 10. *Let X be a compact Kähler manifold, and let $E = \sum E_i$ be a snc divisor. Let (L, h) be a holomorphic line bundle on X where h is a possible singular metric such that $i\Theta_h(L) \geqslant 0$. Let $u \in H^0(X, \Omega_X^1(\log E) \otimes L^\star)$ and $(L^\star, h^\star)$ be the dual metric on (L, h). Then $D'_{h^\star}u = 0$ and $i\Theta_h(L) \wedge u \wedge \overline{u} = 0$ on $X \setminus E$.*

Proof. We follow the notations in Proposition 8. By the step 1 of Proposition (8), we know that

$$\int_X \Xi_N^2 \cdot \{D'_{h^\star}u, D'_{h^\star}u\} \wedge \omega_X^{n-2} \leqslant \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|} \right\} \wedge \omega_X^{n-2} \tag{10}$$

In order to prove the proposition, it is sufficient to show the RHS of (10) tends to zero when $N \to +\infty$.

Since $\frac{ds_{E_i}}{s_{E_i}} \wedge \frac{ds_{E_i}}{s_{E_i}} = 0$ and u is a 1-form, $\{\partial \log|s_E| \wedge u, \partial \log|s_E| \wedge u\} \wedge \omega_X^{n-2}$ is upper bounded by

$$C \cdot \sum_{i \neq j} \frac{\omega_X^n}{|s_{E_i} s_{E_j}|^2}.$$

Then the RHS (10) is controlled by

$$C\sum_{i\neq j}\int_X \frac{(\Xi'_N)^2\omega_X^n}{\log^2(-\log|s_E|)\log^2|s_E|\cdot|s_{E_i}s_{E_j}|^2}. \tag{11}$$

Note that the integral

$$\int_{0\leqslant r_1,r_2\leqslant 1}\frac{\mathrm{d}r_1\wedge\mathrm{d}r_2}{\log^2(-\log|r_1r_2|)\log^2|r_1r_2|\cdot r_1r_2}<+\infty.$$

Therefore (11) converges to zero when $N\to 0$. As a consequence, the RHS of (10) tends to zero when $N\to+\infty$. Therefore $D'_{h^\star}u=0$ on $X\setminus E$. □

4. Lifting subsheaves to the resolution

Let Y be a normal complex space with klt singularities, and let $\nu: Y'\to Y$ be a proper surjective morphism from a normal complex space Y'. Since klt singularities are rational [13, Thm. 5.22], by [12, Thm. 1.10] there exists for every $p\in\mathbb{N}$ a cotangent map

$$\mathrm{d}\nu:\nu^\star\Omega_Y^{[p]}\longrightarrow\Omega_{Y'}^{[p]} \tag{12}$$

If Y has log-canonical singularities we can still combine the proof of [8, Thm. 4.3] with [12, Thm. 1.5] to obtain[3] that there exists for every $p\in\mathbb{N}$ a cotangent map

$$\mathrm{d}\nu:\nu^\star\Omega_Y^{[p]}\longrightarrow\Omega_{Y'}^{[p]}(\log\Delta) \tag{13}$$

where $\Delta\subset Y'$ is the largest reduced Weil divisor contained in ν^{-1}(non-klt locus).

The following statement is well-known to experts and essentially a rewriting of the proof of [8, Thm. 7.2]. We include it for the convenience of the reader:

Lemma 11. *Let Y be a normal complex space with log-canonical singularities, and let $\mathscr{A}\subset\Omega_Y^{[p]}$ be a reflexive subsheaf of rank one that is $\mathbb{Q}$-Cartier, i.e. there exists a $m\in\mathbb{N}$ such that $\mathscr{A}^{[m]}$ is locally free.*

Let $\pi: X\to Y$ be a log resolution, and let E be the exceptional divisor. Let $\mathscr{C}\subset\Omega_X^p(\log E)$ be the saturation of the image of the morphism

$$\pi^\star\mathscr{A}\longrightarrow\pi^\star\Omega_Y^{[p]}\xrightarrow{\mathrm{d}\pi}\Omega_X^p(\log E).$$

Then there exists a non-zero morphism $\pi^\star\mathscr{A}^{[m]}\to\mathscr{C}^{\otimes m}$.

Remark. The morphism $\pi^\star\mathscr{A}^{[m]}\to\mathscr{C}^{\otimes m}$ is an isomorphism in the complement of the exceptional divisor E. Thus, up to multiplication by a holomorphic function that is a pull-back from Y, the morphism is unique.

If Y has klt singularities, we could use (12) and consider $\mathscr{C}'\subset\Omega_X^p$, the saturation of the image of the morphism

$$\pi^\star\mathscr{A}\longrightarrow\pi^\star\Omega_Y^{[p]}\xrightarrow{\mathrm{d}\pi}\Omega_X^p,$$

but in general there will be no morphism $\pi^\star\mathscr{A}^{[m]}\to(\mathscr{C}')^{\otimes m}$. However, in the course of the proof of Lemma 11 we will prove the following remark that will be useful for the proof of Proposition 14:

Remark 12. If Y is klt, let $\tilde{\gamma}:\tilde{Z}\to X$ be the cover induced by a (local) index-one cover $\gamma: Z\to Y$ of $\mathscr{A}$ (cf. Diagram (14)). Then $\pi_Z^\star\gamma^\star\mathscr{A}^{[m]}$ is a subsheaf of $S^{[m]}\Omega_{\tilde{Z}}^{[p]}$.

[3] Note that [12, Thm. 1.10] holds for any morphism, while we only need the simpler case where the morphism is surjective.

For the proof let us recall the notion of index one covers [13, Def. 5.19]: given a normal complex space Y and a reflexive sheaf $\mathscr{A}$ such that some reflexive power $\mathscr{A}^{[m]}$ is trivial, there exists a quasi-étale morphism $\gamma : Z \to Y$ from a normal complex space Z such that the reflexive pull-back $\gamma^{[\star]}\mathscr{A}$ is isomorphic to $\mathscr{O}_Z$.

Proof of Lemma 11. The locally free sheaves coincide in the complement of the exceptional locus $E = \bigcup_i E_i$, so we can write $\mathscr{C}^{\otimes m} \simeq \pi^\star \mathscr{A}^{[m]} \otimes \mathscr{O}_X(\sum a_i E_i)$ with uniquely determined $a_i \in \mathbb{Z}$. We are done if we show that $a_i \geqslant 0$ for all i. This property can be checked locally on the base Y.

Therefore we can replace Y by a Stein neighborhood such that there exists an index-one cover $\gamma : Z \to Y$, and let $\tilde{\gamma} : \tilde{Z} \to X$ be the induced finite map from the normalisation $\tilde{Z}$ of $X \times_Y Z$. We denote by $\pi_Z : \tilde{Z} \to Z$ the bimeromorphic morphism induced by π and summarize the construction in a commutative diagram:

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{\gamma}"] \arrow[d, "\pi_Z"'] & X \arrow[d, "\pi"] \\ Z \arrow[r, "\gamma"] & Y \end{tikzcd} \tag{14}$$

The morphism $\gamma : Z \to Y$ is an index-one cover for $\mathscr{A}$, so γ is étale in codimension one and $\gamma^{[\star]}\mathscr{A} =: \mathscr{B}$ is locally free. In particular Z still has log-canonical singularities [13, Prop. 5.20 (4)]. Denote the exceptional locus of π_Z by E_Z and observe that E_Z is equal to the support of $\tilde{\gamma}^\star E$. In particular E_Z contains the preimage of the non-klt locus of Z, so (13) gives a natural map

$$d\pi_Z : \pi_Z^\star \Omega_Z^{[p]} \longrightarrow \Omega_{\tilde{Z}}^{[p]}(\log E_Z)$$

Since $\mathscr{A} \subset \Omega_Y^{[p]}$ and γ is étale in codimension one we have an inclusion $\mathscr{B} \subset \Omega_Z^{[p]} \simeq \gamma^{[\star]}\Omega_Y^{[p]}$ and hence an induced map

$$\pi_Z^\star \mathscr{B} \longrightarrow \pi_Z^\star \Omega_Z^{[p]} \longrightarrow \Omega_{\tilde{Z}}^{[p]}(\log E_Z).$$

Since $\mathscr{B}$ is locally free, this induces an inclusion

$$\pi_Z^\star \mathscr{B}^{\otimes m} \simeq (\pi_Z^\star \mathscr{B})^{\otimes m} \longrightarrow S^{[m]}\Omega_{\tilde{Z}}^{[p]}(\log E_Z). \tag{15}$$

By assumption $A^{[m]}$ is locally free, so its (non-reflexive !) pull-back $\gamma^\star \mathscr{A}^{[m]}$ is still locally free. Thus $B^{\otimes m} \simeq \gamma^\star A^{[m]}$ since they are both reflexive and coincide in codimension one. Thus we have constructed a morphism

$$\pi_Z^\star \gamma^\star A^{[m]} \longrightarrow S^{[m]}\Omega_{\tilde{Z}}^{[p]}(\log E_Z).$$

We interrupt the proof of the lemma for the *Proof of Remark 12.*

If Y is klt, the index one cover Z also has klt singularities [13, Prop. 5.20 (4)]. Thus we can replace the pull-back with logarithmic poles (13) by the usual pull-back (12) to obtain

$$d\pi_Z : \pi_Z^\star \Omega_Z^{[p]} \longrightarrow \Omega_{\tilde{Z}}^{[p]}$$

As above the inclusion $\gamma^{[\star]}\mathscr{A} \simeq \mathscr{B} \subset \Omega_Z^{[p]} \simeq \gamma^{[\star]}\Omega_Y^{[p]}$ then gives the inclusion

$$\pi_Z^\star \gamma^\star \mathscr{A}^{[m]} \simeq \pi_Z^\star \mathscr{B}^{\otimes m} \simeq (\pi_Z^\star \mathscr{B})^{\otimes m} \longrightarrow S^{[m]}\Omega_{\tilde{Z}}^{[p]}.$$

This proves Remark 12, we now proceed with the proof of Lemma 11.

Since X is smooth, the saturated subsheaf $\mathscr{C} \subset \Omega_X^p(\log E)$ is locally free and a subbundle in codimension one. Thus

$$\mathscr{C}^{\otimes m} \subset S^m \Omega_X^p(\log E) \tag{16}$$

is locally free and a subbundle in codimension one, hence a saturated subsheaf. The finite morphism $\tilde{\gamma}$ is étale in the complement of E and $\Omega_X^p(\log E)$ is locally free, so the tangent map gives an isomorphism

$$\tilde{\gamma}^\star \Omega_X^p(\log E) \simeq \Omega_{\tilde{Z}}^{[p]}(\log E_Z). \tag{17}$$

and hence an isomorphism

$$\widetilde{\gamma}^\star S^m\Omega^p_X(\log E) \simeq S^{[m]}\Omega^{[p]}_{\tilde{Z}}(\log E_Z).$$

Composing the inclusion (16) with this isomorphism we obtain that

$$\widetilde{\gamma}^\star \mathscr{C}^{\otimes m} \longrightarrow S^{[m]}\Omega^{[p]}_{\tilde{Z}}(\log E_Z)$$

is a saturated subsheaf.

Since Y is Stein and $\mathscr{A}^{[m]}$ is invertible we can choose for every point $y \in Y$ a section $\sigma \in H^0(Y, \mathscr{A}^{[m]})$ that does not vanish in y. In particular σ generates $\mathscr{A}^{[m]}$ as an $\mathscr{O}_Y$-module near the point y. Thus it induces a section

$$\pi_Z^\star\gamma^\star\sigma \in H^0(\widetilde{Z}, S^{[m]}\Omega^{[p]}_{\tilde{Z}}(\log E_Z))$$

that generates the image of $\pi_Z^\star\gamma^\star\mathscr{A}^{[m]}$. The pull-back $\pi^\star\sigma$ defines a meromorphic section of $\mathscr{C}^{\otimes m}$ that has poles at most along E, thus $\widetilde{\gamma}^\star\pi^\star\sigma$ defines a meromorphic section of $\widetilde{\gamma}^\star\mathscr{C}^{\otimes m}$ that has poles at most along E_Z. Since $\widetilde{\gamma}^\star\mathscr{C}^{\otimes m}$ is saturated in $S^{[m]}\Omega^{[p]}_{\tilde{Z}}(\log E_Z)$ and

$$\pi_Z^\star\gamma^\star\sigma = \widetilde{\gamma}^\star\pi^\star\sigma \in H^0(\widetilde{Z}, S^{[m]}\Omega^{[p]}_{\tilde{Z}}(\log E_Z))$$

has no poles, we see that

$$\widetilde{\gamma}^\star\pi^\star\sigma \in H^0(\widetilde{Z}, \widetilde{\gamma}^\star\mathscr{C}^{\otimes m}).$$

Thus the local generator of the subsheaf $\pi_Z^\star\gamma^\star\mathscr{A}^{[m]}$ lies in $\widetilde{\gamma}^\star\mathscr{C}^{\otimes m}$ and we have an inclusion

$$\widetilde{\gamma}^\star\pi^\star\mathscr{A}^{[m]} \simeq \pi_Z^\star\gamma^\star\mathscr{A}^{[m]} \hookrightarrow \widetilde{\gamma}^\star\mathscr{C}^{\otimes m}.$$

Thus we see that

$$\widetilde{\gamma}^\star\mathscr{O}_X(\sum a_i E_i) \simeq \widetilde{\gamma}^\star(\mathscr{C}^{\otimes m} \otimes \pi^\star\mathscr{A}^{[-m]})$$

is represented by an effective divisor with support in the exceptional locus of π_Z. Since $\widetilde{\gamma}^\star(\sum a_i E_i)$ is linearly equivalent to an effective, exceptional divisor and has also support in the exceptional locus of π_Z, it is effective. Thus we have shown that $a_i \geqslant 0$ for all i. □

As in immediate application we obtain a variant of [8, Thm. 7.2], [6, Cor. 1.3] for pseudoeffective line bundles.

Corollary 13. *Let Y be a normal compact complex space with log-canonical singularities, and let $\mathscr{A} \subset \Omega^{[p]}_Y$ be a reflexive subsheaf of rank one that is $\mathbb{Q}$-Cartier, i.e. there exists a $m \in \mathbb{N}$ such that $\mathscr{A}^{[m]}$ is locally free. Let $\mathscr{C} \subset \Omega^p_X(\log E)$ be the saturation of $\pi^\star\mathscr{A}$. If $\mathscr{A}^{[m]}$ is pseudoeffective, then $\mathscr{C}$ is pseudoeffective.*

Proof. Since pseudoeffectivity of a line bundle is invariant under taking tensor powers, it is sufficient to show that $\mathscr{C}^{\otimes m}$ is pseudoeffective. Yet this follows from the non-zero morphism $\pi^*\mathscr{A}^{[m]} \to \mathscr{C}^{\otimes m}$ constructed in Lemma 11. □

We need the following proposition.

Proposition 14. *In the situation of Lemma 11, write*

$$\mathscr{C}^{\otimes m} = \pi^\star\mathscr{A}^{[m]} \otimes \mathscr{O}_X\Big(\sum a_i E_i\Big), \tag{18}$$

where $a_i \geqslant 0$ and $E = \sum E_i$ is the exceptional locus.

Assume that Y has klt singularities, and let E_i be an irreducible component of the exceptional locus. Let $\mathrm{Res}_{E_i}(\mathscr{C})$ be the residue of the image of $\mathscr{C}$ in $\Omega^p_X(\log E)$. If $\mathrm{Res}_{E_i}(\mathscr{C}) \neq 0$, then $a_i > 0$.

Proof. The claim is local on Y, so we will use the construction from the proof of Lemma 11 summarized in the commutative diagram (14).

Fix a prime divisor $\widetilde{E}_i \subset \widetilde{Z}$ that maps onto $E_i \subset X$, and choose a general point $\tilde{x} \in \widetilde{E}_i \cap \widetilde{Z}_{\mathrm{nons}}$ such that $\widetilde{E}_i$ (resp. E_i) is smooth in $\tilde{x}$ (resp. smooth in $x := \tilde{\gamma}(\tilde{x})$). Since $\tilde{x}$ is general, the finite morphism $\tilde{\gamma}$ has constant rank in an analytic neighborhood of $\tilde{\gamma}$, hence we can find local coordinates on $\widetilde{Z}$ and X such that

$$E_i = \{z_1 = 0\}$$

and $\tilde{\gamma}$ is given locally by

$$\tilde{\gamma} : (t, z_2, ..z_n) \longrightarrow (t^d, z_2, .., z_n).$$

The exterior power $\Omega_X^p(\log E)_x$ is generated by $\left\{\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_J, \mathrm{d}z_I\right\}$ where $J \subset \{2,\ldots,n\}$ has length $p-1$ and $I \subset \{2,\ldots,n\}$ has length p. Thus we obtain a basis $\{e_1,\ldots,e_k\}$ of $S^m\Omega_X(\log E)_x$ by taking products of length m, where each e_i is of type:

$$e_i = \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_1}\right) \otimes \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_2}\right) \otimes \cdots \otimes \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_q}\right) \otimes \mathrm{d}z_{I_1} \otimes \cdots \otimes \mathrm{d}z_{I_{m-q}}.$$

In our local coordinates the pull-back becomes

$$\tilde{\gamma}^\star(e_i) = \left(\frac{\mathrm{d}t}{t} \wedge \mathrm{d}z_{J_1}\right) \otimes \left(\frac{\mathrm{d}t}{t} \wedge \mathrm{d}z_{J_2}\right) \otimes \cdots \otimes \left(\frac{\mathrm{d}t}{t} \wedge \mathrm{d}z_{J_q}\right) \otimes \mathrm{d}z_{I_1} \otimes \cdots \otimes \mathrm{d}z_{I_{m-q}}.$$

In particular, the pull back $\{\tilde{\gamma}^\star(e_i)\}_{i=1}^k$ is a basis of $S^m\Omega_{\widetilde{Z}}(\log E_Z)$ at $\tilde{x}$.

Let σ be a generator of $\mathscr{A}^{[m]}$ at $\pi(x) \in Y$. Then $\pi^\star\sigma \in \pi^\star\mathscr{A}^{[m]} \subset S^m\Omega_X(\log E)$ is a local generator near x. We can write

$$\pi^\star\sigma = \sum f_i e_i,$$

where f_i are holomorphic functions near x. Now recall that by Remark 12

$$\pi_Z^\star \mathscr{B}^{\otimes m} \simeq \pi_Z^\star \gamma^\star \mathscr{A}^{[m]} \simeq \tilde{\gamma}^\star \pi^\star \mathscr{A}^{[m]}$$

is a subsheaf of $S^{[m]}\Omega_{\widetilde{Z}}^{[p]}$. In particular, since $\widetilde{Z}$ is smooth in $\tilde{x}$, we have

$$(\tilde{\gamma} \circ \pi)^\star \sigma \in (S^m\Omega_{\widetilde{Z}}^p)_{\tilde{x}}.$$

As a consequence, $f_i(x) = 0$ when e_i is of type

$$e_i = \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_1}\right) \otimes \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_2}\right) \otimes \cdots \otimes \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_m}\right),$$

since this generator of $(S^m\Omega_{\widetilde{Z}}^p(\log E_Z))_{\tilde{x}}$ is not contained in $(S^m\Omega_{\widetilde{Z}}^p)_{\tilde{x}}$.

Now we can prove the proposition. Near a general point $x \in E_i$, we suppose that $\mathscr{C}_x \subset (\Omega_{\widetilde{Z}}^p)_{\tilde{x}}$ is generated by

$$\sum g_i \cdot \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_i}\right) + \sum h_i \cdot \mathrm{d}z_{I_i},$$

where g_i, h_i are holomorphic functions. Thanks to Lemma 11, we have

$$F \cdot \left(\sum g_i \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_i}\right) + \sum h_i \mathrm{d}z_{I_i}\right)^{\otimes m} = \left(\sum f_i e_i\right),$$

where F is a holomorphic function near x. If $\mathrm{Res}_{E_i}(\mathscr{C}) \neq 0$, we know that there is one i_0 such that $g_{i_0}(x) \neq 0$. Set

$$e_{i_0} := \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_{i_0}}\right)^{\otimes m}.$$

Then $F \cdot g_{i_0}^m = f_{i_0}$. By the above paragraph, we know that $f_{i_0}(x) = 0$. Then $F(x) = 0$. The proposition is thus proved. □

We are now in the position to verify the technical condition in Proposition 8:

Theorem 15. *In the setting of Theorem 2, let $\pi : X \to Y$ be a log-resolution and denote by E the exceptional locus. Let $L \subset \Omega_X^p(\log E)$ be the saturation of $\pi^\star \mathscr{A}$, and let $\widetilde{u} \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$ the corresponding section. Then there exists a metric h_1 on L such that we have $D'_{h_1^\star} \widetilde{u} = 0$ on $X \setminus E$*

Proof. By Lemma 11, we know that

$$c_1(L) = \frac{1}{m} \pi^\star c_1(\mathscr{A}^{[m]}) + \sum_{i \in I} a_i E_i + \sum_{i \in I'} a_i E_i, \tag{19}$$

such that all the coefficients $a_i \geqslant 0$ and the $i \in I$ correspond to the exceptional divisors E_i such that $\mathrm{Res}_{E_i}(\mathscr{C}) \neq 0$ and $i \in I'$ corresponds to $\mathrm{Res}_{E_i}(\mathscr{C}) = 0$. By Proposition 14 we have $a_i > 0$ when $i \in I$. Let h_0 be a possibly singular metric on $\pi^\star \mathscr{A}^{[m]}$ such that $i\Theta_{h_0}(\pi^\star \mathscr{A}^{[m]}) \geqslant 0$. By (19) this induces a metric h_1 on L. Thanks to Proposition 8, the theorem is proved. □

5. Proof of the main result

The setup for the proof of Theorem 2 is as follows: the non-zero section u determines an injective morphism of sheaves

$$\mathscr{A} \hookrightarrow \Omega_Y^{[p]}.$$

Let $\pi : X \to Y$ be a log-resolution of Y, and denote by E the exceptional locus. Since Y is log-canonical, we have the tangent map (13)

$$d\pi : \pi^\star \Omega_Y^{[p]} \longrightarrow \Omega_X^p(\log E),$$

and we denote by $L \subset \Omega_X^p(\log E)$ the saturation of $\pi^\star \mathscr{A}$. By Lemma 11 there exists a morphism $\pi^\star \mathscr{A}^{[m]} \to L^{\otimes m}$, so L is a pseudoeffective line bundle on X. The inclusion $L \subset \Omega_X^p(\log E)$ corresponds to a non-zero holomorphic section

$$\widetilde{u} \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$$

which coincides with u on $X \setminus E \simeq Y_{\mathrm{nons}}$. In particular the subsheaf $S_{\widetilde{u}} \subset T_X$ defined by contraction with $\widetilde{u}$ coincides with $S_u \subset T_Y$ on a Zariski open set. Thus we are left to show the integrability of $S_{\widetilde{u}} \subset T_X$ on $X \setminus E$. By the formula for the exterior derivative of p-forms (cf. [3, p. 97]) the integrability of $S_{\widetilde{u}}$ follows if we find a metric h on L such that $D'_{h^\star} \widetilde{u} = 0$ on $X \setminus E$.

Assume that we are in the first case of Theorem 2: Since Y is klt, the existence of the metric h is guaranteed by Theorem 15.

Assume that we are in the second case of Theorem 2: Since $p = 1$ we know by Proposition 10 that any singular metric with positive curvature current will suffice. Since L is pseudoeffective, such a metric exists. □

References

[1] F. Campana and H. Flenner, "Contact singularities", *Manuscr. Math.* **108** (2002), no. 4, pp. 529–541.

[2] P. Cascini and C. Spicer, "MMP for co-rank one foliations on threefolds", *Invent. Math.* **225** (2021), no. 2, pp. 603–690.

[3] J.-P. Demailly, "On the Frobenius integrability of certain holomorphic p-forms", in *Complex geometry (Göttingen, 2000)*, Springer, 2002, pp. 93–98.

[4] J.-P. Demailly, *Analytic methods in algebraic geometry*, International Press; Higher Education Press, 2012, pp. viii+231.

[5] S. Druel, "Codimension 1 foliations with numerically trivial canonical class on singular spaces", *Duke Math. J.* **170** (2021), no. 1, pp. 95–203.

[6] P. Graf, "Bogomolov-Sommese vanishing on log canonical pairs", *J. Reine Angew. Math.* **702** (2015), pp. 109–142.

[7] P. Graf and S. J. Kovács, "An optimal extension theorem for 1-forms and the Lipman-Zariski conjecture", *Doc. Math.* **19** (2014), pp. 815–830.

[8] D. Greb, S. Kebekus, S. J. Kovács and T. Peternell, "Differential forms on log canonical spaces", *Publ. Math., Inst. Hautes Étud. Sci.* (2011), no. 114, pp. 87–169.

[9] D. Greb, S. Kebekus and T. Peternell, "Reflexive differential forms on singular spaces. Geometry and cohomology", *J. Reine Angew. Math.* **697** (2014), pp. 57–89.

[10] D. Greb, S. Kebekus and T. Peternell, "Projectively flat klt varieties", *J. Éc. Polytech., Math.* **8** (2021), pp. 1005–1036.

[11] R. Hartshorne, *Algebraic geometry*, Springer, 1977, pp. xvi+496.

[12] S. Kebekus and C. Schnell, "Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities", *J. Am. Math. Soc.* **34** (2021), no. 2, pp. 315–368.

[13] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press, 1998, pp. viii+254. with the collaboration of C. H. Clemens and A. Corti.

[14] F. Loray, J. V. Pereira and F. Touzet, "Singular foliations with trivial canonical class", *Invent. Math.* **213** (2018), no. 3, pp. 1327–1380.

[15] J. Noguchi, "A short analytic proof of closedness of logarithmic forms", *Kodai Math. J.* **18** (1995), no. 2, pp. 295–299.

[16] W. Ou, "Singular rationally connected surfaces with nonzero pluri-forms", *Mich. Math. J.* **63** (2014), no. 4, pp. 725–745.

[17] J. V. Pereira and F. Touzet, "Foliations with vanishing Chern classes", *Bull. Braz. Math. Soc. (N.S.)* **44** (2013), no. 4, pp. 731–754.

[18] R. Śmiech, "Singular contact varieties", 2022. https://arxiv.org/abs/2212.07930.

[19] F. Touzet, "On the structure of codimension 1 foliations with pseudoeffective conormal bundle", in *Foliation theory in algebraic geometry. Proceedings of the conference, New York, NY, USA, September 3–7, 2013*, Springer, 2016, pp. 157–216.