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Multiplicative reduced bases for hyperelasticity

Bases réduites multiplicatives en hyperélasticité

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Abstract. The present paper introduces multiplicative reduced bases for hyperelasticity relying on a truncated version of the Baker–Campbell–Hausdorff's expansion. We show such a construction is equally interpolatory (in a multiplicative way) for the fields of deformation gradients, surfacic and volumetric deformation measures involved in large deformation mechanics. The method is naturally derived from a fully consistent variational setting and we establish an upper bound of the error in the energy norm. From a computational standpoint, the approach achieves efficient Kolmogorov n-width decay when very large rotations and incompressibility are involved.

Résumé. Cet article introduit un principe de bases réduites multiplicatives en hyperélasticité s'appuyant sur le développement de Baker–Campbell–Hausdorff. Nous montrons que cette construction produit des interpolations identiques pour les gradients de déformation, et les mesures de déformation surfaciques et volumiques en grandes déformations. Cette méthode est établie dans un cadre variationel consistant et nous montrons une borne supérieure pour l'erreur en norme de l'énergie. Numériquement, l'approche se distingue par une décroissance efficace de la n-épaisseur de Kolmogorov, particulièrement en présence de grandes rotations pour des comportements incompressibles.

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1. Introduction

Reduced bases approaches rely on low dimensional spaces to represent the solutions of a given model over its full range of parameters. They are particularly relevant whenever the manifold of solutions, as parameters vary, possesses a Kolmogorov n-width whose decay is sufficiently fast as $n \to +\infty$; cf. [7, 11, 15]. Diverse applications have been developed overtime for elliptic problems [26, 29, 32, 37], Navier–Stokes equations [16] or Quantum Mechanics [13, 31] in order to diminish the computational cost associated to the exploration of solutions. Once a set of solutions is computed for selected parameters, reduced bases can be obtained by greedy procedures [15], proper decompositions [1] including the Karhunen–Loeve decomposition [9]. See [27] and the references therein for a global overview.

This being said, the reduced problem is potentially difficult to resolve when the dependance in the parameters is highly nonlinear, which has led to empirical interpolation procedures and "Magic Points" identification [5, 28]. Besides, some highly nonlinear problems exhibit poor Kolmogorov n-width decay, such as turbulent flows (cf. Farhat [33]) or contact problems (cf. [6, 18]), for which local enrichment is required to achieve the desired accuracy. Also, some geometric properties may not be well approximated, such as energy conservation for Hamiltonian systems [22].

Large deformation solid mechanics possesses several geometric features that are not naturally respected by classical reduced bases. Here are several facts that inspire the present work.

- Deformation gradients naturally exhibit a multiplicative character that is in general exploited to formulate inelastic material laws [24, 25, 34, 35].
- The set of large rotations is a Lie manifold that is poorly approximated through vector spaces.
- The preservation of volumes and surfaces is hardly achieved by classical reduced models.

The key idea, as F_1 and F_2 stand for two deformation gradients fields, is to use the multiplicative field $F_1^{\alpha_1} F_2^{\alpha_2}$ as an approximation space; its appeal lies in that surface and volume deformation measures preserve the same structure: $\operatorname{cof}\left(F_1^{\alpha_1} F_2^{\alpha_2}\right) = \operatorname{cof}(F_1)^{\alpha_1} \operatorname{cof}(F_2)^{\alpha_2}$ and $\det\left(F_1^{\alpha_1} F_2^{\alpha_2}\right) = \det(F_1)^{\alpha_1} \det(F_2)^{\alpha_2}$. In order to circumvent the combinatorics implied by this idea for large bases, we rely on the Baker–Campbell–Hausdorff's expansion [3, 12, 17, 21], that $F_1^{\alpha_1} F_2^{\alpha_2} = \exp\left(\alpha_1 \log F_1 + \alpha_2 \log F_2 + \alpha_1 \alpha_2 [\log F_1, \log F_2] + \dots\right)$. It is issued from non-commutative geometry and possesses many applications from operator splitting to physics [8, 36]. Herein, we will restrict ourselves to the first order variant, i.e. without commutators whose added value proves to be in general limited for practical applications while being computationally quite expensive.

The paper is organized as follows. Section 2 presents the basics of hyperelasticity and introduces a mixed formulation involving deformation gradients as a distinct field; the curl-free condition is enforced under weak form and displacements reconstruction naturally follows. Section 3 proposes a multiplicative reduced based on Baker–Campbell–Hausdorff's expansion; this construction is proved to be equally interpolatory (in a multiplicative way) for the fields of deformation gradients, surfacic and volumetric deformation measures. It facilitates the obtention of an upper bound for the error in energy norm. Section 4 provides a simple numerical illustration that testify the benefit from the approach when large local rotations and incompressibility are concerned, on the case of a purely twisted bar.

2. From the problem setting to a mixed formulation involving deformation gradients

2.1. *Hyperelasticity framework*

Let $\Phi_D = \left\{\varphi \in W^{1,p}(\Omega)^3,\ \varphi \equiv \varphi_D \text{ on } \Gamma_D,\ D\varphi \in \mathscr{F}\right\}$ be the set of admissible deformation fields, where $\Gamma_D \subset \partial\Omega$ has positive measure and

$$\mathscr{F} = \left\{F \in L^p(\Omega)^{3\times 3},\ \operatorname{cof} F \in L^q(\Omega)^{3\times 3},\ \det F \in L^r(\Omega)^{3\times 3},\ \det F > 0 \text{ a.e. in } \Omega\right\}$$

stands for the set of admissible deformation gradients; $\varphi_D \in W^{1-1/p,p}(\Gamma_D)^3$ denotes the Dirichlet boundary condition. For every deformation field $\varphi \in \Phi_D$, the total energy is defined as

$$J(\varphi) = \int_\Omega W(x, D\varphi(x))\,\mathrm{d}x - L(\varphi),$$

where $L \in L^{p^*}(\Omega)$ expresses external forces. We assume the stored energy density $W : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}$ is polyconvex in the sense that for almost every $x \in \Omega$ and every $F \in \mathbb{R}^{3\times 3}$, one can write

$$W(x, F) = \mathbb{W}(x, F, \operatorname{cof} F, \det F)$$

where the function $\mathbb{W} : \Omega \times \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \to \mathbb{R}$ is convex with respect to the extended deformation measure

$$\mathbb{F} = (F, \operatorname{cof} F, \det F) \in \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R}.$$

Ball's existence theorem [4, 14, 25] establishes the existence of minimisers for J.

Remark 1. The set of admissible deformation gradients can be restricted to incompressible fields, i.e.

$$\mathscr{F} = \left\{F \in L^p(\Omega)^{3\times3}, \quad \operatorname{cof} F \in L^q(\Omega)^{3\times3}, \quad \det F \equiv 1 \text{ a.e. in } \Omega\right\},$$

and the existence result still holds.

2.2. *Discrete Euler's problem*

Since Φ_D is not a manifold in general, one cannot properly define Euler's problem associated to the minimisation of J. Still, as a minimizer, the solution φ is well approximated by a sequence $(\varphi_h)_{h>0}$ of discrete solutions built over the meshes $(\mathscr{T}_h)_{h>0}$ of the domain Ω, for which Euler's equation holds.

Compressible setting. Let $U_h \subset W^{1,s}(\Omega)^3$ be a finite element space for the displacement field built over $\mathscr{T}_h$ and

$$U_{0,h} = \{v_h \in U_h, \quad v_h \equiv 0 \text{ on } \Gamma_D\}.$$

For simplicity, we will assume that φ_D can be represented exactly by the traces of functions in U_h over Γ_D; we also denote as φ_D one of the functions in U_h admitting φ_D as a trace over Γ_D. Under mild technical assumptions (e.g. [25]), one can prove there exists $\varphi_h \in \varphi_D \oplus U_{0,h}$ such that

$$\int_\Omega \frac{\partial W}{\partial F}(x, D\varphi_h(x)) : D\delta\varphi_h(x)\,\mathrm{d}x = L(\delta\varphi_h), \quad \forall\, \delta\varphi_h \in U_{0,h}. \tag{1}$$

Let us define, for every $\varphi, \psi \in \Phi_D$,

$$\Delta(\varphi,\psi) = \|\varphi - \psi\|_{L^p(\Omega)^3} + \|\operatorname{cof} D\varphi - \operatorname{cof} D\psi\|_{L^q(\Omega)^{3\times3}} + \|\det D\varphi - \det D\psi\|_{L^r(\Omega)},$$

and assume J admits an isolated minimizer $\varphi \in \Phi_D$ in the sense that for some $\rho > 0$,

$$J(\varphi) < J(\psi), \quad \forall\, \psi \in \Phi_D \text{ such that } \Delta(\varphi,\psi) \le \rho.$$

Then, there exists a sequence $(\varphi_h)_{h>0}$ with $\varphi_h \in \varphi_D \oplus U_{0,h}$ such that

$$\lim_{h\to0} \Delta(\varphi,\varphi_h) = 0. \tag{2}$$

We refer to [25] for a complete proof.

Incompressible setting. In order to solve the incompressible problem, one has to define spaces P_h of hydrostatic pressures such that the following inf-sup condition is satisfied [2, 10, 23]:

$$\sup_{v_h \in U_{0,h}\setminus\{0\}} \frac{\int_\Omega q_h \operatorname{cof}(D\varphi_h) : Dv_h}{\|v_h\|_{W^{1,s}(\Omega)^3}} \ge \beta(h) > 0,$$

for every $\varphi_h \in \varphi_D \oplus U_{0,h}$, $h > 0$. The associated problem looks for $\varphi_h \in \varphi_D \oplus U_{0,h}$ and $p_h \in P_h$ such that

$$\int_\Omega \frac{\partial W}{\partial F}(D\varphi_h) : D\delta\varphi_h + \int_\Omega p_h \operatorname{cof}(D\varphi_h) : D\delta\varphi_h = L(\delta\varphi_h), \qquad \forall\, \delta\varphi_h \in U_{0,h}, \tag{3}$$

$$\int_\Omega q_h \left(\det D\varphi_h - 1\right) = 0, \qquad \forall\, q_h \in P_h. \tag{4}$$

Under some technical assumptions (see [25]), and for every isolated minimizer $\varphi \in \Phi_D$, there exists a sequence $(\varphi_h)_{h>0}$ with $\varphi_h \in \varphi_D \oplus U_{0,h}$ such that (2) holds.

2.3. *Mixed formulation involving deformation gradients*

Let us introduce a mixed formulation derived from the discrete problem (1) where both the deformation map $\varphi_h \in \varphi_D \oplus U_{0,h}$ and the deformation gradient are involved. The Lagrange–Hu–Washizu principle leads us to look for $F_h \in \mathscr{F}_h$, $\varphi_h \in \varphi_D \oplus U_{0,h}$, $\Pi_h \in \mathscr{P}_h$, which stationnarize

$$\mathscr{L}(F_h, \varphi_h, \Pi_h) = \int_\Omega W(F_h) - L(\varphi_h) - \int_\Omega \Pi_h : (F_h - D\varphi_h). \tag{5}$$

As a consequence,

$$\int_\Omega \Pi_h : \delta F_h = \int_\Omega \partial_F W(F_h) : \delta F_h, \qquad \forall\, \delta F_h \in \mathscr{F}_h, \tag{6}$$

$$\int_\Omega D\varphi_h : \delta\Pi_h = \int_\Omega F_h : \delta\Pi_h, \qquad \forall\, \delta\Pi_h \in \mathscr{P}_h, \tag{7}$$

$$\int_\Omega \Pi_h : D\delta\varphi_h = L(\delta\varphi_h), \qquad \forall\, \delta\varphi_h \in U_{0,h}. \tag{8}$$

Let us introduce for every $\Pi_h \in \mathscr{P}_h$, the following L^2-orthogonal decomposition

$$\Pi_h = Dw_h + \theta_h, \tag{9}$$

where $w_h \in U_{0,h}$ is uniquely defined as the solution of

$$\int_\Omega Dw_h : D\delta w_h = \int_\Omega \Pi_h : D\delta w_h, \quad \forall\, \delta w_h \in U_{0,h}.$$

Relying on decomposition (9), Equation (8) reduces to finding $w_h \in U_{0,h}$ such that

$$\int_\Omega Dw_h : D\delta\varphi_h = L(\delta\varphi_h), \quad \forall\, \delta\varphi_h \in U_{0,h}, \tag{10}$$

and we denote $L^\sharp := Dw_h$. We therefore look for $\Pi_h = L^\sharp + \theta_h$ for some $\theta_h \in \Theta_h$ where

$$\Theta_h = \left\{\theta_h \in \mathscr{P}_h, \quad \int_\Omega Dw_h : \theta_h = 0, \quad \forall\, w_h \in U_{0,h}\right\}.$$

The mixed problem of interest boils down to determining

$$F_h \in \mathscr{F}_{D,h} := \left\{G_h \in \mathscr{F}_h, \quad G_h \times n = D\varphi_D \times n \text{ a.e. on } \Gamma_D\right\},$$

where n stands for the unit normal vector on Γ_D, and $\theta_h \in \Theta_h$ such that

$$\int_\Omega \partial_F W(F_h) : \delta F_h = \int_\Omega \left(L^\sharp + \theta_h\right) : \delta F_h, \qquad \forall\, \delta F_h \in \mathscr{F}_h, \tag{11}$$

$$\int_\Omega F_h : \delta\theta_h = \int_\Omega D\varphi_D : \delta\theta_h, \qquad \forall\, \delta\theta_h \in \Theta_h. \tag{12}$$

The reconstruction of $\varphi_h \in \varphi_D \oplus U_{0,h}$ is naturally performed by the resolution of the Laplace problem

$$\int_\Omega D\varphi_h : D\delta w_h = \int_\Omega F_h : D\delta w_h, \qquad \forall\, \delta w_h \in U_{0,h}, \tag{13}$$

which we denote as $\varphi_h = \mathscr{H}(F_h)$.

The computation of $L^\sharp$ and the reconstruction of φ_h involve the resolution of the same Laplace problem with different right-hand-sides, and this Laplace problem can be factorized once for all independently of the implicit parameters $\lambda \in \Lambda$ of the problem (material parameters, domain shape…). We therefore focus on the resolution of the mixed problem (11)–(12).

The formulation presented in this section will be exclusively employed with reduced bases throughout the paper, the sampled solutions being exclusively computed from formulations (1) or (3)–(4).

3. Multiplicative Reduced Bases

3.1. *Preliminaries*

For every matrix $A \in \mathbb{R}^{n\times n}$, the exponential and logarithm operators are defined as

$$\exp(A) = \sum_{k=0}^{+\infty} \frac{A^k}{k!}, \qquad \log(I - A) = -\sum_{k=1}^{+\infty} \frac{A^k}{k}, \tag{14}$$

with respective convergence radii $+\infty$ and 1, in the norm $\|A\| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} |Ax|/|x|$. Introducing the complex valued diagonalization $A = P^{-1}DP$, one has $\log A = P^{-1}\log(D)P$ where the diagonal elements read $\log(D)_{mm} = \log(D_{mm}) = \log(\varrho) + i\psi \mod 2i\pi$, $1 \le m \le n$, for $D_{mm} = \varrho\exp(i\psi)$ in which $\varrho \in \mathbb{R}_+$ and $\psi \in [0, 2\pi)$. We arbitrarily choose the principal value $\log(D)_{mm} = \log(\varrho) + i\psi$.

The following theorem is fundamental for the sequel. It relies on the work of Baker [3], Campbell [12] and Hausdorff [21]; the explicit form hereunder is due to Dynkin [17]. This result underlies multiple applications from linear PDEs to Numerical Analysis and several monographs are dedicated to it (Bonfiglioli & Fulci [8], Varadarajan [36], see also the developments in [20]).

Theorem 2 (BCH). *Let $X, Y \in \mathbb{R}^{n\times n}$ be square matrices. There exists a matrix $Z \in \mathbb{R}^{n\times n}$ such that*

$$\exp(X)\exp(Y) = \exp(Z),$$

which is given by the following formal series

$$Z = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} \sum_{\substack{r_i+s_i>0\\ 1\le i\le k}} \frac{\left[X^{(r_1)}, Y^{(s_1)}, \dots, X^{(r_k)}, Y^{(s_k)}\right]}{\left(\sum_{i=1}^k (r_i+s_i)\right)\prod_{i=1}^k r_i! s_i!}, \tag{15}$$

where the last sum is over the indices $(r_i) \in \mathbb{N}$ and $(s_i) \in \mathbb{N}$, $1 \le i \le k$, such that $r_i + s_i$ has positive value for every $1 \le i \le k$. Besides, the following notation holds:

$$[A_1, \dots, A_m] = [A_1, [\dots, [A_{m-2}, [A_{m-1}, A_m]]]],$$

where $[A, B] = AB - BA$ is the matrix commutator and $A^{(r)} = \underbrace{A, A, \dots, A}_{r\ times}$. The convergence of the series (15) *holds whenever $\|X\| + \|Y\| < \log 2$.*

Let us now proceed to the estimation of the coefficients from the BCH formula at all orders. It can be rewritten as

$$Z = X + Y + \sum_{n\ge2}\sum_{m\ge1}\sum_{w\in\mathscr{W}_{m,n}(X,Y)} \frac{1}{n} g_w\,[w],$$

where words $w \in \mathscr{W}_{m,n}(X,Y)$ of length n with m parts are expressions of the form

$$X^{\sigma_1}Y^{\sigma_2}\dots(X\vee Y)^{\sigma_m} \quad \text{or} \quad Y^{\sigma_1}X^{\sigma_2}\dots(X\vee Y)^{\sigma_m},$$

that end with X or Y; $(\sigma_i)_{1\le i\le m} \in \mathbb{N}$ are such that $\sum_{i=1}^m \sigma_i = n$. The notation $[w]$ refers to the iterated commutators, for instance $[X^{(\sigma_1)}, Y^{(\sigma_2)}, \dots, (X\vee Y)^{(\sigma_m)}]$, as defined in Theorem 2.

Following Goldberg [19] and So [30], one has

$$\gamma_n = \frac{1}{n}\sum_{m=1}^{n}\sum_{w\in\mathscr{W}_{m,n}(X,Y)} |g_w| \le \frac{2}{n}. \tag{16}$$

Observe that the estimate (16) is suboptimal because, cf. [20], $\gamma_1 = 1/2$, $\gamma_2 = 1/6$, $\gamma_3 = 1/24$, $\gamma_4 = 1/40$.

As a result, assuming that $\|X\| \le M$ and $\|Y\| \le M$, one has

$$\|Z\| \le 2M + \sum_{n\ge2} \frac{1}{n}(2M)^n = -\log(1-2M)$$

since $\|[w]\| \leq 2^{n-1}M^n$. The upper bound imposes the condition $2M < 1$ to comply with the convergence radius of the log series. The resulting condition $M < 1/2$ is slightly worse than the $(\log 2)/2$ stated in Theorem 2.

3.2. *Reduced bases for deformation gradients*

Herein, we define low dimensional manifolds based on the multiplicative combinations of the deformation gradient fields, relying on the truncated Baker–Campbell–Hausdorff's [3, 12, 17, 21] expansion.

Definition 3. *Let $(\lambda_i)_{1\leq i\leq I} \in \Lambda$ be a collection of selected parameters for the system, and $F_{\lambda_i} = D\varphi_{\lambda_i} \in \mathscr{F}_h$, $1 \leq i \leq I$ be the solutions; in order to filter out the boundary conditions, we introduce*

$$\mathfrak{F}_{\lambda_i} = F_{\lambda_i}\, D\varphi_D^{-1}.$$

Let us define the following reduced manifold

$$\mathscr{F}_I = \left\{ \exp\left(\sum_{1\leq i\leq I} \alpha_i \log \mathfrak{F}_{\lambda_i} \right) D\varphi_D;\ (\alpha_i) \in \mathbb{R} \right\}.$$

Owing to the identity $\log\det = \operatorname{tr}\log$ and its counterpart $\det\exp = \exp\operatorname{tr}$, the multiplicative structure conveys to surface and volume deformation measures:

Lemma 4. *For every $F \in \mathscr{F}_I$ with the above notation, one has*

$$\begin{aligned}
\operatorname{cof} F &= \exp\left(\sum_{1\leq i\leq I} \alpha_i \log \operatorname{cof} \mathfrak{F}_{\lambda_i} \right) \operatorname{cof} D\varphi_D,\\
\det F &= \exp\left(\sum_{1\leq i\leq I} \alpha_i \log \det \mathfrak{F}_{\lambda_i} \right) \det D\varphi_D = \left(\prod_{1\leq i\leq I} \det \mathfrak{F}_{\lambda_i}^{\alpha_i} \right) \det D\varphi_D.
\end{aligned}$$

3.3. *Reduced formulation*

In order to control the compatibility of reduced deformation gradient fields, let us define

$$\theta_{\lambda_i} = \partial_F W(F_{\lambda_i}) - L^{\#}_{\lambda_i}, \quad 1 \leq i \leq I,$$

that span the space $\Theta_I = \operatorname{span}\{\theta_{\lambda_i}; 1 \leq i \leq I\}$. The mixed problem of interest will determine $\widetilde{F} \in \mathscr{F}_I$ and $\widetilde{\theta} \in \Theta_I$, such that

$$\int_\Omega \partial_F W(\widetilde{F}) : \delta\widetilde{F} = \int_\Omega \left(L^\sharp + \widetilde{\theta} \right) : \delta\widetilde{F}, \qquad \forall\, \delta\widetilde{F} \in T_{\widetilde{F}}\mathscr{F}_I, \tag{17}$$

$$\int_\Omega \widetilde{F} : \delta\widetilde{\theta} = 0, \qquad \forall\, \delta\widetilde{\theta} \in \Theta_I. \tag{18}$$

3.4. *Kolmogorov n-width, choice of bases vectors and convergence*

In the present setting, the notion of Kolmogorov width naturally extends to the following definition

$$d_I(F(\Lambda)) = \inf_{(F_{\lambda_i})_{1\leq i\leq I}} \sup_{\lambda\in\Lambda} \inf_{\widetilde{F}\in\mathscr{F}_I} \Delta\left(\widetilde{F}, F_\lambda\right),$$

where

$$\Delta(\widetilde{F}, F_\lambda) = \left\| \widetilde{F} F_\lambda^{-1} - \mathrm{Id} \right\|_{L^p} + \left\| \operatorname{cof}\left(\widetilde{F} F_\lambda^{-1}\right) - \mathrm{Id} \right\|_{L^q} + \left\| \det\left(\widetilde{F} F_\lambda^{-1}\right) - 1 \right\|_{L^r}.$$

It represents the best approximation of the solution manifold $F(\Lambda) = \{F_\lambda; \lambda \in \Lambda\}$ by our multiplicative reduced bases in the distance implied by the stored energy functions. This definition naturally inspires a greedy algorithm to select the bases vectors:

$$\lambda_1 = \arg\max_{\lambda\in\Lambda} \Delta(\mathrm{Id}, F_\lambda),$$
$$\lambda_{i+1} = \arg\max_{\lambda\in\Lambda} \inf_{\tilde F\in\mathscr{F}_i} \Delta\left(\widetilde{F}, F_\lambda\right), \quad i \geq 1.$$

In practice, the selection of λ_{i+1} can be performed among a large set of precomputed solutions.

Notation. Let us introduce

$$\mathscr{M}_1 = \sup_{1\le i\le I} \left\|\log \mathfrak{F}_i\right\|, \qquad \mathscr{M}_k = \sup_{1\le i_1,\dots,i_k\le I} \left\|\left[\log\mathfrak{F}_{i_1}, \log\mathfrak{F}_{i_2}, .., \log\mathfrak{F}_{i_k}\right]\right\|, \qquad \forall\, k>1,$$
$$\mathscr{M}_1^{\mathrm{cof}} = \sup_{1\le i\le I} \left\|\log\mathrm{cof}\,\mathfrak{F}_i\right\|, \quad \mathscr{M}_k^{\mathrm{cof}} = \sup_{1\le i_1,\dots,i_k\le I} \left\|\left[\log\mathrm{cof}\,\mathfrak{F}_{i_1}, \log\mathrm{cof}\,\mathfrak{F}_{i_2}, .., \log\mathrm{cof}\,\mathfrak{F}_{i_k}\right]\right\|, \quad \forall\, k>1,$$

and assume $\mathscr{M}_k, \mathscr{M}_k^{\mathrm{cof}} \in L^\infty(\Omega)$ for every $k \geq 1$. It is clear that $\mathscr{M}_k \leq 2^{k-1}(\mathscr{M}_1)^k$ and $\mathscr{M}_k^{\mathrm{cof}} \leq 2^{k-1}(\mathscr{M}_1^{\mathrm{cof}})^k$ a.e. on Ω.

Proving that $d_I(F(\Lambda))$ decays significantly faster than its classical linear counterpart as I grows is a difficult task, even though the numerics shows it is the case for highly nonlinear problems. Still, owing to the multiplicative structure, the error can be bounded in the energy norm, as exemplified in the following exercise.

Theorem 5. *With the above notation, for every $F \in \mathscr{F}_h$ which decomposes as*

$$F = \exp\left(\sum_{i=1}^{+\infty} \alpha_i \log\mathfrak{F}_{\lambda_i}\right) D\varphi_D, \quad (\alpha_i) \in \mathbb{R}, \tag{19}$$

one has

$$\inf_{\tilde F\in\mathscr{F}_I} \Delta(\widetilde{F}, F) \leq \left\|\exp(\mathscr{R}_I) - 1\right\|_{L^p} + \left\|\exp(\mathscr{R}_I^{\mathrm{cof}}) - 1\right\|_{L^q} + \left\|\mathscr{R}_I^{\mathrm{det}} - 1\right\|_{L^r},$$

where for almost every $x \in \Omega$,

$$\mathscr{R}_I(x) \leq \left(\sum_{i'=I+1}^{+\infty} |\alpha_{i'}|\right)\left(\mathscr{M}_1(x) + \sum_{n=2}^{+\infty} \frac{2}{n}\left(\sum_{i=1}^{+\infty}|\alpha_i|\right)^{n-1} \mathscr{M}_n(x)\right).$$

The upper bound on $\mathscr{R}_I^{\mathrm{cof}}$ is identical, $(\mathscr{M}_k)$ being replaced by $(\mathscr{M}_k^{\mathrm{cof}})$, and $\mathscr{R}_I^{\mathrm{det}} \leq \prod_{i'=I+1}^{+\infty}\left(\det\mathfrak{F}_{\lambda_{i'}}\right)^{|\alpha_{i'}|}$.

The proof of the above theorem relies on Baker–Campbell–Hausdorff's expansion ; cf. Section 3.1.

Remark 6 (Convergence of the method). The convergence of the multiplicative reduced basis approximation requires that

- $\sum_{i'=I+1}^{+\infty} |\alpha_{i'}| \to 0$ as I increases, i.e. the efficiency of the selected basis (F_{λ_i}),
- the amplitude of the iterated commutators $\mathscr{M}_k(x)$ decays exponentially fast as $k \to \infty$.

The same naturally holds for $\mathscr{R}_I^{\mathrm{cof}}(x)$, whereas $\mathscr{R}_I^{\mathrm{det}}(x)$ does not involve any restriction on commutators. Whenever these conditions are satisfied, convergence is achieved for the method and no restriction holds on the norms of F and $\widetilde{F}$ in order to respect the convergence radius of the general BCH formula.

3.5. *Summary of the method*

In order to solve efficiently problem (1) over a wide range of parameters, the proposed method proceeds with the following steps.

- Offline stage
  - Solve problem (1) for selected parameters $(\lambda_i)_{1\le i\le I}$.
  - Assemble and factorise the matrix $\mathbb{A}_h$ of the Laplace operator from Eq. (10) that reads $\mathbb{A}_h w_h = L$.
- Online stage; for every targeted state of the system, described by the parameter $\lambda\in\Lambda$,
  - *Lift the right hand side* – Compute $w_{\lambda,h}\in U_{0,h}$ such as $\mathbb{A}_h w_{\lambda,h} = L_\lambda$, as in Eq. (10).
  - *Solve the reduced problem* – Determine $\widetilde{F}\in\mathscr{F}_I$ and $\widetilde{\theta}\in\Theta_I$ according to Eqs. (17)-(18), i.e.

$$\int_\Omega \partial_F W_\lambda(\widetilde{F}):\delta\widetilde{F} = \int_\Omega \left(Dw_{\lambda,h}+\widetilde{\theta}\right):\delta\widetilde{F}, \qquad \forall\,\delta\widetilde{F}\in T_{\widetilde{F}}\mathscr{F}_I,$$
$$\int_\Omega \widetilde{F}:\delta\widetilde{\theta} = 0, \qquad \forall\,\delta\widetilde{\theta}\in\Theta_I.$$

  - *Reconstruct displacements* – The displacement field $\varphi_{\lambda,h}\in\varphi_{\lambda,D}\oplus U_{0,h}$ reconstructs as in Eq. (13) as we solve $\mathbb{A}_h\varphi_{\lambda,h} = D^*\widetilde{F}$, with the right hand side

$$D^*\widetilde{F}\cdot w_h = \int_\Omega \widetilde{F}:D\delta w_h, \qquad \forall\,\delta w_h\in U_{0,h}.$$

4. A simple numerical illustration: twisted bar

In this section, we illustrate on a simple but very nonlinear case, the characteristics and the main assets of the proposed reduced basis approach.

4.1. *The manifold of purely twisted states*

We compute herein the torsion of a bar $(1m\times 3m\times 20m)$ along its main axis e_2, by the imposition of pure rotations $R(\beta)$ and $R(-\beta)$ of its tips, $\beta\in[0,\pi]$. The employed material obeys the Mooney–Rivlin constitutive law $W(F) = |F|^2 + |\mathrm{cof}\,F|^2$, under the incompressibility constraint $\det F\equiv 1$. The finite element discretisation is built on a tetrahedral mesh (mesh-size $h = 0.5m$) using $\mathbb{P}_2$ displacements fields and $\mathbb{P}_1$ continuous hydrostatic pressures. Figure 1 displays the corresponding deformation states, as β increases from 0 to π. We arbitrarily select the deformation field φ_{β_0} with $\beta_0 = \frac{\pi}{4}$ and compare the classical reduced basis line $\left\{\overline{\phi}_\alpha := \mathrm{id}+\alpha\left(\varphi_{\beta_0}-\mathrm{id}\right);\alpha\in\mathbb{R}\right\}$, with the smallest reduced manifold built in this paper,

$$\left\{\widetilde{\phi}_\alpha := \mathscr{H}\left(F_{\beta_0}^\alpha\right);\alpha\in\mathbb{R}\right\}.$$

Figure 2 displays the reduced bases approximations $\overline{\phi}_\alpha$ and $\widetilde{\phi}_\alpha$ for $\alpha = 0.5, 2, 4$. The multiplicative version described herein clearly shows a better representation of deformation fields, since α describes the torsion level of the bar whereas for classical reduced bases, α only parameterizes the amplitude.

We denote $\widetilde{F}_\alpha = F_{\beta_0}^\alpha$ and $\overline{F}_\alpha = \mathrm{id}+\alpha\left(F_{\beta_0}-\mathrm{id}\right)$. Table 1 shows that $\widetilde{F}_{\beta/\beta_0}$ provides a good approximation of F_β, even when approximated by piecewise-constant values. From a numerical standpoint, the best approximation $\widetilde{F}_\alpha$ of F_β is computed as the minimizer of the penalized problem,

$$\inf_\alpha \int_\Omega W(\widetilde{F}_\alpha) + K\int_{\Gamma_D}\left|\left(\widetilde{F}_\alpha - R(\pm\beta)\right)\times n\right|^2, \qquad K = 10^6\,J/m^2, \tag{20}$$

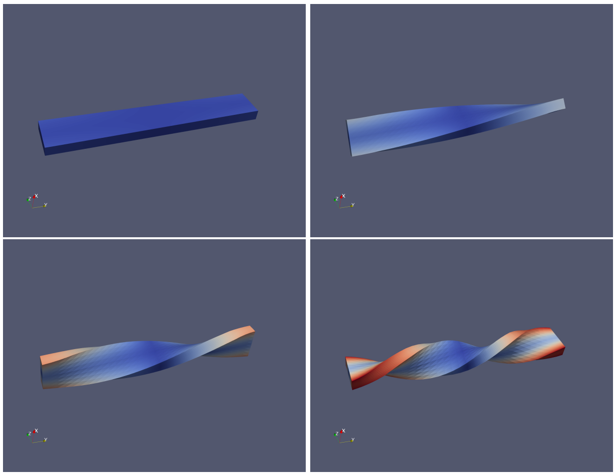

Figure 1. Deformation states of a bar undergoing torsion along its main axis, by the imposition of pure rotations $R(\beta)$ and $R(-\beta)$ of its tips. Snapshots correspond to $\beta = 0, \pi/4, \pi/2, \pi$.

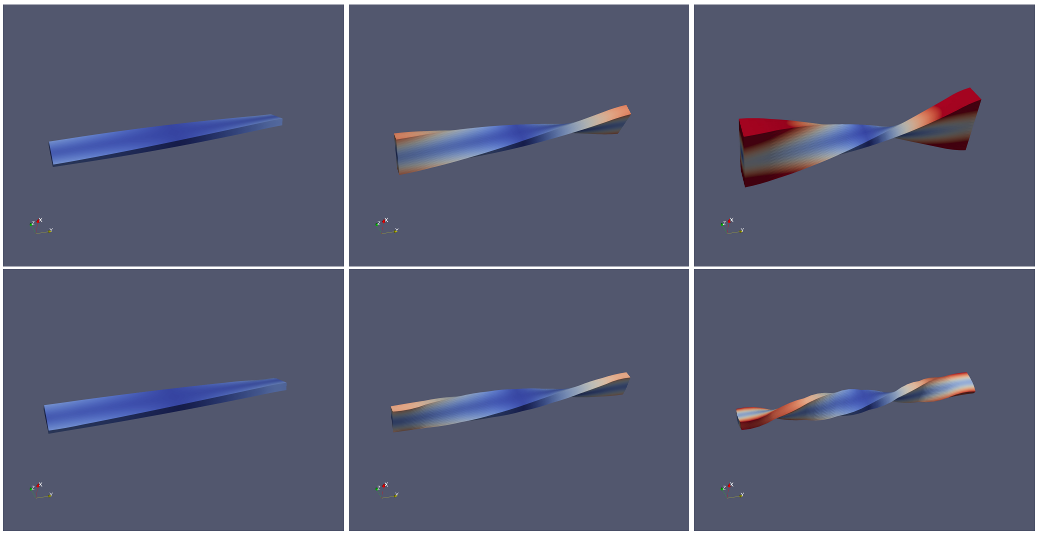

Figure 2. Deformations spanned by the reduced bases ϕ_α (top) and $\widetilde{\phi}_\alpha$ (bottom) for $\alpha = 0.5, 2, 4$.

which shows indeed that $\alpha = \beta/\beta_0$ provides the best approximation, with excellent numerical accuracy. In Table 1, we also display the norm $\|D\mathscr{H}(\widetilde{F}_\alpha) - \widetilde{F}_\alpha\|_{L^2(\Omega)}$ as an indicator of the curl-free constraint violation for $\widetilde{F}_\alpha$; it is where the most part of the approximation error is concentrated in the $\mathbb{P}_0$ case. In the $\mathbb{P}_2$ case, this reconstruction error does not introduce any noticeable error in the energy norm of the solution. Table 2 displays the situation for the classical $1D$ vector reduced basis approach. It reveals the impossibility for the $1D$ classical reduced basis, to provide a reasonable approximation of the deformation states and the boundary condition at the same time.

Table 1. Approximation provided by the $1D$ multiplicative reduced basis relying on the pre-computed solution $F_{\pi/4}$ for $\mathbb{P}_0$ (left) and $\mathbb{P}_2$ (right) approximations of deformation gradients.

Approx. space for $F_{\beta_0}, \beta_0 = \pi/4$	$\mathbb{P}_0$			$\mathbb{P}_2$		
β / α	$\pi/8$ / 0.5	$\pi/2$ / 2.0	π / 4.0	$\pi/8$ / 0.5	$\pi/2$ / 2.0	π / 4.0
$\Vert\widetilde{F}_\alpha - F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$2,636\,10^{-03}$	$2,069\,10^{-02}$	$1,121\,10^{-01}$	$2,953\,10^{-03}$	$2,309\,10^{-02}$	$1,243\,10^{-01}$
$\Vert\mathrm{cof}\,\widetilde{F}_\alpha - \mathrm{cof}\,F_\beta\Vert_{0,\Omega}/\Vert\mathrm{cof}\,F_\beta\Vert_{0,\Omega}$	$2,638\,10^{-03}$	$2,070\,10^{-02}$	$1,121\,10^{-01}$	$2,956\,10^{-03}$	$2,314\,10^{-02}$	$1,244\,10^{-01}$
$\Vert\det\widetilde{F}_\alpha - \det F_\beta\Vert_{0,\Omega}/\Vert\det F_\beta\Vert_{0,\Omega}$	$1,664\,10^{-04}$	$1,248\,10^{-03}$	$6,315\,10^{-03}$	$1,907\,10^{-04}$	$1,419\,10^{-03}$	$7,288\,10^{-03}$
$\Vert D\mathscr{H}(\widetilde{F}_\alpha) - \widetilde{F}_\alpha\Vert_{0,\Omega}/\Vert\widetilde{F}_\alpha\Vert_{0,\Omega}$	$2,225\,10^{-02}$	$8,852\,10^{-02}$	$1,824\,10^{-01}$	$.2,147\,10^{-03}$	$1,670\,10^{-02}$	$8,967\,10^{-02}$
$\int_\Omega(\det\widetilde{F}_\alpha - \det F_\beta)/\int_\Omega \det F_\beta$	$1,654\,10^{-04}$	$-1,238\,10^{-03}$	$-6,224\,10^{-03}$	$-2,667\,10^{-07}$	$2,087\,10^{-05}$	$3,526\,10^{-04}$
$\Vert(F - F_D)\times N\Vert_{0,\Gamma_D}/\Vert F_D \times N\Vert_{0,\Gamma_D}$	$6,039\,10^{-06}$	$3,780\,10^{-04}$	$1,146\,10^{-02}$	$8,316\,10^{-06}$	$5,151\,10^{-04}$	$1,533\,10^{-02}$
$\Vert D\widetilde{\phi}_\alpha - F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$2,228\,10^{-02}$	$8,891\,10^{-02}$	$1,867\,10^{-01}$	$2,028\,10^{-03}$	$1,595\,10^{-02}$	$8,618\,10^{-02}$
$\Vert\mathrm{cof}\,D\widetilde{\phi}_\alpha - \mathrm{cof}\,F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$2,230\,10^{-02}$	$8,960\,10^{-02}$	$1,908\,10^{-01}$	$2,185\,10^{-03}$	$1,701\,10^{-02}$	$9,029\,10^{-02}$
$\Vert\det D\widetilde{\phi}_\alpha - \det F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$1,686\,10^{-03}$	$2,071\,10^{-02}$	$8,586\,10^{-02}$	$1,412\,10^{-03}$	$1,036\,10^{-02}$	$4,944\,10^{-02}$

Table 2. Approximation provided by the $1D$ classical reduced basis relying on the pre-computed solution $F_{\pi/4}$ for $\mathbb{P}_0$ (left) and $\mathbb{P}_2$ (right) approximations of the deformation gradient fields.

Approx. space for $F_{\beta_0}, \beta_0 = \pi/4$	$\mathbb{P}_0$			$\mathbb{P}_2$		
β / α	$\pi/8.$ / $\alpha = 0,5$	$\pi/2$ / $\alpha = 1,80050302$	$\pi.$ / $\alpha = 1,87869848$	$\pi/8$ / $\alpha = 0,5004963$	$\pi/2$ / $\alpha = 1,7998276$	π / $\alpha = 1,86640859$
$\Vert\overline{F}_\alpha - F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$3,072\,10^{-02}$	$2,244\,10^{-01}$	$9,276\,10^{-01}$	$3,097\,10^{-02}$	$2,261\,10^{-01}$	$9,305\,10^{-01}$
$\Vert\mathrm{cof}\,\overline{F}_\alpha - \mathrm{cof}\,F_\beta\Vert_{0,\Omega}/\Vert\mathrm{cof}\,F_\beta\Vert_{0,\Omega}$	$5,189\,10^{-02}$	$3,304\,10^{-01}$	$9,817\,10^{-01}$	$5,221\,10^{-02}$	$3,315\,10^{-01}$	$9,781\,10^{-01}$
$\Vert\det\overline{F}_\alpha - \det F_\beta\Vert_{0,\Omega}/\Vert\det F_\beta\Vert_{0,\Omega}$	$7,248\,10^{-02}$	$4,200\,10^{-01}$	$4,894\,10^{-01}$	$7,285\,10^{-02}$	$4,204\,10^{-01}$	$4,723\,10^{-01}$
$\Vert D\mathscr{H}(\overline{F}_\alpha) - \overline{F}_\alpha\Vert_{0,\Omega}/\Vert\overline{F}_\alpha\Vert_{0,\Omega}$	$2,258\,10^{-02}$	$7,237\,10^{-02}$	$7,457\,10^{-02}$	$2,688\,10^{-15}$	$8,473\,10^{-15}$	$8,729\,10^{-15}$
$\int_\Omega(\det\overline{F}_\alpha - \det F_\beta)/\int_\Omega \det F_\beta$	$5,554\,10^{-02}$	$-3,220\,10^{-01}$	$-3,772\,10^{-01}$	$5,570\,10^{-02}$	$-3,214\,10^{-01}$	$-3,611\,10^{-01}$
$\Vert(F - F_D)\times N\Vert_{0,\Gamma_D}/\Vert F_D \times N\Vert_{0,\Gamma_D}$	$1,416\,10^{-03}$	$7,600\,10^{-02}$	$1,314$	$1,440\,10^{-03}$	$7,710\,10^{-02}$	$1,316$
$\Vert D\overline{\phi}_\alpha - F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$3,764\,10^{-02}$	$2,386\,10^{-01}$	$9,306\,10^{-01}$	$3,097\,10^{-02}$	$2,261\,10^{-01}$	$9,305\,10^{-01}$
$\Vert\mathrm{cof}\,D\overline{\phi}_\alpha - \mathrm{cof}\,F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$5,609\,10^{-02}$	$3,359\,10^{-01}$	$9,823\,10^{-01}$	$5,221\,10^{-02}$	$3,315\,10^{-01}$	$9,781\,10^{-01}$
$\Vert\det D\overline{\phi}_\alpha - \det F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$7,218\,10^{-02}$	$4,082\,10^{-01}$	$4,761\,10^{-01}$	$7,285\,10^{-02}$	$4,204\,10^{-01}$	$4,723\,10^{-01}$

4.2. *Efficient approximation of purely twisted states*

Finally, we assess the ability of multiplicative reduced bases to approximate accurately the whole range of solutions as β varies. We make the choice of the most relevant bases vectors by the greedy procedure presented in Section 3.4 on the basis of the $\mathbb{P}_2$ approximation of gradients. Table 3 displays the convergence of relative errors; they prove to achieve better values, by almost 2 orders of magnitude, and a better convergence rate than the one achieved by the classical reduced basis method.

Table 3. Convergence of the multiplicative approach (left) as I increases for the bar torsion case as compared to the classical approach (right). The table displays the worse relative errors obtained for the considered reduced space.

	$\widetilde{\mathscr{F}}_1$	$\widetilde{\mathscr{F}}_2$	$\widetilde{\mathscr{F}}_3$	$\overline{\mathscr{F}}_1$	$\overline{\mathscr{F}}_2$	$\overline{\mathscr{F}}_3$
$\Vert\widetilde{F}_\alpha - F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$4,85\,10^{-02}$	$5,73\,10^{-03}$	$5,45\,10^{-04}$	$3,83\,10^{-01}$	$5,80\,10^{-02}$	$1,98\,10^{-02}$
$\Vert\operatorname{cof}\widetilde{F}_\alpha - \operatorname{cof}F_\beta\Vert_{0,\Omega}/\Vert\operatorname{cof}F_\beta\Vert_{0,\Omega}$	$4,86\,10^{-02}$	$5,66\,10^{-03}$	$5,47\,10^{-04}$	$5,19\,10^{-01}$	$6,97\,10^{-02}$	$2,66\,10^{-02}$
$\Vert\det\widetilde{F}_\alpha - \det F_\beta\Vert_{0,\Omega}/\Vert\det F_\beta\Vert_{0,\Omega}$	$2,14\,10^{-03}$	$4,36\,10^{-04}$	$2,50\,10^{-05}$	$6,11\,10^{-01}$	$6,61\,10^{-02}$	$3,07\,10^{-02}$
$\Vert D\mathscr{H}(\widetilde{F}_\alpha) - \widetilde{F}_\alpha\Vert_{0,\Omega}/\Vert\widetilde{F}_\alpha\Vert_{0,\Omega}$	$3,43\,10^{-02}$	$4,08\,10^{-03}$	$3,59\,10^{-04}$	0	0	0
$\int_\Omega(\det\overline{F}_\alpha - \det F_\beta)/\int_\Omega \det F_\beta$	$-1,69\,10^{-04}$	$-1,44\,10^{-05}$	$-5,55\,10^{-07}$	$5.02\,10^{-01}$	$-5,59\,10^{-02}$	$2,61\,10^{-02}$
$\Vert(F-F_D)\times N\Vert_{0,\Gamma_D}/\Vert F_D\times N\Vert_{0,\Gamma_D}$	$2,55\,10^{-03}$	$3,73\,10^{-05}$	$5,47\,10^{-07}$	$2,21\,10^{-01}$	$4,94\,10^{-03}$	$5,80\,10^{-04}$
$\Vert D\widetilde{\phi}_\alpha - F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$3,42\,10^{-02}$	$4,02\,10^{-03}$	$4,10\,10^{-04}$	$3,83\,10^{-01}$	$5,80\,10^{-02}$	$1,98\,10^{-02}$
$\Vert\operatorname{cof}D\widetilde{\phi}_\alpha - \operatorname{cof}F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$3,69\,10^{-02}$	$4,19\,10^{-03}$	$4,59\,10^{-04}$	$5,19\,10^{-01}$	$6,97\,10^{-02}$	$2,66\,10^{-02}$
$\Vert\det D\widetilde{\phi}_\alpha - \det F_\beta\Vert_{0,\Omega}/\Vert F_\beta\Vert_{0,\Omega}$	$2,36\,10^{-02}$	$2,39\,10^{-03}$	$3,67\,10^{-04}$	$6,11\,10^{-01}$	$6,61\,10^{-02}$	$3,07\,10^{-02}$

5. Conclusion

The present paper derives and exemplifies the use of multiplicative reduced bases for nonlinear elasticity. The benefit is particularly obvious when local rotations or incompressibility phenomena are involved due to the respect of the Lie group structure of solutions. Natural extensions to inelasticity or shape optimisation can be envisioned as additional tensor fields play multiplicative roles. Finally, extension to elastodynamics can be derived similarly through the classical action

$$S(\varphi) = \int_0^T \int_\Omega \left[\frac{1}{2}\varrho\dot{\varphi}^2 - W\left(\frac{\partial\varphi}{\partial x}\right)\right] - \int_0^T L(t,\varphi(t))\,\mathrm{d}t$$

to stationarize, where the following space-time mapping $\boldsymbol{\phi} : [0,\mathscr{T}]\times\Omega \to [0,T]\times\mathbb{R}^3$ can be plugged-in, and the weak equality $\mathbb{F} \equiv \boldsymbol{D\phi}$ can be enforced. All the rest follows readily, the lifting and reconstruction operations being chosen so as to involve the wave equation.

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