ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Mumtaz Hussain and Johannes Schleischitz

The Baker–Schmidt problem for dual approximation and some classes of manifolds

Volume 362 (2024), p. 817-828

Online since: 7 October 2024

https://doi.org/10.5802/crmath.585

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2024, Vol. 362, p. 817-828
https://doi.org/10.5802/crmath.585

Research article / *Article de recherche*
Number theory / *Théorie des nombres*

The Baker–Schmidt problem for dual approximation and some classes of manifolds

Le problème de Baker–Schmidt pour l'approximation duale et quelques classes de variétés

Mumtaz Hussain [*,a] **and Johannes Schleischitz** [b]

[a] Mumtaz Hussain, Department of Mathematical and Physical Sciences, La Trobe University, Bendigo 3552, Australia

[b] Middle East Technical University, Northern Cyprus Campus, Kalkanli, Güzelyurt, Cyprus

E-mails: m.hussain@latrobe.edu.au, johannes@metu.edu.tr, jschleischitz@outlook.com

Abstract. The Generalised Baker–Schmidt Problem (1970) concerns the Hausdorff f-measure of the set of Ψ-approximable points on a nondegenerate manifold. We refine and extend our previous work [Int. Math. Res. Not. IMRN 2021, no. 12, 8845–8867] in which we settled the problem (for dual approximation) for hypersurfaces. We verify the GBSP for certain classes of nondegenerate submanifolds of codimension greater than 1. Concretely, for codimension two or three, we provide examples of manifolds where the dependent variables can be chosen as quadratic forms. Our method requires the manifold to have even dimension at least the minimum of four and half the dimension of the ambient space. We conjecture that these restrictions on the dimension of the manifold are sufficient to provide similar examples in general.

Résumé. Le problème de Baker–Schmidt généralisé (1970) concerne la mesure de Hausdorff f de l'ensemble des points Ψ-approximables sur une variété non dégénérée. Nous affinons et étendons notre travail précédent [Int. Math. Res. Not. IMRN 2021, no. 12, 8845-8867] dans lequel nous avons résolu le problème (pour l'approximation duale) pour les hypersurfaces. Nous vérifions le GBSP pour certaines classes de sous-variétés non dégénérées de codimension supérieure à 1. Concrètement, pour la codimension deux ou trois, nous donnons des exemples de variétés où les variables dépendantes peuvent être choisies comme des formes quadratiques. Notre méthode exige que la variété ait une dimension paire au moins égale au minimum de quatre et à la moitié de la dimension de l'espace ambiant.

Nous conjecturons que ces restrictions sur la dimension de la variété sont suffisantes pour fournir des exemples similaires en général.

Keywords. Baker–Schmidt Problem, Hausdorff measure and dimension, Jarnik theorem.

Mots-clés. Problème de Baker–Schmidt, mesure et dimension de Hausdorff, théorème de Jarnik.

2020 Mathematics Subject Classification. 28A78, 58C35.

Funding. The research of Mumtaz Hussain is supported by the Australian Research Council Discovery Project (200100994).

Manuscript received 24 February 2023, revised 15 September 2023, accepted 6 November 2023.

* Corresponding author

1. Dual Diophantine approximation on manifolds

Let $n \geq 1$ be a fixed integer, $\mathbf{q} := (q_1,\ldots,q_n) \in \mathbb{Z}^n$, and $\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$. Let $\Psi : \mathbb{Z}^n \to [0,\infty)$ be a *multivariable approximating function*, that is, Ψ has the property that $\Psi(\mathbf{q}) \to 0$ as $\|\mathbf{q}\| := \max(|q_1|,\ldots,|q_n|) \to \infty$. Let θ be an arbitrary real number[1]. Consider the set

$$\mathscr{D}_n^\theta(\Psi) := \left\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{q}\cdot\mathbf{x} + p + \theta| < \Psi(\mathbf{q}) \text{ for i.m. } (p,\mathbf{q}) \in \mathbb{Z}\times\mathbb{Z}^n\right\},$$

where "i.m." stands for "infinitely many". A vector $\mathbf{x} \in \mathbb{R}^n$ will be called (Ψ,θ)*-approximable* if it lies in the set $\mathscr{D}_n^\theta(\Psi)$. When $\theta = 0$, the problem reduces to the *homogeneous* setting. We are interested in the "size" of the set $\mathscr{D}_n^\theta(\Psi)$ with respect to the f-dimensional Hausdorff measure $\mathscr{H}^f$ for some dimension function f. By a dimension function f we mean an increasing continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$.

Diophantine approximation on manifolds concerns the study of approximation properties of points in $\mathbb{R}^n$ which are functionally related or in other words restricted to a submanifold $\mathscr{M}$ of $\mathbb{R}^n$. To estimate the size of sets of points $\mathbf{x} \in \mathbb{R}^n$ which lie on a k-dimensional, non-degenerate[2], analytic submanifold $\mathscr{M} \subseteq \mathbb{R}^n$ is an intricate and challenging problem. The fundamental aim is to estimate the size of the set $\mathscr{M} \cap \mathscr{D}_n^\theta(\Psi)$ in terms of Lebesgue measure, Hausdorff measure, and Hausdorff dimension. When asking such questions it is natural to phrase them in terms of a suitable measure supported on the manifold, since when $k < n$ the n-dimensional Lebesgue measure is zero irrespective of the approximating functions. For this reason, results in the dependent Lebesgue theory (for example, Khintchine–Groshev type theorems for manifolds) are posed in terms of the k-dimensional Lebesgue measure (equivalent to the Hausdorff measure) on $\mathscr{M}$.

In full generality, a complete Hausdorff measure treatment for manifolds $\mathscr{M}$ represents a deep open problem referred to as the Generalised Baker–Schmidt Problem (GBSP) inspired by the pioneering work of Baker and Schmidt [3]. There are two variants of this problem, concerning simultaneous and dual approximation. In this paper we are concerned with the dual approximation only. Ideally one would want to solve the following problem in full generality.

Generalised Baker–Schmidt Problem for Hausdorff Measure: dual setting. *Let $\mathscr{M}$ be a non-degenerate submanifold of $\mathbb{R}^n$ with $\dim\mathscr{M} = k$ and $n \geq 2$. Let Ψ be a multivariable approximating function. Let f be a dimension function such that $r^{-k}f(r) \to \infty$ as $r \to 0$. Assume that $r \mapsto r^{-k}f(r)$ is decreasing and $r \mapsto r^{1-k}f(r)$ is increasing. Prove that*

$$\mathscr{H}^f(\mathscr{D}_n^\theta(\Psi)\cap\mathscr{M}) = \begin{cases} 0 & \text{if } \sum\limits_{\mathbf{q}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \|\mathbf{q}\|^k\Psi(\mathbf{q})^{1-k} f\left(\frac{\Psi(\mathbf{q})}{\|\mathbf{q}\|}\right) < \infty; \\[2ex] \infty & \text{if } \sum\limits_{\mathbf{q}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \|\mathbf{q}\|^k\Psi(\mathbf{q})^{1-k} f\left(\frac{\Psi(\mathbf{q})}{\|\mathbf{q}\|}\right) = \infty. \end{cases}$$

Note that $\mathscr{H}^f$ is proportional to the standard Lebesgue measure when $f(r) = r^n$. In fact, the GBSP is stated in the most idealistic format and solving it in this form is extremely challenging. The main difficulties lie in the convergence case and therein constructing a suitable nice cover for the set $\mathscr{D}_n^\theta(\Psi)\cap\mathscr{M}$. Recently (2021), the authors with David Simmons settled the GBSP for hypersurfaces for both homogeneous and inhomogeneous settings with non-monotonic multivariable approximating functions [10]. We also proved several results for the one-dimensional manifolds such as non-degenerate planar curves or for Veronese curves in [9] under some regularity conditions on the dimension function. In this paper we refine our framework set out in [10] and extend those results to certain classes of nondegenerate manifolds of higher codimension.

[1] We remark that, in fact, our results still apply for any sufficiently smooth function $\theta(\mathbf{x})$, as in [10], see Remark 2 below.
[2] In this context "non-degenerate" means suitably curved, see [4, 11] for precise formulations

We refer the reader to [10, Subsection 1.1] for a description of the historical progression towards the GBSP. Specifically, we refer the reader to [2, 3, 6–9, 11]. For the recent developments of the GBSP for the simultaneous approximation, we refer the reader to [10, Remark 1.1] or [6].

1.1. *Setup and main result*

We first state some regularity conditions on the manifold $\mathscr{M}$ and the dimension function f. Let $\mathscr{M}$ be a manifold of dimension $n-l \geq 1$ in $\mathbb{R}^n$, with $n \geq 2$ and $l \geq 1$. Assume more precisely that $\mathscr{M}$ is a graph of a C^2-map $g : U \to \mathbb{R}^l$, where $U \subset \mathbb{R}^{n-l}$ is a connected bounded open set. We denote this as $\mathscr{M} = \Gamma(g)$. By default we will assume that vectors are line vectors, and use superscript t for their transpose. Let $\mathscr{M}$ be parametrised by

$$\mathscr{M} = \{(x_1, \ldots, x_{n-l}, g_1(\mathbf{x}), \ldots, g_l(\mathbf{x})) : \mathbf{x} = (x_1, \ldots, x_{n-l}) \in U \subseteq \mathbb{R}^{n-l}\}. \tag{1}$$

Our conditions read as follows:

(I) Let f be a dimension function satisfying

$$f(xy) \lesssim x^s f(y), \qquad \text{for all } y < 1 < x \tag{2}$$

for some $s < 2(n-l-1)$.

(II) The real symmetric $(n-l) \times (n-l)$ matrix $\Lambda = \Lambda(g, \boldsymbol{s}, \mathbf{x})$ with entries

$$\Lambda_{j,i} = \sum_{u=1}^{l} s_u \cdot \frac{\partial^2 g_u}{\partial x_i \partial x_j}(\mathbf{x}), \qquad 1 \leq i, j \leq n-l,$$

for g as above is regular for any choice of real $\boldsymbol{s} = (s_1, s_2, \ldots, s_l) \neq \mathbf{0}$ and all $\mathbf{x} \in U \setminus S_{\mathscr{M}}$ outside a set $S_{\mathscr{M}}$ of f-measure 0.

While (II) turns out to be rather complicated, we point out that (I) holds as soon as the manifold has dimension at least three. This follows from the assumption that $r \to r^{-k} f(f)$ is decreasing, as in [10], see paragraph 2.1 below. Section 2 of this paper is reserved for a more detailed discussion on the conditions, followed by examples of manifolds satisfying the hypotheses in Section 3. Our new criterion for the convergence part of the GBSP to be proved in Section 4 reads as follows.

Theorem 1. *Let Ψ be a multivariable approximating function. Let f be a dimension function satisfying* (I) *and let $g : \mathbb{R}^{n-l} \to \mathbb{R}^l$ be a C^2-function satisfying* (II). *Then if $\mathscr{M}$ is given via g by* (1), *then*

$$\mathscr{H}^f(\mathscr{D}_n^\theta(\Psi) \cap \mathscr{M}) = 0$$

if the series

$$\sum_{\mathbf{q} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} \|\mathbf{q}\|^{n-l} \Psi(\mathbf{q})^{l+1-n} f\left(\frac{\Psi(\mathbf{q})}{\|\mathbf{q}\|}\right)$$

converges.

Remark 2. It is possible to formulate a variant of Theorem 1 with a functional inhomogenity $\theta(\mathbf{x})$ as in [10], however condition (II) becomes more technical and less natural.

1.2. *Corollaries (from a combination with the divergence results)*

In this section, we detail some of the corollaries of our theorem along with some of the consequences. We begin by summarising the notation used.

Notation. In the case where the dimension function is of the form $f(r) := r^s$ for some $s < k$, $\mathscr{H}^f$ is simply denoted as $\mathscr{H}^s$. On occasions we will consider functions of the form $\Psi(\mathbf{q}) = \psi(\|\mathbf{q}\|)$, and in this case we use $W_n^\theta(\psi)$ as a shorthand for $W_n^\theta(\Psi)$. The function $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is called a *single-variable approximating function.* $B_n(\mathbf{x},r)$ denotes a ball centred at the point $\mathbf{x}\in\mathbb{R}^n$ of radius r. For real quantities A,B and a parameter t, we write $A\lesssim_t B$ if $A\le c(t)B$ for a constant $c(t)>0$ that depends on t only (while A and B may depend on other parameters). We write $A\asymp_t B$ if $A\lesssim_t B\lesssim_t A$. If the constant $c>0$ depends only on parameters that are constant throughout a proof, we simply write $A\lesssim B$ and $B\asymp A$.

The divergence part of the GBSP was proved by Badziahin–Beresnevich–Velani [1] for the s-dimensional Hausdorff measure[3] and for multivariable approximating functions satisfying a certain property **P** for any non-degenerate manifolds. Following the terminology of [1], we say that an approximating function Ψ satisfies property **P** if it is of the form $\Psi(\mathbf{q})=\psi(\|\mathbf{q}\|_{\mathbf{v}})$ for a monotonically decreasing function $\psi:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ (single-variable approximation function), $\mathbf{v}=(v_1,\dots,v_n)$ with $v_i>0$ and $\sum_{1\le i\le n}v_i=n$, and $\|\cdot\|_{\mathbf{v}}$ defined as the quasi-norm $\|\mathbf{q}\|_{\mathbf{v}}=\max_i|q_i|^{1/v_i}$. When combined with Theorem 1 we obtain the following implication on the GBSP problem for non-degenerate (see [1]) manifolds for dimensions not less than 3.

Corollary 3. *Let Ψ be a decreasing multivariable approximating function satisfying property* **P**. *Let f be a dimension function satisfying* (I) *and let g be a C^2-function satisfying* (II). *Let $\mathcal{M}$ be a non-degenerate manifold in $\mathbb{R}^n$ of dimension $n-l$, given via g by* (1). *Then*

$$\mathcal{H}^f(\mathcal{D}_n^\theta(\Psi)\cap\mathcal{M})=\begin{cases}0 & \text{if } \sum\limits_{\mathbf{q}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}}\|\mathbf{q}\|^{n-l}\Psi(\mathbf{q})^{l+1-n}f\left(\frac{\Psi(\mathbf{q})}{\|\mathbf{q}\|}\right)<\infty;\\[2ex] \infty & \text{if } \sum\limits_{\mathbf{q}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}}\|\mathbf{q}\|^{n-l}\Psi(\mathbf{q})^{l+1-n}f\left(\frac{\Psi(\mathbf{q})}{\|\mathbf{q}\|}\right)=\infty.\end{cases}$$

We emphasise again that only the divergence case, treated in [1], assumes monotonicity on the approximating function Ψ.

Corollary 4. *Let $\mathcal{M}$ and Ψ be as in Corollary 3 and let s be a real number satisfying $s<2(n-l-1)$. Then*

$$\mathcal{H}^s(\mathcal{D}_n^\theta(\Psi)\cap\mathcal{M})=\begin{cases}0 & \text{if } \sum\limits_{\mathbf{q}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}}\|\mathbf{q}\|\left(\frac{\Psi(\mathbf{q})}{\|\mathbf{q}\|}\right)^{s+l+1-n}<\infty;\\[2ex] \infty & \text{if } \sum\limits_{\mathbf{q}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}}\|\mathbf{q}\|\left(\frac{\Psi(\mathbf{q})}{\|\mathbf{q}\|}\right)^{s+l+1-n}=\infty.\end{cases}$$

For $l=1,n=2$ where $\mathcal{M}$ is a planar curve, while the corollary and its proof are formally valid, the involved parameter range for s is empty. We refer the reader to [9] for the GBSP type results on non-degenerate curves such as Veronese curves, i.e. sets of the form $\{(x,x^2,\dots,x^n):x\in\mathbb{R}\}$. We remark that in view of Corollary 4, Conjecture 7 below would imply the GBSP for a large class of manifolds and any Ψ with property **P**.

One of the consequences of Corollary 4 is the following Hausdorff dimension result. Let τ_Ψ be the lower order at infinity of $1/\Psi$, that is,

$$\tau_\Psi:=\liminf_{t\to\infty}\frac{\log(1/\Psi(t))}{\log t},\quad\text{where}\quad \Psi(t)=\inf_{\mathbf{x}\in\mathbb{Z}^n:\|\mathbf{x}\|\le t}\Psi(\mathbf{x}),$$

and we may assume $\tau_\Psi\ge n>0$. Then from the definition of Hausdorff measure, Theorem 1 implies that for any approximating function Ψ with lower order at infinity τ_Ψ and for $\mathcal{M}$ as above (that is, with property (II) of dimension $\dim\mathcal{M}\ge 3$, we have

$$\dim_{\mathcal{H}}(W_n^\theta(\Psi)\cap\mathcal{M})\le\dim\mathcal{M}-1+\frac{n+1}{\tau_\Psi+1}.$$

This Hausdorff dimension result was previously only known for either the planar curve [2, 6, 8], Veronese curve [5], or for the hypersurface [10].

[3] It is to be noted that the techniques used in proving [1, Theorem 2] also work for the f-dimensional Hausdorff measure situation.

2. On conditions (I) and (II)

2.1. *On Condition* (I)

Note that in the statement of the GBSP, the standard condition on the dimension function f that $f(q)q^{-(n-l)}$ (recall $n-l=\dim\mathscr{M}$) is decreasing is assumed. With this condition in hand, the condition (I) is satisfied as soon as

$$n-l\geq 3.$$

From the aforementioned decay property we see that $f(xy)\leq x^{n-l}f(y)$ for $y<1<x$. Since $n-l<2(n-l-1)$ as soon as $n-l\geq 3$, indeed we infer that (2) holds in the non-empty parameter range $s\in[n-l,2(n-l-1))$. On the other hand, condition (I) excludes all curves (one dimensional manifolds), as well as all two dimensional manifolds, for many interesting functions f.

2.2. *On Condition* (II)

Condition (II) is more delicate. It replaces and generalises the non-vanishing of the determinant of the Hessian $\nabla^2 g\in\mathbb{R}^{(n-1)\times(n-1)}$ condition within U from [10] when $l=1$. The determinant of Λ in (II) becomes a multivariate, homogeneous polynomial P of total degree $n-l$ in the variables $s_1,\dots,s_l$; with coefficients (as functions) in the $\ell\cdot\binom{n-l+1}{2}$ second order partial derivatives of g (using symmetry of Λ), evaluated at $\mathbf{x}\in U$. In other words, for (II) we need that some form of degree $n-l$ in l variables is positive (or negative) definite at any point in U. This definiteness problem of forms seems related to Hilbert's XVII-th problem, whose positive answer in particular implies that P above must have a presentation as a sum of squares. It is evident that the complexity of the problem increases fast as l and n grow.

A priori, it is not clear if, generally, g satisfying hypothesis (II) exists when $l\geq 2$. However, for certain pairs (l,n) we provide examples below. It is evident from the form of Λ that, provided that examples exists at all, we can choose the coordinate functions g_j as quadratic forms, defined on the entire space $U=\mathbb{R}^{n-l}$. Moreover, we can perturb any such example by adding any functions with small enough second order derivatives by absolute value uniformly on $\mathbb{R}^{n-l}$ to the g_j. For simplicity of presentation, we introduce the following notion.

Definition 5. *Call a pair of integers* (l,n) *with* $n>l\geq 1$ *a* good pair *if* (II) *holds for some* $g:\mathbb{R}^{n-l}\to\mathbb{R}^l$ *as in the introduction.*

By the above observation, the induced g of any good pair can be taken a quadratic form defined globally. It is obvious that any pair $(1,n)$ is good. For larger l, the following general going-up and going-down properties for good pairs are straightforward.

Proposition 6. *If* (l,n) *is a good pair, then so is*

(i) *the pair* $(l-t,n-t)$ *for any integer* $0\leq t\leq l-1$.
(ii) *the pair* $(l,\tilde{n})$ *with* $\tilde{n}=n+t(n-l)$ *for any integer* $t\geq 0$.

Claim (i) can be seen by specialisation of t variables, for example via putting $s_1=\cdots=s_{t+1}$. The proof of (ii) will become apparent from the examples in Section 3. On the other hand, it is in general unclear if (l,n) being a good pair will imply the same for $(l,\tilde{n})$ with a general larger $\tilde{n}>n$, even if $\tilde{n}-l$ is even (see Obstruction 1 below).

On the other hand, condition (II) has some natural limitations, as captured in the following obstructions.

Obstruction 1. *If $l \geq 2$ and $n-l$ is odd, then (l,n) is not a good pair, that is, the condition* (II) *cannot hold.*

While a short proof seems to follow directly from the positive answer to Hilbert's XVII'th problem, we want to explicitly explain how this special case can be handled. First assume $l=2$. Indeed, if either the coefficient of x_1^{n-l} or x_2^{n-l} of the polynomial $P(x_1,x_2)$ defined above vanishes, then we may take x_1 arbitrary and $x_2=0$ or vice versa, hence we get non-trivial solutions for $\det\Lambda = P(x_1,x_2)=0$. *If otherwise both coefficients are not* 0, *then by choosing any $x_2\neq 0$ we get a single-variable polynomial in x_1 of odd degree with non-zero constant term, again inducing a non-trivial solution for* $\det\Lambda = P(x_1,x_2)=0$. *Finally if $l>2$, by specialising $l-2$ variables $x_3=\cdots=x_l=0$, we get a polynomial $Q(x_1,x_2)$ in two variables. Regardless if $Q\equiv 0$ is the constant* 0 *polynomial or not, by the observations for $l=2$ above, we may choose at least one of x_1,x_2 (almost) arbitrary for a solution of* $\det\Lambda=0$. *Thus we again find a non-trivial solution for* $\det\Lambda=0$ *in $s_1,\dots,s_l$.*

Obstruction 2. *We require $n\geq 2l$ for (l,n) being a good pair.*

Otherwise if $n-l<l$ then we can annihilate a line of Λ on a linear subspace of dimension at least 1 *of $\mathbf{s}\in\mathbb{R}^l$, regardless of the choice of $g,\mathbf{x}$. More precisely, in case of $n-l<l$, we find a subspace of dimension at least one in $\mathbb{R}^l$ consisting of $\mathbf{s}=(s_1,\dots,s_l)$ so that for the first line of Λ we have*

$$\Lambda_{1,i}=\sum_{u=1}^{l} s_u\cdot\sigma_{u,i}=0, \qquad 1\leq i\leq n-l, \tag{3}$$

where we have put

$$\sigma_{u,i}:=\frac{\partial^2 g_u}{\partial x_i\partial x_1}(\mathbf{x}).$$

Indeed, then (3) *is a homogeneous linear equation system in more variables than equations, hence it has a non-trivial solution in $\mathbf{s}$. Clearly any resulting matrix Λ is singular.*

Obstruction 2 means that the manifold must have at least half the dimension of the ambient space. We believe that these are the only obstructions. In view of Theorem 1 and the observations in Section 2.1 we therefore go on to state the following conjecture.

Conjecture 7. *The following claims hold*

(i) *If $l\geq 2$ and $n\geq 2l$ and $n-l$ is even, then (l,n) is a good pair.*
(ii) *If $l\geq 2$ and $n\geq 2l$ and $n-l\geq 4$ is even, then there exists g with coordinate functions g_j quadratic forms such that* (I), (II) *holds on $\mathbb{R}^{n-l}$, hence the convergence part of the GBSP holds for the induced manifolds.*

Clearly it would suffice to verify (i), claim (ii) is just stated for completeness. We will verify the conjecture for $l=2$ in Section 3. Moreover, for $l=3$ we establish the partial result that $n-l$ being a positive multiple of 4 is sufficient. The simplest cases where Conjecture 7 remains open are $l=3, n=9$ and $l=4, n=8$.

3. Examples

3.1. *Special case $l=2$*

We start with an example to illustrate condition (II) in the case $n=4, l=2$. Unfortunately, since $\dim\mathscr{M}=2$, condition (I) does not hold in this context, see Section 2.1.

Example 8. Let $l=2$, $n=4$, so that $\mathscr{M}$ is a two-dimensional manifold with codimension two. Using multilinearity of the determinant and after some calculations, we see that the polynomial representing $\det\Lambda$ becomes

$$P(s_1,s_2)=s_1^2A_1(g,\mathbf{x})+s_2^2A_2(g,\mathbf{x})+s_1s_2A_3(g,\mathbf{x})$$

where

$$A_1(g,\mathbf{x}) = \frac{\partial^2 g_1}{\partial^2 x_1}(\mathbf{x}) \cdot \frac{\partial^2 g_1}{\partial^2 x_2}(\mathbf{x}) - (\frac{\partial^2 g_1}{\partial x_1 \partial x_2}(\mathbf{x}))^2,$$

$$A_2(g,\mathbf{x}) = \frac{\partial^2 g_2}{\partial^2 x_1}(\mathbf{x}) \cdot \frac{\partial^2 g_2}{\partial^2 x_2}(\mathbf{x}) - (\frac{\partial^2 g_2}{\partial x_1 \partial x_2}(\mathbf{x}))^2,$$

and

$$A_3(g,\mathbf{x}) = \frac{\partial^2 g_1}{\partial^2 x_1}(\mathbf{x}) \cdot \frac{\partial^2 g_2}{\partial^2 x_2}(\mathbf{x}) - 2\frac{\partial^2 g_1}{\partial x_1 \partial x_2}(\mathbf{x}) \cdot \frac{\partial^2 g_2}{\partial x_1 \partial x_2}(\mathbf{x}) + \frac{\partial^2 g_2}{\partial^2 x_1}(\mathbf{x}) \cdot \frac{\partial^2 g_1}{\partial^2 x_2}(\mathbf{x}).$$

By the criterion of minors to test definiteness of a quadratic form with respect to the corresponding symmetric matrix with rows $(A_1(g,\mathbf{x}), A_3(g,\mathbf{x})/2)$ and $(A_3(g,\mathbf{x})/2, A_2(g,\mathbf{x}))$, then the criterion of condition (II) that P is positive (or negative) definite becomes

$$A_3(g,\mathbf{x})^2 < 4A_1(g,\mathbf{x})A_2(g,\mathbf{x}). \tag{4}$$

A sufficient criterion for second order derivatives is given by

$$\frac{\partial^2 g_1}{\partial^2 x_1}(\mathbf{x}) \cdot \frac{\partial^2 g_1}{\partial^2 x_2}(\mathbf{x}) = \frac{\partial^2 g_2}{\partial^2 x_1}(\mathbf{x}) \cdot \frac{\partial^2 g_2}{\partial^2 x_2}(\mathbf{x}) < 0, \qquad \frac{\partial^2 g_1}{\partial x_1 \partial x_2}(\mathbf{x}) \neq \frac{\partial^2 g_2}{\partial x_1 \partial x_2}(\mathbf{x})$$

as then we may write inequality (4) equivalently as

$$4\frac{\partial^2 g_2}{\partial^2 x_1}(\mathbf{x}) \cdot \frac{\partial^2 g_2}{\partial^2 x_2}(\mathbf{x}) \cdot \left(\frac{\partial^2 g_1}{\partial x_1 \partial x_2}(\mathbf{x}) - \frac{\partial^2 g_2}{\partial x_1 \partial x_2}(\mathbf{x})\right)^2 < 0.$$

Specialising further, if for example, we can take at some $\mathbf{x} \in U$ the derivates

$$\frac{\partial^2 g_1}{\partial^2 x_1}(\mathbf{x}) = \frac{\partial^2 g_2}{\partial^2 x_1}(\mathbf{x}) = 2,$$

$$\frac{\partial^2 g_1}{\partial^2 x_2}(\mathbf{x}) = \frac{\partial^2 g_2}{\partial^2 x_2}(\mathbf{x}) = -2,$$

$$\frac{\partial^2 g_1}{\partial x_1 \partial x_2}(\mathbf{x}) \neq \frac{\partial^2 g_2}{\partial x_1 \partial x_2}(\mathbf{x}),$$

then it will be true in some neighbourhood of $\mathbf{x}$. For example when $n = 4, l = 2$ and $\delta \neq 0$, then

$$g(\mathbf{x}) = g(x, y) = (x^2 - y^2 + \delta xy, x^2 - y^2)$$

satisfies this for all (x, y) in $\mathbb{R}^2$. The arising parametrised manifolds becomes

$$\mathcal{M}_\delta = \{(x, y, x^2 - y^2 + \delta xy, x^2 - y^2) : x, y \in \mathbb{R}\}, \text{ where } \delta \neq 0. \tag{5}$$

We remark that $\delta \neq 0$ is necessary for the convergence part of GBSP as otherwise the manifold lies in the rational subspace of $\mathbb{R}^4$ defined by $x_3 = x_4$.

We now present some examples satisfying both (I) and (II). Keeping $l = 2$, we can extend the previous example to general even $n \geq 6$ by essentially building Cartesian products. A possible class of manifolds derived from this method that verifies Conjecture 7 for $l = 2$ is captured in the following example.

Example 9. Let $l = 2, n - l = 2t$ for $t \geq 2$. Then for any $\delta_1, \ldots, \delta_t \neq 0$ the manifold

$$\mathcal{M} = \left\{\left(x_1, y_1, \ldots, x_t, y_t, \sum_{u=1}^{t} x_u^2 - y_u^2 + \delta_u x_u y_u, \sum_{u=1}^{t} x_u^2 - y_u^2\right) : \ x_i, y_i \in \mathbb{R}\right\},$$

satisfies (I) and (II). Hence the GBSP holds for any decreasing Ψ with property $\mathbf{P}$.

Indeed, the critical determinant $\det(\Lambda)$ in Example 9 decomposes as a product of t determinants as in Example 8, which we found all to be non-zero, so (II) holds. Since $n - l \geq 4$ condition (I) holds too. Clearly Example 9 can be generalised in terms of parameter ranges for the coefficients of the quadratic forms. By Obstruction 1, the condition that $n - l$ is even is necessary for (II) (unless $n - l = 1$, the hypersurface case). A similar argument proves Proposition 6.

3.2. *The case $l > 2$*

Now let us consider $l > 2$. Together with the restrictions

$$n - l \equiv 0 \bmod 2, \qquad n - l \geq 3, \qquad n \geq 2l,$$

from Section 2.1 and Obstructions 1 and 2, the easiest example is $l = 3, n = 7$.

The following general criterion for (II), or good pairs, involving "definite determinants" essentially comes from specialising certain variables (second order derivatives of g).

Lemma 10. *Let $n > l \geq 1$ be integers. Assume there exists a symmetric $(n-l) \times (n-l)$ matrix $M_{l,n}$ with the entries*

$$M_{l,n}(i,j) \in \{0, \pm z_1, \dots, \pm z_l\}, \qquad 1 \leq i, j \leq n - l,$$

for formal variables z_v, $1 \leq v \leq l$, so that $\det M_{l,n} \neq 0$ *for any choice of real z_v not all* 0. *Then (l, n) is a good pair.*

Hence the existence of $M_{l,n}$ as in the lemma implies that there exist $(n-l)$-dimensional submanifolds of $\mathbb{R}^n$, defined as in (1) via $g = (g_1, \dots, g_l)$ with $g_j(\mathbf{x})$ real quadratic forms, that satisfy (II) on $U = \mathbb{R}^{n-l}$. Again, the hypothesis of Lemma 10 forces $n \geq 2l$ and $n - l$ to be even.

Remark 11. More generally, instead of $M_{l,n}(i,j) = \pm z_v$ or 0, we may take the matrix entries arbitrary linear combinations of the z_v, for the same conclusion. This generalised condition is in fact equivalent to (II). The proof is essentially the same as below.

Proof of Lemma 10. Choose any set of l linearly independent (over $\mathbb{R}$) vectors $L_v = (a_{1,v}, \dots, a_{l,v})$, $1 \leq v \leq l$, in $\mathbb{R}^l$ and define the linear forms in l variables $s_1, \dots, s_l$

$$L_v \cdot \boldsymbol{s} = a_{1,v}s_1 + \cdots + a_{l,v}s_l, \quad 1 \leq v \leq l.$$

Given $M_{l,n}$ as in the lemma, we construct a matrix $M'_{l,n}$ with entries in formal variables $s_1, \dots, s_l$ as follows: We identify z_v with $L_v \cdot \boldsymbol{s}$, that is, if for a pair (i,j) the index $v = v(i,j)$ is so that $M_{l,n}(i,j) = \pm z_v$, then we choose the entry at position (i,j) of $M'_{l,n}$ equal to $M'_{l,n}(i,j) = \pm L_v \cdot \boldsymbol{s}$, with the same sign choice as above. Else if $M_{l,n}(i,j) = 0$ then keep the value $M'_{l,n}(i,j) = 0$. Then $M'_{l,n}$ is well-defined and depends on $s_1, \dots, s_l$. By linear independence of the L_v and the hypothesis of the lemma, for any non-trivial choice of real numbers $s_1, \dots, s_l$, the matrix $M'_{l,n}$ has non-zero determinant. We finally notice that there is a one-to-one correspondence between any such collection of coefficient vectors L_v and a collection of quadratic forms $g_1, \dots, g_l$ via identifying for $v = v(i,j)$ as above $a_{u,v} = \partial^2 g_u / \partial x_i \partial x_j$ for $1 \leq u \leq l$. In other words, $a_{u,v}x_i x_j$ if $i \neq j$ and $(a_{u,v}/2)x_i^2$ if $i = j$ is the term containing $x_i x_j$ resp. x_i^2 in the quadratic form $g_u(\mathbf{x}) = \sum_{i,j} a_{u,v} x_i x_j$. Hence (II) holds for the globally defined function $g = (g_1, \dots, g_l)$, so (l, n) is a good pair. □

In the proof, we may just let L_v the canonical base vector in $\mathbb{R}^l$ for $1 \leq v \leq l$ so that essentially z_v equals s_v, however the proof contains more information.

Hence the problem (II) can be relaxed to finding suitable matrices $M_{l,n}$ as in the lemma. For $l = 2, n = 4$ we can take the matrix

$$M_{2,4} = \begin{pmatrix} z_1 & z_2 \\ z_2 & -z_1 \end{pmatrix}$$

which leads to Example 8. More generally, for $l = 2$ and $n = 2v$ with $v \geq 2$ an even integer, we may take a matrix consisting of $v - 1$ copies of 2×2 "diagonal" blocks as above and zeros elsewhere, similar to Example 9 and Proposition 6.

For $l = 3, n = 7$ we can take

$$M_{3,7} = \begin{pmatrix} z_1 & z_2 & z_3 & -z_3 \\ z_2 & -z_1 & z_3 & z_3 \\ z_3 & z_3 & z_1 & z_2 \\ -z_3 & z_3 & z_2 & -z_1 \end{pmatrix} \tag{6}$$

which has determinant $\det(M_{3,7}) = (z_1^2 + z_2^2)^2 + 4z_3^4$. Our construction leads to the following example.

Example 12. Let $l = 3, n = 7$. Taking the linear forms L_v from the proof of Lemma 10 the canonical base vectors $(1,0,0), (0,1,0), (0,0,1)$ of $\mathbb{R}^3$ and inserting for the z_v in $M_{3,7}$ from (6), leads to $g = (g_1, g_2, g_3)$ with quadratic form entries $g_u(\mathbf{x}) = g_u(x_1, x_2, x_3, x_4)$ given by

$$g_1(\mathbf{x}) = \frac{x_1^2 - x_2^2 + x_3^2 - x_4^2}{2},$$
$$g_2(\mathbf{x}) = x_1x_2 + x_3x_4,$$
$$g_3(\mathbf{x}) = x_1x_3 + x_2x_3 + x_2x_4 - x_1x_4.$$

So, the manifold becomes

$$\mathscr{M} = \{(x_1, x_2, x_3, x_4, g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x}) : x_i \in \mathbb{R}\},$$

which satisfies (I) and (II), and thus GBSP for any decreasing Ψ with property **P**.

For $l = 3$ we may take any $n \in \{7, 11, 15, 19, \ldots\}$ again by repeating this 4×4-block matrix $M_{3,7}$ along the "diagonal". It is unclear if $l = 3$ and $n \in \{9, 13, 17, \ldots\}$ can be achieved. It would suffice to verify this for $n = 9$ to infer the claim for all n in the list by considering matrices decomposing into two types of diagonal blocks, $M_{3,7}$ and the vacant $M_{3,9}$. This would confirm Conjecture 7 for $l = 3$ as well. Unfortunately, we are unable to find a suitable matrix $M_{3,9}$.

The above discussion on Lemma 10 motivates the following problem implicitly stated within Conjecture 7.

Problem 13. *Given $l \geq 3$, what is the minimum n so that a matrix $M_{l,n}$ as in Lemma 10 exists? Equivalently, given even $n - l$, what is the largest l for which the hypothesis holds for some matrix.*

It is unclear if such n exists at all if $l \geq 4$. As remarked in Section 2.2, in all examples of Section 3, we can manipulate the manifold by adding any functions with uniformly (in absolute value) small enough second order derivatives to the l functionally dependent variables. In particular, for any analytic functions defined on a neighbourhood of $\mathbf{0} \in \mathbb{R}^{n-l}$ with quadratic terms as in our examples above, GBSP holds upon possibly shrinking the neighbourhood.

4. Proof of Theorem 1

Let us first clarify some notation. With g as above, write

$$g(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_l(\mathbf{x})) \in \mathbb{R}^l, \qquad \mathbf{x} \in U,$$

which we will assume as a row vector. Furthermore, write ∇g^t for the $(n-l) \times l$ matrix with i-th partial derivative vector in $\mathbb{R}^l$ of the transpose of g (thus a column vector) in the i-th column $(1 \leq i \leq n-l)$. Denote by $\nabla^2 g_u^t \in \mathbb{R}^{(n-l)\times(n-l)}$ the Hessians of the (transposed) coordinate functions g_u^t, $1 \leq u \leq l$, where again the i-th column consists of the partial derivative vector $\partial g_u / \partial x_i$, $1 \leq i \leq n-l$.

The proof is a refinement and extension of arguments presented in our previous paper [10]. Here we only detail some modifications and other necessary details.

For any $\mathbf{q} \in \mathbb{Z}^n$ and $p \in \mathbb{Z}$, analogously to [10] we define

$$S(\mathbf{q}, p) = S_{\Psi,\theta}(\mathbf{q}, p) = \{\mathbf{x} \in K : |\mathbf{q} \cdot (\mathbf{x}, g(\mathbf{x}))^t - p - \theta| < \Psi(\mathbf{q})\},$$

where K is a compact subset of U. However, notice that our $g(\mathbf{x})$ is a vector here. Write $\mathbf{q} = \widetilde{q} \cdot (\mathbf{r}, \boldsymbol{s})$ for

$$\widetilde{q} = \max_{n-l+1 \leq i \leq n} |q_i|$$

for $\mathbf{r} \in \mathbb{Q}^{n-l}$ and $\boldsymbol{s} \in \mathbb{Q}^l$. Let $a = a(\mathbf{q}) = (p+\theta)/\widetilde{q}$ and $\rho = \Psi(\mathbf{q})/|\widetilde{q}|$ if $\widetilde{q} \neq 0$ (assume this for now). Then $\|\boldsymbol{s}\| = 1$, and $\widetilde{q} \leq \|\mathbf{q}\|$ by definition. For any $\mathbf{q} \in \mathbb{Z}^n$ and $p \in \mathbb{Z}$, we will bound the size of this set which is equivalently given as

$$S(\mathbf{q}, p) = \{\mathbf{x} \in K : |\mathbf{r}\cdot\mathbf{x}^t + \boldsymbol{s}\cdot g(\mathbf{x})^t - a| < \rho\}.$$

Similarly as in [10], for fixed p and $\mathbf{q}$, define a function $h : \mathbb{R}^{n-l} \to \mathbb{R}$ by

$$h(\mathbf{x}) = \mathbf{r}\cdot\mathbf{x}^t + \boldsymbol{s}\cdot g(\mathbf{x})^t - a,$$

for $\mathbf{r}, \boldsymbol{s}, a$ induced by $\mathbf{q}, p$ as above. Finally in case $\widetilde{q} = 0$, we instead let

$$\mathbf{r} = (q_1, \dots, q_{n-l}), \qquad h(\mathbf{x}) = \mathbf{r}\cdot\mathbf{x}^t - p - \theta, \qquad \rho = \Psi(\mathbf{q}).$$

As in [10] we see that in either case

$$S(\mathbf{q}, p) = \{\mathbf{x} \in K : |h(\mathbf{x})| < \rho\}.$$

We now identify $\nabla^2 h$ with the matrix Λ of the theorem by the following calculation: For $1 \leq i \leq n-l$, the i-th entry of the vector $\nabla h(\mathbf{x})$ equals $r_i + \boldsymbol{s}\cdot G_i(\mathbf{x})^t$ with

$$G_i(\mathbf{x})^t = (\partial g^t/\partial x_i)(\mathbf{x}) = (\partial g_1/\partial x_i(\mathbf{x}), \dots, \partial g_l/\partial x_i(\mathbf{x}))^t$$

the i-th column of ∇g^t evaluated at $\mathbf{x}$ if $\widetilde{q} \neq 0$, and $\nabla h(\mathbf{x})_i = r_i$ if $\widetilde{q} = 0$. In the sequel we assume $\widetilde{q} \neq 0$, else similarly the argument is analogous to [10]. So this i-th entry of $\nabla h(\mathbf{x})$, denote it by $\nabla h(\mathbf{x})_i$, reads

$$\nabla h(\mathbf{x})_i = r_i + \boldsymbol{s}\cdot\left(\frac{\partial g}{\partial x_i}(\mathbf{x})\right)^t = r_i + \sum_{u=1}^{l} s_u \cdot (\partial g_u/\partial x_i(\mathbf{x})), \qquad 1 \leq i \leq n-l.$$

Then

$$\nabla h(\mathbf{x}) - \sum_{i=1}^{n-l} r_i \mathbf{e}_i = \sum_{i=1}^{n-l} \boldsymbol{s}\cdot\left(\frac{\partial g}{\partial x_i}(\mathbf{x})\right)^t \cdot \mathbf{e}_i = \sum_{i=1}^{n-l}\sum_{u=1}^{l} s_u \cdot (\partial g_u/\partial x_i(\mathbf{x}))\cdot\mathbf{e}_i,$$

with $\mathbf{e}_i$ the canonical base vectors in $\mathbb{R}^{n-l}$. Hence the quadratic matrix $\nabla^2 h(\mathbf{x})$ with j-th line $\partial\nabla h(\mathbf{x})/\partial x_j$ has entries

$$\nabla^2 h(\mathbf{x})_{j,i} = \sum_{u=1}^{l} s_u \cdot \frac{\partial^2 g_u}{\partial x_i \partial x_j}(\mathbf{x}), \qquad 1 \leq i, j \leq n-l, \tag{7}$$

(and vanishes if $\widetilde{q} = 0$). Thus indeed we may identify $\nabla^2 h$ with Λ. In contrast to [10], the matrix $\nabla^2 h(\mathbf{x})$ now also depends on $\mathbf{q} \in \mathbb{Z}^n$ via its dependence on the induced rational vector $\boldsymbol{s}$, however by a compactness argument we will deal with this issue.

By assumption of (II), for any unit vector $\boldsymbol{s} \in \mathbb{R}^l$ and any $\mathbf{x} \in K$, the determinant of $\nabla^2 h(\mathbf{x})$ does not vanish. For simplicity assume for the moment $\mathbf{q}$ and thus $\boldsymbol{s}$ are fixed. Then as in [10], for some $\varepsilon > 0$, the matrices $\nabla^2 h(\mathbf{x})$ with $\mathbf{x} \in K$ belong to a compact, convex set of matrices (identified with $\mathbb{R}^{(n-l)\times(n-l)}$) with determinants at least ε. Now the same argument as in [10] based on the mean value inequality gives the analogue of [10, Claim 2.3], which reads:

Claim. If the norm of $\mathbf{q} \in \mathbb{Z}^n$ is large enough, then either there exists $\mathbf{v} \in \mathbb{R}^{n-l}$ such that

$$\|\nabla h(\mathbf{x})\| \asymp \|\mathbf{x} - \mathbf{v}\|, \qquad \mathbf{x} \in K,$$

or

$$\|\nabla h(\mathbf{x})\| \asymp \|(\mathbf{r}, \boldsymbol{s})\| \asymp \|(\mathbf{r}, 1)\|, \qquad \mathbf{x} \in K.$$

Either way, we have $\|\nabla h(\mathbf{x})\| \ll 1$ for all $\mathbf{x} \in K$.

We remark that the last claim is obvious by $\|\boldsymbol{s}\| = 1$ and (7). It should be pointed out that following the proof in [10], the implied constants in the lemma will still depend on $\mathbf{q}$ in the form of the dependence on $\boldsymbol{s}$. However, by the continuous dependence of $\nabla^2 h(\mathbf{x})$ on $\boldsymbol{s}$ and the compactness of the set $\{\boldsymbol{s} \in \mathbb{R}^l : \|\boldsymbol{s}\| = 1\}$, it is easily seen that we can find uniform constants.

The remainder of the proof of Theorem 1 works analogously as in [10], where we consider two cases and we replace n by $n-l+1$ consistently and omit $\nabla\theta(\mathbf{x})$ and $\nabla^2\theta(\mathbf{x})$, as we consider θ constant. It is worth noticing that in Case 1 we apply the analogue of [10, Lemma 2.4], which we again want to state explicitly for convenience:

Lemma 14 (Hussain, Schleischitz, Simmons). *Assume n, l are positive integers satisfying $n > l+1$. Let $\phi : U \subset \mathbb{R}^{n-l} \to \mathbb{R}$ be a C^2 function. Fix $\alpha > 0$, $\delta > 0$, and $\mathbf{x} \in U$ such that $B_{n-l}(\mathbf{x},\alpha) \subset U$. There exists a constant $C > 0$ depending only on n such that if*

$$\|\nabla\phi(\mathbf{x})\| \geq C\alpha \sup_{\mathbf{z}\in U} \|\nabla^2\phi(\mathbf{z})\|, \tag{8}$$

then the set

$$S(\phi,\delta) = \{\mathbf{y} \in B_{n-l}(\mathbf{x},\alpha) : |\phi(\mathbf{y})| < \|\nabla\phi(\mathbf{x})\|\delta\}$$

can be covered by $\asymp (\alpha/\delta)^{n-l-1}$ balls of radius δ.

Indeed, this is precisely [10, Lemma 2.4], upon replacing n by $n-l+1$. Rest of the proof of Theorem follows just like [10] with obvious adaptations mentioned above. Specifically, by taking into account the observation that the implied constants in the Claim above are independent of $p, \mathbf{q}$ (or $\boldsymbol{s}$). That ultimately leads to the sufficient condition $s < 2(n-l-1)$ in place of $s < 2n-2$ from [10], which agrees with condition (I).

Acknowledgments

We are thankful to the referee for some useful comments.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] D. Badziahin, V. Beresnevich and S. Velani, "Inhomogeneous theory of dual Diophantine approximation on manifolds", *Adv. Math.* **232** (2013), pp. 1–35.

[2] D. Badziahin, S. Harrap and M. Hussain, "An inhomogeneous Jarník type theorem for planar curves", *Math. Proc. Camb. Philos. Soc.* **163** (2017), no. 1, pp. 47–70.

[3] A. Baker and W. M. Schmidt, "Diophantine approximation and Hausdorff dimension", *Proc. Lond. Math. Soc.* **21** (1970), pp. 1–11.

[4] V. Beresnevich, "A Groshev type theorem for convergence on manifolds", *Acta Math. Hung.* **94** (2002), no. 1-2, pp. 99–130.

[5] V. I. Bernik, "Application of the Hausdorff dimension in the theory of Diophantine approximations", *Acta Arith.* **42** (1983), no. 3, pp. 219–253.

[6] J.-J. Huang, "Hausdorff theory of dual approximation on planar curves", *J. Reine Angew. Math.* **740** (2018), pp. 63–76.

[7] J.-J. Huang, "The density of rational points near hypersurfaces", *Duke Math. J.* **169** (2020), no. 11, pp. 2045–2077.

[8] M. Hussain, "A Jarník type theorem for planar curves:everything about the parabola", *Math. Proc. Camb. Philos. Soc.* **159** (2015), no. 1, pp. 47–60.

[9] M. Hussain, J. Schleischitz and D. Simmons, "Diophantine approximation on curves", 2019. https://arxiv.org/abs/1902.02094.

[10] M. Hussain, J. Schleischitz and D. Simmons, "The generalized Baker–Schmidt problem on hypersurfaces", *Int. Math. Res. Not.* **2021** (2021), no. 12, pp. 8845–8867.

[11] D. Y. Kleinbock and G. A. Margulis, "Flows on homogeneous spaces and Diophantine approximation on manifolds", *Ann. Math.* **148** (1998), no. 1, pp. 339–360.