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Infinite volume and atoms at the bottom of the spectrum

Volume infini et atomes au bas du spectre

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Abstract. Let G be a higher rank simple real algebraic group, or more generally, any semisimple real algebraic group with no rank one factors and X the associated Riemannian symmetric space. For any Zariski dense discrete subgroup $\Gamma < G$, we prove that $\mathrm{Vol}(\Gamma\backslash X) = \infty$ if and only if no positive Laplace eigenfunction belongs to $L^2(\Gamma\backslash X)$, or equivalently, the bottom of the L^2-spectrum is not an atom of the spectral measure of the negative Laplacian. This contrasts with the rank one situation where the square-integrability of the base eigenfunction is determined by the size of the critical exponent relative to the volume entropy of X.

Résumé. Soit G un groupe algébrique réel simple de rang supérieur, ou plus généralement un groupe algébrique réel semi-simple sans facteurs de rang un et X l'espace symétrique riemannien associé. Pour tout sous-groupe discret dense de Zariski $\Gamma < G$, on prouve que $\mathrm{Vol}(\Gamma\backslash X) = \infty$ si et seulement si aucune fonction propre de Laplacien positive appartient à $L^2(\Gamma\backslash X)$, ou de manière équivalente, le bas du spectre L^2 n'est pas un atome de la mesure spectrale du Laplacien négatif. Cela contraste avec la situation de rang un où l'intégrabilité au carré de la fonction propre de base est déterminée par la taille de l'exposant critique par rapport à l'entropie volumique de X.

Keywords. Laplace eigenfunction, locally symmetric manifolds, infinite volume, Patterson–Sullivan measure.

Mots-clés. Fonctions propres de l'opérateur de Laplace–Beltrami, espaces localement symétriques, volume infini, mesures de Patterson–Sullivan.

2020 Mathematics Subject Classification. 22F30, 43A85.

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1. Introduction

Let $\mathscr{M}$ be a complete Riemannain manifold and let Δ denote the Laplace–Beltrami operator on $\mathscr{M}$. Define the real number $\lambda_0(\mathscr{M}) \in [0,\infty)$ by

$$\lambda_0(\mathscr{M}) := \inf\left\{ \frac{\int_{\mathscr{M}} \|\operatorname{grad} f\|^2 \, \mathrm{d}\mathrm{vol}}{\int_{\mathscr{M}} |f|^2 \, \mathrm{d}\mathrm{vol}} : f \in C_c^\infty(\mathscr{M}) \right\}, \tag{1.1}$$

where $C_c^\infty(\mathscr{M})$ denotes the space of all smooth functions with compact support. This number $\lambda_0(\mathscr{M})$ is known as the bottom of the L^2-spectrum of the negative Laplacian $-\Delta$ and separates the L^2-spectrum and the positive spectrum [24, p. 329] (Figure 1).

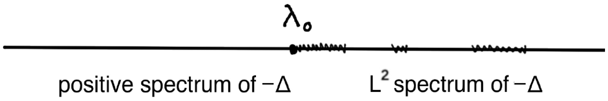

Figure 1. λ_0 separates the L^2 and positive spectrum

More precisely, let $L^2(\mathscr{M})$ denote the space of all square-integrable functions with respect to the inner product $\langle f_1, f_2\rangle = \int_{\mathscr{M}} f_1 \overline{f_2} \, \mathrm{d}\mathrm{vol}$. Let $W^1(\mathscr{M}) \subset L^2(\mathscr{M})$ denote the closure of $C_c^\infty(\mathscr{M})$ with respect to the norm

$$\|f\|_{W^1} = \left(\int_{\mathscr{M}} f^2 \, \mathrm{d}\mathrm{vol} + \int_{\mathscr{M}} \|\operatorname{grad} f\|^2 \, \mathrm{d}\mathrm{vol} \right)^{1/2}.$$

There exists a unique self-adjoint operator on the space $W^1(\mathscr{M})$ extending the Laplacian Δ on $C_c^\infty(\mathscr{M})$, which we also denote by Δ (cf. [11, Chapter 4.2]). The L^2-spectrum of $-\Delta$ is the set of all $\lambda \in \mathbb{C}$ such that $\Delta + \lambda$ does not have a bounded inverse $(\Delta+\lambda)^{-1} : L^2(\mathscr{M}) \to W^1(\mathscr{M})$. Sullivan showed that the L^2-spectrum of $-\Delta$ contains $\lambda_0(\mathscr{M})$ and is contained in the positive ray $[\lambda_0(\mathscr{M}),\infty)$, that is, $\lambda_0(\mathscr{M})$ is the bottom of the L^2-spectrum, and moreover, there are no positive eigenfunctions with eigenvalue strictly bigger than $\lambda_0(\mathscr{M})$ [24, Theorem 2.1 and 2.2] (see Figure 1). We will call an eigenfunction with eigenvalue $\lambda_0(\mathscr{M})$ a *base eigenfunction*. Note that the absence of a base eigenfunction in $L^2(\mathscr{M})$ is the same as the absence of a positive eigenfunction in $L^2(\mathscr{M})$ [24, Corollary 2.9].

In this paper, we are concerned with locally symmetric spaces. Let G be a connected semisimple real algebraic group and (X,d) the associated Riemannian symmetric space. Let $\Gamma < G$ be a discrete torsion-free subgroup and let $\mathscr{M} = \Gamma\backslash X$ the corresponding locally symmetric manifold.

For a rank one locally symmetric manifold $\mathscr{M} = \Gamma\backslash X$, the relation between $\lambda_0(\mathscr{M})$ and the critical exponent[1] δ_Γ is well-known: if we denote by $D = D_X$ the volume entropy of X, then

$$\lambda_0(\mathscr{M}) = \begin{cases} D^2/4 & \text{if } \delta_\Gamma \le D/2 \\ \delta_\Gamma(D-\delta_\Gamma) & \text{otherwise} \end{cases}$$

([3, 7–9, 17–19, 24]). We refer to ([1, 2, 5, 15, 27]) for extensions of these results to higher ranks. We remark that when G has Kazhdan's property (T) (cf. [28, Theorem 7.4.2]), we have $\mathrm{Vol}(\mathscr{M}) = \infty$ if and only if $\lambda_0(\mathscr{M}) > 0$ ([3], [15]).

The goal of this article is to study the square-integrability of a base eigenfunction of locally symmetric manifolds. The space of square-integrable base eigenfunctions is at most one dimensional and generated by a *positive* function when non-trivial [24]. Based on this positivity property and using their theory of conformal measures on the geometric boundary, Patterson and Sullivan showed that if $\mathscr{M}$ is a geometrically finite real hyperbolic $(n+1)$-manifold, then $\mathscr{M}$ has

[1] the abscissa of convergence of the Poincare series $s \mapsto \sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(o,\gamma o)}$, $o \in X$.

a square-integrable base eigenfunction if and only if the critical exponent δ_Γ is strictly greater than $n/2$ ([20], [23], [24, Theorem 2.21]). More generally, the formula for $\lambda_0(\mathscr{M})$ given above, together with [12, Corollary 3.2] (cf. also [16]) and [26, Theorem 1.1], implies that any rank one geometrically finite manifold $\mathscr{M}$ has a square-integrable base eigenfunction if and only if the critical exponent δ_Γ is strictly greater than $D_X/2$.

The main theorem of this paper is the following surprising higher rank phenomenon that contrasts with the rank one situation:

Theorem 1. *Let G be a connected semisimple real algebraic group with no rank one factors. For any Zariski dense discrete torsion-free subgroup $\Gamma < G$, we have $\mathrm{Vol}(\Gamma\backslash X) = \infty$ if and only if $\Gamma\backslash X$ does not possess any square-integrable positive Laplace eigenfunction, that is, $\lambda_0(\Gamma\backslash X) > 0$ is not an atom for the spectral measure of $-\Delta$.*

In other words, when $\mathrm{Vol}(\Gamma\backslash X) = \infty$, no base eigenfunction is square-integrable (see also Theorem 10 for a more general version). A special case of this theorem for Anosov subgroups of higher rank semisimple Lie groups was proved in [5, Theorem 1.8]. See Theorem 10 for a more general version.

Our proof of Theorem 1 is based on the higher rank version of Patterson–Sullivan theory introduced by Quint [21], with a main new input being the recent theorem of Fraczyk and Lee (Theorem 8, [10]). Suppose that $\mathrm{Vol}(\Gamma\backslash X) = \infty$ and a base eigenfunction is square-integrable. Using Sullivan's work [24], it was then shown by Edwards and Oh [5] that there exists a Γ-conformal density $\{\nu_x : x \in X\}$ on the Furstenberg boundary of G (see Definition 3) such that any such base eigenfunction is proportional to the function E_ν given by

$$E_\nu(x) = |\nu_x| \quad \text{for all } x \in X. \tag{1.2}$$

Moreover, the following higher rank version of the smearing theorem of Thurston and Sullivan ([23, 25]) was also obtained by Edwards–Oh [5] (see Theorem 6):

$$|\mathsf{m}_{\nu,\nu}| \ll \int_{\Gamma\backslash X} |E_\nu|^2 \, d\mathrm{vol},$$

where $\mathsf{m}_{\nu,\nu}$ is a generalized Bowen–Margulis–Sullivan measure on $\Gamma\backslash G$ corresponding to the pair (ν,ν); see Definition 3.3. On the other hand, the recent theorem of Fraczyk and Lee (Theorem 8, [10]) which describes all discrete subgroups admitting finite BMS measures implies that $|\mathsf{m}_{\nu,\nu}| = \infty$, and consequently, $E_\nu \notin L^2(\Gamma\backslash X)$, yielding a contradiction. We remark that the integrand on the right hand side of (1.2) can be replaced by an $O(1)$-neighborhood of the support of $\mathsf{m}_{\nu,\nu}$ and Sullivan used the rank one version of this to deduce the finiteness of the BMS measure $\mathsf{m}_{\nu,\nu}$ attached to the (unique) Patterson–Sullivan measure ν from the the growth control of the base eigenfunction for Γ geometrically finite [23].

We close the introduction by presenting a related question on the L^2-spectrum. When $\Gamma < G$ is geometrically finite in a rank one Lie group and there is no positive square-integrable eigenfunction, there are no Laplace eigenfunctions in $L^2(\Gamma\backslash X)$ and the quasi-regular representation $L^2(\Gamma\backslash G)$ is tempered[2] ([4, 14, 17, 23]). In view of this, we ask the following question: let G be a semisimple real algebraic group with no rank one factors and $\Gamma < G$ be a Zariski dense discrete subgroup.

Question 2. *When $\Gamma < G$ is not a lattice, can there exist any Laplace eigenfunction in $L^2(\Gamma\backslash X)$?*

[2]This means that $L^2(\Gamma\backslash G)$ is weakly contained in $L^2(G)$, or equivalently, every matrix coefficient of $L^2(\Gamma\backslash G)$ is $L^{2+\varepsilon}(G)$-integrable for any $\varepsilon > 0$.

2. Positive eigenfuntions and conformal measures

Let G be a connected semisimple real algebraic group. We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively the Weyl-group invariant inner product and norm on $\mathfrak{a}$ induced from the Killing form on $\mathfrak{g}$. We denote by $X = G/K$ the corresponding Riemannian symmetric space equipped with the Riemannian metric d induced by the Killing form on $\mathfrak{g}$. The Riemannian volume form on X is denoted by $\mathrm{d}\,\mathrm{vol}$. We also use $\mathrm{d}x$ to denote this volume form, as well as for the Haar measure on G.

Let $A := \exp \mathfrak{a}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M, and we set N to be the maximal horospherical subgroup for A so that $\log(N)$ is the sum of all positive root subspaces for our choice of $\mathfrak{a}^+$. We set $P = MAN$, which is a minimal parabolic subgroup of G. The quotient

$$\mathscr{F} = G/P$$

is known as the Furstenberg boundary of G, and since K acts transitively on $\mathscr{F}$ and $K \cap P = M$, we may identify $\mathscr{F}$ with K/M.

Let Σ^+ denote the set of all positive roots for $(\mathfrak{g}, \mathfrak{a}^+)$. We also write $\Pi \subset \Sigma^+$ for the set of all simple roots. For any $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$ such that $g \in K \exp \mu(g) K$. The map $\mu : G \to \mathfrak{a}^+$ is called the Cartan projection. Setting $o = [K] \in X$, we then have $\|\mu(g)\| = d(go, o)$ for all $g \in G$. Throughout the paper we will identify functions on X with right K-invariant functions on G. For each $g \in G$, we define the following *visual* maps:

$$g^+ := gP \in \mathscr{F} \quad \text{and} \quad g^- := g w_0 P \in \mathscr{F}, \tag{2.1}$$

where w_0 denotes the longest Weyl group element, i.e. the Weyl group element such that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The unique open G-orbit $\mathscr{F}^{(2)}$ in $\mathscr{F} \times \mathscr{F}$ under the diagonal G-action is given by $\mathscr{F}^{(2)} = G(e^+, e^-) = \{(g^+, g^-) \in \mathscr{F} \times \mathscr{F} : g \in G\}$. Let $G = KAN$ be the Iwasawa decomposition, and define the Iwasawa cocycle $H : G \to \mathfrak{a}$ by the relation:

$$g \in K \exp\big(H(g)\big) N.$$

The $\mathfrak{a}$-valued Busemann map is defined using the Iwasawa cocycle as follows: for all $g \in G$ and $[k] \in \mathscr{F}$ with $k \in K$, define

$$\beta_{[k]}(g(o), h(o)) := H(g^{-1}k) - H(h^{-1}k) \in \mathfrak{a} \quad \text{for all } g, h \in G.$$

Conformal measures. We denote by $\mathfrak{a}^*$ the space of all real-valued linear forms on $\mathfrak{a}$. In the rest of this section, let $\Gamma < G$ be a discrete subgroup. The following notion of conformal densities was introduced by Quint [21, Section 1.2], generalizing Patterson–Sullivan densities for rank one groups ([20, Section 3], [22, Section 1]).

Definition 3. *Let $\psi \in \mathfrak{a}^*$.*

(1) *A finite Borel measure ν on $\mathscr{F} = K/M$ is said to be a (Γ, ψ)-conformal measure (for the basepoint o) if for all $\gamma \in \Gamma$ and $\xi = [k] \in K/M$,*

$$\frac{\mathrm{d}\gamma_* \nu}{\mathrm{d}\nu}(\xi) = \mathrm{e}^{-\psi(\beta_\xi(\gamma o, o))},$$

where $\gamma_* \nu(Q) = \nu(\gamma^{-1} Q)$ for any Borel subset $Q \subset \mathscr{F}$.

(2) *A collection $\{\nu_x : x \in X\}$ of finite Borel measures on $\mathscr{F}$ is called a (Γ, ψ)-conformal density if, for all $x, y \in X$, $\xi \in \mathscr{F}$ and $\gamma \in \Gamma$,*

$$\frac{\mathrm{d}\nu_x}{\mathrm{d}\nu_y}(\xi) = \mathrm{e}^{-\psi(\beta_\xi(x,y))} \quad \text{and} \quad \mathrm{d}\gamma_* \nu_x = \mathrm{d}\nu_{\gamma(x)}. \tag{2.2}$$

A (Γ,ψ)-conformal measure ν defines a (Γ,ψ)-conformal density $\{\nu_x : x \in X\}$ by the formula:

$$\mathrm{d}\nu_x(\xi) = \mathrm{e}^{-\psi(\beta_\xi(x,o))}\,\mathrm{d}\nu(\xi),$$

and conversely any (Γ,ψ)-conformal density $\{\nu_x\}$ is uniquely determined by its member ν_o by (2.2). By a Γ-conformal measure on $\mathscr{F}$, we mean a (Γ,ψ)-conformal measure for some $\psi \in \mathfrak{a}^*$.

Definition 4. *Let $\psi \in \mathfrak{a}^*$. Associated to a (Γ,ψ)-conformal measure ν on $\mathscr{F}$, we define the following function E_ν on G: for $g \in G$,*

$$E_\nu(g) := |\nu_{g(o)}| = \int_{\mathscr{F}} \mathrm{e}^{-\psi\left(H(g^{-1}k)\right)}\,\mathrm{d}\nu([k]). \tag{2.3}$$

Since $|\nu_{\gamma(x)}| = |\nu_x|$ for all $\gamma \in \Gamma$ and $x \in X$, the left Γ-invariance and right K-invariance of E_ν are clear. Hence we may consider E_ν as a K-invariant function on $\Gamma\backslash G$, or, equivalently, as a function on $\Gamma\backslash X$.

Let $\mathscr{D} = \mathscr{D}(X)$ denote the ring of all G-invariant differential operators on X. For each (Γ,ψ)-conformal measure ν, E_ν is a joint eigenfunction of $\mathscr{D}$ and conversely, any *positive* joint eigenfunction on $\Gamma\backslash X$ arises as E_ν for some (Γ,ψ)-conformal measure ν [5, Proposition 3.3].

Let Δ denote the Laplace–Beltrami operator on X or on $\Gamma\backslash X$. Since Δ is an elliptic differential operator, an eigenfunction is always smooth. We say a smooth function f is λ-harmonic if

$$-\Delta f = \lambda f.$$

Define the real number $\lambda_0 = \lambda_0(\Gamma\backslash X) \in [0,\infty)$ as follows:

$$\lambda_0 := \inf\left\{ \frac{\int_{\Gamma\backslash X} \|\mathrm{grad}\, f\|^2\,\mathrm{d}\mathrm{vol}}{\int_{\Gamma\backslash X} |f|^2\,\mathrm{d}\mathrm{vol}} : f \in C_c^\infty(\Gamma\backslash X),\ f \neq 0 \right\}. \tag{2.4}$$

We call a λ_0-harmonic function on $\Gamma\backslash X$ a base eigenfunction. In general, a λ-harmonic function need not be a joint eigenfunction for the ring $\mathscr{D}(X)$. However, a square-integrable λ_0-harmonic function turns out to be a *positive* joint eigenfunction, up to a constant multiple. The following is obtained in [5, Corollary 6.6, Theorem 6.5] using Sullivan's work [24] and [13].

Theorem 5 ([5]). *If a base eigenfunction ϕ_0 belongs to $L^2(\Gamma\backslash X)$, then there exists $\psi \in \mathfrak{a}^*$ and a (Γ,ψ)-conformal measure ν on $\mathscr{F}$ such that ϕ_0 is proportional to E_ν.*

Here the space $L^2(\Gamma\backslash X)$ consists of square-integrable functions with respect to the inner product $\langle f_1, f_2\rangle = \int_{\Gamma\backslash X} f_1 \overline{f_2}\,\mathrm{d}\mathrm{vol}$.

3. Higher rank smearing theorem

Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a discrete subgroup. We recall the definition of a generalized Bowen–Margulis–Sullivan measure, as was defined in [6, Section 3].

Fix a pair of linear forms $\psi_1, \psi_2 \in \mathfrak{a}^*$. Let ν_1 and ν_2 be respectively (Γ,ψ_1) and (Γ,ψ_2) conformal measures on $\mathscr{F}$. Using the homeomorphism (called the Hopf parametrization) $G/M \to \mathscr{F}^{(2)} \times \mathfrak{a}$ given by $gM \mapsto (g^+, g^-, b = \beta_{g^-}(o, go))$, define the following locally finite Borel measure $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$ on G/M as follows: for $g = (g^+, g^-, b) \in \mathscr{F}^{(2)} \times \mathfrak{a}$,

$$d\tilde{\mathsf{m}}_{\nu_1,\nu_2}(g) = \mathrm{e}^{\psi_1(\beta_{g^+}(o,go)) + \psi_2(\beta_{g^-}(o,go))}\,\mathrm{d}\nu_1(g^+)\mathrm{d}\nu_2(g^-)\,\mathrm{d}b, \tag{3.1}$$

where $\mathrm{d}b = \mathrm{d}\ell(b)$ is the Lebesgue measure on $\mathfrak{a}$ induced from the inner product $\langle\cdot,\cdot\rangle$. The measure $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$ is left Γ-invariant and right A-semi-invariant: for all $a \in A$,

$$a_* \tilde{\mathsf{m}}_{\nu_1,\nu_2} = \mathrm{e}^{(-\psi_1 + \psi_2 \circ \mathrm{i})(\log a)}\,\tilde{\mathsf{m}}_{\nu_1,\nu_2}, \tag{3.2}$$

where i denotes the opposition involution[3] $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ (cf. [6, Lemma 3.6]). The measure $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$ gives rise to a left Γ-invariant and right M-invariant measure on G by integrating along the fibers of $G \to G/M$ with respect to the Haar measure on M. By abuse of notation, we will also denote this measure by $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$. We denote by

$$\mathsf{m}_{\nu_1,\nu_2} \tag{3.3}$$

the measure on $\Gamma\backslash G$ induced by $\tilde{\mathsf{m}}_{\nu_1,\nu_2}$, and call it the generalized BMS-measure associated to the pair (ν_1, ν_2).

The following theorem was proved in [5, Theorem 7.4], extending the smearing argument due to Sullivan and Thurston ([23, Proposition 5], [4, Proof of Theorem 4.1]) to the higher rank setting.

Theorem 6 (Edwards–Oh, [5]). *Let $\psi_1, \psi_2 \in \mathfrak{a}^*$. There exists a constant $c = c(\psi_1,\psi_2) > 0$ such that for any pair (ν_1, ν_2) of (Γ,ψ_1) and (Γ,ψ_2)-conformal measures on $\mathscr{F}$ respectively,*

$$|\mathsf{m}_{\nu_1,\nu_2}| \le c \int_{1\text{-neighborhood of} \operatorname{supp} \mathsf{m}_{\nu_1,\nu_2}} E_{\nu_1}(x) E_{\nu_2}(x)\, \mathrm{d}x.$$

Although [5, Theorem 7.4] was stated so that c depends on ν_1, ν_2, the formula for c given in its proof shows that c depends only on the associated linear forms ψ_1, ψ_2.

An immediate corollary is as follows:

Corollary 7. *Let ν be a Γ-conformal measure on $\mathscr{F}$. If $|\mathsf{m}_{\nu,\nu}| = \infty$, then*

$$E_\nu \notin L^2(\Gamma\backslash X).$$

4. Proof of Main theorem

As in Theorem 1, let G be a connected semisimple real algebraic group with no rank one factors and $\Gamma < G$ be a Zariski dense discrete torsion-free subgroup. We recall the following recent theorem:

Theorem 8 (Fraczyk–Lee, [10]). *Suppose that* $\mathrm{Vol}(\Gamma\backslash X) = \infty$. *Then for any pair (ν_1, ν_2) of (Γ,ψ) and $(\Gamma, \psi \circ \mathrm{i})$-conformal measures for some $\psi \in \mathfrak{a}^*$,*

$$\mathsf{m}_{\nu_1,\nu_2}(\Gamma\backslash G) = \infty.$$

Corollary 9. *If* $\mathrm{Vol}(\Gamma\backslash X) = \infty$, *then for any pair (ν_1, ν_2) of Γ-conformal measures,*

$$\mathsf{m}_{\nu_1,\nu_2}(\Gamma\backslash G) = \infty.$$

Proof. For $k = 1, 2$, let ν_k be a (Γ, ψ_k)-conformal measure with $\psi_k \in \mathfrak{a}^*$. Suppose $|\mathsf{m}_{\nu_1,\nu_2}| < \infty$. Since $a_* \mathsf{m}_{\nu_1,\nu_2} = e^{\psi_1(\log a) - \psi_2(\mathrm{i} \log a)} \mathsf{m}_{\nu_1,\nu_2}$ for all $a \in A$ by (3.2), it follows that

$$|\mathsf{m}_{\nu_1,\nu_2}| = \mathrm{e}^{\psi_1(\log a) - \psi_2(\mathrm{i}\log a)} |\mathsf{m}_{\nu_1,\nu_2}|.$$

Since $|\mathsf{m}_{\nu_1,\nu_2}| < \infty$, we must have

$$\psi_2 = \psi_1 \circ \mathrm{i}.$$

Therefore the claim follows from Theorem 8. □

Proof of Theorem 1. Suppose that $\mathrm{Vol}(\Gamma\backslash X) = \infty$ and ϕ_0 is a base eigenfunction in $L^2(\Gamma\backslash X)$. By Proposition 5, we may assume that $\phi_0 = E_\nu$ for some Γ-conformal measure ν on $\mathscr{F}$. Now by Theorem 6 and Corollary 9,

$$\infty = |\mathsf{m}_{\nu,\nu}| \ll \|E_\nu\|_2^2.$$

This is a contradiction. □

[3] It is defined by $\mathrm{i}(u) = -\mathrm{Ad}_{w_0}(u)$, where w_0 is the longest Weyl element.

Indeed, using a more precise version of the main theorem of [10] in replacement of Theorem 8, we obtain the following without the hypothesis on no rank one factors.

Theorem 10. *Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. If $\Gamma\backslash X$ admits a square-integrable base eigenfunction, then $G = G_1G_2$, Γ is commensurable with $\Gamma_1\Gamma_2$ where G_1 (resp. G_2) is an almost direct product of rank one (resp. higher rank) factors of G, $\Gamma_1 < G_1$ is a discrete subgroup and $\Gamma_2 < G_2$ is a lattice.*

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