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On representation of solutions to the heat equation

Sur la représentation des solutions de l'équation de la chaleur

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Abstract. We propose a simple method to obtain semigroup representation of solutions to the heat equation using a local L^2 condition with prescribed growth and a boundedness condition within tempered distributions. This applies to many functional settings and, as an example, we consider the Koch and Tataru space related to BMO^{-1} initial data.

Résumé. Nous proposons une méthode simple pour obtenir une représentation par semi-groupe des solutions de l'équation de la chaleur utilisant une condition L^2 à poids et un contrôle dans des distributions tempérées. Cette méthode s'applique à de nombreux espaces fonctionnels. À titre d'exemple, nous considérons l'application aux solutions dans l'espace de Koch et Tataru lié aux données initiales dans BMO^{-1}.

1. Introduction

The purpose of this note is to investigate representation for solutions to the heat equation

$$\partial_t u - \Delta u = 0 \tag{1}$$

on the upper-half space $\mathbb{R}^{1+n}_+ := (0,\infty) \times \mathbb{R}^n$ or on a strip $(0, T) \times \mathbb{R}^n$. That is, when can we assert that u can be represented by the heat semigroup acting on a data, i.e.,

$$u(t) := u(t,\cdot) = e^{t\Delta} u_0 \tag{2}$$

for some u_0 and all $t \in (0, T)$?

The topic is not new, of course, so let us first briefly comment on some classical results in the literature.

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The most general framework for such a representation is via tempered distributions. More precisely, given $u_0 \in \mathscr{S}'(\mathbb{R}^n)$, then $t \mapsto e^{t\Delta}u_0$ lies in $C^\infty([0,\infty); \mathscr{S}'(\mathbb{R}^n))$. Conversely, it has been shown in [12, Chap. 3, Prop. 5.1] that any $u \in C^\infty([0,\infty); \mathscr{S}'(\mathbb{R}^n))$ solving the heat equation is represented by the heat semigroup applied to its initial value. Certainly, the argument still works in $C^1((0,\infty); \mathscr{S}'(\mathbb{R}^n)) \cap C([0,\infty); \mathscr{S}'(\mathbb{R}^n))$, which seems to close the topic. But it uses Fourier transform, so it is not transposable to more general equations (e.g., parabolic equations with coefficients). Thus, one may wonder whether different concrete knowledge, like a growth condition, on the solution could lead to a representation, not using Fourier transform. Yet, one can observe that growth exceeding the inverse of a Gaussian when $|x| \to \infty$ is forbidden for the representation.

Another framework is that of non-negative solutions. A classical result by D. Widder [15, Thm. 6] shows that in one-dimensional case, any non-negative C^2-solution u in the strip must be of the form (2) for some non-negative Borel measure u_0. It has been generalized to higher dimensions and classical solutions of parabolic equations with smooth coefficients by M. Krzyzanski [10], via internal representation and a limiting argument. We are also going to use this idea below, but we want to remove the sign condition. D.G. Aronson later extended it to non-negative weak solutions of real parabolic equations, see [1, Thm. 11].

Next, the uniqueness problem is tied with representation but they are different issues. For instance, let us mention the pioneering work on non-uniqueness by A. Tychonoff [14], and two works giving sufficient criteria on strips for uniqueness. One by S. Täcklind [11] provides the optimal pointwise growth condition, and the other by A. Gushchin [7] provides a local L^2 condition with prescribed growth, also optimal but more amenable to more general equations. In these results, the growth can be faster than the inverse of a Gaussian when $|x| \to \infty$, which hence excludes usage of tempered distributions, so uniqueness can hold without being able to represent general solutions.

With these observations in mind, it seems that we have two very different theories to approach representation (and uniqueness): one only using distributions and Fourier transform; one not using them at all. The goal of this note is to make a bridge between them, i.e., to obtain tempered distributions, not just measurable functions or measures, as initial data from local integrability conditions. Such conditions may only include integrability conditions in the interior, completed by a uniform control.

Let us state our result. A sequence (T_k) of tempered distributions is bounded if $(\langle T_k, \varphi \rangle)$ is bounded for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Recall that any distributional solution to the heat equation on strips is in fact smooth by hypoellipticity, see for instance [8, § 4.4].

Theorem 1. *Let $0 < T \le \infty$. Let $u \in \mathscr{D}'((0,T) \times \mathbb{R}^n)$ be a distributional solution to the heat equation. Suppose that:*

(i) *(Size condition) For $0 < a < b < T$, there exist $C(a,b) > 0$ and $0 < \gamma < 1/4$ such that for any $R > 0$,*

$$\left(\int_a^b \int_{B(0,R)} |u(t,x)|^2 \, dt\, dx \right)^{1/2} \le C(a,b) \exp\left(\frac{\gamma R^2}{b-a} \right); \tag{3}$$

(ii) *(Uniform control) There exists a sequence (t_k) tending to 0 such that $(u(t_k))$ is bounded in $\mathscr{S}'(\mathbb{R}^n)$.*

Then there exists a unique $u_0 \in \mathscr{S}'(\mathbb{R}^n)$ so that $u(t) = e^{t\Delta}u_0$ for all $0 < t < T$, where the heat semigroup is understood in the sense of tempered distributions.

Let us first give some remarks. The L^2 condition (i) is only assumed on interior strips, and its growth is in the order of the inverse of a Gaussian. It can be proved that this condition alone implies that $u(t)$ is a tempered distribution for each $t > 0$. The precise behavior of $C(a,b)$ is not

required, and it could blow up as $a \to 0$. Condition (ii) provides us with necessary uniformity to define the initial data, which a priori does not follow from (i).

The conclusion also shows that u has a natural extension to a solution of the heat equation on $(0, \infty) \times \mathbb{R}^n$ if $T < \infty$. And if T was already ∞, then we could get a control on u when $t \to \infty$ via any knowledge we might get on u_0, e.g., if $u_0 \in L^2(\mathbb{R}^n)$ then $u(t)$ is bounded in $L^2(\mathbb{R}^n)$.

As an interesting remark, the argument is not using Fourier transform at all, and it indeed extends with an analogous strategy to more general parabolic equations with bounded measurable, real or complex coefficients at the expense of working in more restrictive spaces than $\mathscr{S}'(\mathbb{R}^n)$. The reader can refer to [4, 16], and [2] for the proof and applications of the general result in the context of measurable initial data. Applications towards distributional data for more general equations will appear in the forthcoming work [3].

The consequence for uniqueness is as follows.

Corollary 2. *Let $0 < T \le \infty$. Let $u \in \mathscr{D}'((0, T) \times \mathbb{R}^n)$ be a distributional solution to the heat equation. Suppose that* (i) *holds and that there exists a sequence (t_k) tending to 0 such that $(u(t_k))$ converges to 0 in $\mathscr{S}'(\mathbb{R}^n)$. Then $u = 0$.*

Remark 3. Convergence in $\mathscr{S}'(\mathbb{R}^n)$ cannot be replaced by convergence in $\mathscr{D}'(\mathbb{R}^n)$. The reader can refer to [6] for a non-identically zero solution $u \in C^\infty(\mathbb{R}^2_+) \cap C([0, \infty) \times \mathbb{R})$ with $u(0, x) = 0$ everywhere and $|u(t, x)| \le C(\epsilon) e^{\epsilon/t}$ for any $\epsilon > 0$. The continuity implies uniform convergence of $u(t)$ to 0 on compact intervals as $t \to 0$, and hence convergence in distributional sense.

2. Proof of the theorem

Our main lemma asserts the L^2-growth on rectangles of caloric function implies internal semi-group representation, which is an interesting fact in its own sake that we did not find in the literature. It can be seen as a particular case of [4, Thm. 5.1] obtained for general parabolic equations with time-dependent, bounded measurable and elliptic coefficients. For the heat equation, an elementary proof based on Green's formula and basic estimates will be given.

Lemma 4. *Let $u \in \mathscr{D}'((a, b) \times \mathbb{R}^n)$ be a distributional solution to the heat equation. Suppose that* (3) *holds. Then when $a < s < t < b$, for any $h \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} u(t, x) h(x) \, dx = \int_{\mathbb{R}^n} u(s, x) \left(e^{(t-s)\Delta} h \right)(x) \, dx \tag{4}$$

where the right-hand integral absolutely converges.

The identity (4) can heuristically be called *homotopy identity*, as it formally shows $u(t) = e^{(t-s)\Delta} u(s)$ in the sense of distributions. In fact, once (4) is shown, one can extend it to all $h \in \mathscr{S}(\mathbb{R}^n)$, so that this holds in the sense of tempered distributions.

Proof. First observe that u in fact lies in $C^\infty((a, b) \times \mathbb{R}^n)$ by hypoellipticity, as we pointed out before Theorem 1. Thus, all integrals and integration by parts below are justified.

Then for fixed $h \in C_c^\infty(\mathbb{R}^n)$ and $t \in \mathbb{R}$, define

$$\varphi(\tau, x) := \left(e^{(t-\tau)\Delta} h \right)(x),$$

which satisfies $\partial_\tau \varphi + \Delta \varphi = 0$ on $(-\infty, t) \times \mathbb{R}^n$. Green's formula implies for any $r > 0$ and $\tau \in (0, t)$,

$$\int_{B(0,r)} \left(u \Delta \varphi - \varphi \Delta u \right)(\tau) = \int_{\partial B(0,r)} \left(u \nabla \varphi - \varphi \nabla u \right)(\tau) \cdot \mathbf{n} \, d\sigma, \tag{5}$$

where $\mathbf{n}$ is the outer unit normal vector and $d\sigma$ is the sphere volume form. Here and in the sequel, unspecified measures are Lebesgue measures. Newton–Leibniz formula yields

$$\int_s^t \int_{B(0,r)} \left(\varphi \partial_\tau u + u \partial_\tau \varphi \right) = \int_{B(0,r)} \left(u(t)\varphi(t) - u(s)\varphi(s) \right). \tag{6}$$

Integrating (5) over $s < \tau < t$ and adding it to (6), we have

$$\int_{B(0,r)} u(t)\varphi(t) - \int_{B(0,r)} u(s)\varphi(s) = \int_s^t \int_{\partial B(0,r)} \left(\varphi \nabla u - u \nabla \varphi \right) \cdot \mathbf{n} \, d\sigma \, d\tau. \tag{7}$$

Note that $u(t)\varphi(t) \in L^1(\mathbb{R}^n)$ as $\varphi(t) = h$ has compact support. We claim $u(s)\varphi(s)$ also lies in $L^1(\mathbb{R}^n)$. Indeed, pick $\rho > 0$ such that $\operatorname{supp}(h) \subset B(0, \rho)$. Let $\kappa > 1$ be a constant to be determined. Denote by C_0 the ball $B(0, \kappa \rho)$ and by C_j the annulus $\{x \in \mathbb{R}^n : \kappa^j \rho \le |x| < \kappa^{j+1} \rho\}$ for $j \ge 1$. We have

$$\int_{\mathbb{R}^n} |u(s)| \left| e^{(t-s)\Delta} h \right| \le \sum_{j \ge 0} \| u(s) \|_{L^2(C_j)} \left\| e^{(t-s)\Delta} h \right\|_{L^2(C_j)}. \tag{8}$$

Note that only terms with $j \gg 1$ are of concern as both $u(s)$ and $\varphi(s)$ are bounded on compact sets. Caccioppoli's inequality[1] and (i) imply

$$\| u(s) \|_{L^2(C_j)}^2 \lesssim_\kappa \left(\frac{1}{\left(\kappa^{j+1} \rho \right)^2} + \frac{1}{s-a} \right) \int_a^b \| u(\tau) \|_{L^2(B(x, \kappa^{j+2}\rho))}^2 \, d\tau$$

$$\lesssim_{\rho,a,b} \frac{1}{s-a} \exp\left(\frac{2\gamma}{b-a} \left(\kappa^{j+2} \rho \right)^2 \right).$$

The heat kernel representation implies

$$\left\| e^{(t-s)\Delta} h \right\|_{L^2(C_j)} \lesssim \exp\left(-\frac{c d_j^2}{t-s} \right) \| h \|_2$$

for $0 < c < 1/4$ and $d_j := \operatorname{dist}(C_j, \operatorname{supp}(h))$, which asymptotically equals to $\kappa^j \rho$. Pick c close to $1/4$ and κ close to 1 so that $\gamma < c/\kappa^4$, and thus for $j \gg 1$,

$$\gamma < \frac{c d_j^2}{\left(\kappa^{j+2} \rho \right)^2} < \frac{c d_j^2}{\left(\kappa^{j+2} \rho \right)^2} \cdot \frac{b-a}{t-s}.$$

It ensures that the sum in (8) converges and the claim hence follows.

Thus, it suffices to prove that there exists an increasing sequence (r_m) tending to ∞ such that

$$\lim_{m \to +\infty} \int_s^t \int_{\partial B(0, r_m)} \left(\left| \varphi \nabla u \right| + \left| u \nabla \varphi \right| \right) d\tau \, d\sigma = 0.$$

Indeed, suppose so, and then applying (7) for (r_m) and taking limits on m imply (4) holds. Let us show the existence of such sequence. Let $0 < \lambda < 1$ be a constant to be determined. Define

$$\Phi(R) := \int_{\lambda R}^R r^{n-1} \int_s^t \int_{\partial B(0,r)} \left(\left| \varphi \nabla u \right| + \left| u \nabla \varphi \right| \right) d\tau \, d\sigma \, dr$$

$$= \int_s^t \int_{\lambda R < |x| < R} \left(\left| \varphi \nabla u \right| + \left| u \nabla \varphi \right| \right) d\tau \, dx,$$

and denote by $\Phi_i(R)$ the ith term for $i = 1, 2$. Then, it is enough to show that $\Phi(R)$ is bounded. For $\Phi_1(R)$, Caccioppoli's inequality, kernel representation of the heat semigroup, and (i) altogether imply

$$\Phi_1(R) \le \left(\int_s^t \int_{\lambda R < |x| < R} |\nabla u|^2 \right)^{1/2} \left(\int_s^t \int_{\lambda R < |x| < R} |\varphi|^2 \right)^{1/2}$$

$$\lesssim_\kappa \left[\frac{1}{s-a} \left(1 + \frac{t-a}{R^2} \right) \int_a^b \int_{|x| < \kappa R} |u|^2 \right]^{1/2} \left(\int_s^t e^{-\frac{2c d^2}{t-\tau}} \, d\tau \right)^{1/2} \| h \|_2$$

$$\lesssim_{a,b,\kappa} \left(\frac{t-s}{s-a} \right)^{1/2} \exp\left(-\frac{c d^2}{t-s} + \frac{\gamma (\kappa R)^2}{b-a} \right) \| h \|_2$$

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[1] By this we mean the energy estimates obtained by multiplying u with proper cut-off and seeing this as a solution to the heat equation with localised source term.

for sufficiently large R. Here, $\kappa > 1$ and $0 < c < 1/4$ are constants to be determined, and $d := \text{dist}(\text{supp}(h), \{\lambda R < |x| < R\})$. Then, pick κ close to 1, λ close to 1, and c close to $1/4$ so that $\gamma < c\lambda^2/\kappa^2$, and thus

$$\gamma < \frac{cd^2}{(\kappa R)^2} < \frac{cd^2}{(\kappa R)^2} \cdot \frac{b-a}{t-s} \tag{9}$$

for sufficiently large R. We hence conclude that $\Phi_1(R)$ is bounded.

For $\Phi_2(R)$, the kernel representation of $(t^{1/2}\nabla e^{t\Delta})_{t>0}$ and (i) yield

$$\begin{aligned}
\Phi_2(R) &\le \left(\int_a^b \int_{\lambda R < |x| < R} |u|^2\right)^{1/2} \left(\int_s^t \int_{\lambda R < |x| < R} |\nabla \varphi|^2\right)^{1/2} \\
&\lesssim e^{\frac{\gamma R^2}{b-a}} \left(\int_s^t e^{-\frac{2cd^2}{t-\tau}} \frac{d\tau}{t-\tau}\right)^{1/2} \|h\|_2 \\
&\lesssim \frac{(t-s)^{1/2}}{c^{1/2}d} \exp\left(-\frac{cd^2}{t-s} + \frac{\gamma R^2}{b-a}\right) \|h\|_2.
\end{aligned}$$

The same choice for λ and c as above implies (9), and hence the boundedness of $\Phi_2(R)$. This completes the proof. $\qquad\square$

Let us recall some topological facts about tempered distributions.

Lemma 5. *Let X be a Fréchet space and Y be a normed space. Let I be an index set and $\{\psi_\alpha\}_{\alpha \in I}$ be a collection of continuous linear maps from X to Y. If $\sup_{\alpha \in I} \|\psi_\alpha(x)\|_Y$ is bounded for any $x \in X$, then the family $\{\psi_\alpha\}_{\alpha \in I}$ is equicontinuous.*

Proof. It is a direct consequence of a generalized version of Banach-Steinhaus theorem on barrelled spaces, due to the fact that Fréchet spaces are barrelled spaces, see [5, § III.4]. $\qquad\square$

One can easily obtain two corollaries. The pairings below are all understood in the sense of tempered distributions.

Corollary 6. *Let (φ_k) be a sequence converging to φ in $\mathscr{S}(\mathbb{R}^n)$ and (u_k) be a sequence converging to u in $\mathscr{S}'(\mathbb{R}^n)$. Then $(\langle u_k, \varphi_k \rangle)$ converges to $\langle u, \varphi \rangle$.*

Corollary 7. *Any bounded sequence in $\mathscr{S}'(\mathbb{R}^n)$ has a convergent subsequence.*

Let us provide the proof of Theorem 1.

Proof of Theorem 1. Given $0 < s < t < T$, pick a, b so that $0 < a < s < t < b < T$. Applying Lemma 4, we get for any $h \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} u(t,x)h(x)\,dx = \int_{\mathbb{R}^n} u(s,x)\left(e^{(t-s)\Delta}h\right)(x)\,dx. \tag{10}$$

Moreover, Corollary 7 implies there is a subsequence (t_{k_j}) so that $(u(t_{k_j}))$ converges to some u_0 in $\mathscr{S}'(\mathbb{R}^n)$ as $j \to \infty$. One can also easily verify that $e^{(t-s)\Delta}h$ converges to $e^{t\Delta}h$ in $\mathscr{S}(\mathbb{R}^n)$ as $s \to 0$. Thus, we infer from Corollary 6 that

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} u(t_{k_j}, x)\left(e^{\left(t-t_{k_j}\right)\Delta}h\right)(x)\,dx = \langle u_0, e^{t\Delta}h \rangle.$$

Applying (10) for $s = t_{k_j}$ and taking limits on j yield $u(t) = e^{t\Delta}u_0$ in $\mathscr{D}'(\mathbb{R}^n)$. As the right-hand side belongs to $\mathscr{S}'(\mathbb{R}^n)$, so does $u(t)$ for all $0 < t < T$. In particular, it implies $u(t)$ converges to u_0 in $\mathscr{S}'(\mathbb{R}^n)$ as $t \to 0$, so u_0 is unique. This completes the proof. $\qquad\square$

3. Applications

Our statement covers many functional spaces of common use in analysis, even when one expects that the initial data is a distribution. To keep this note short, let us illustrate the result with an example related to the famous work of H. Koch and D. Tataru on Navier–Stokes equations [9]. More applications will be studied in forthcoming works. Define the *tent space* T^∞ as the collection of measurable functions u for which

$$\|u\|_{T^\infty} := \sup_B \left(\frac{1}{|B|} \int_0^{r(B)^2} \int_B |u(t,y)|^2 \, dt dy \right)^{1/2} < \infty,$$

where B describes balls in $\mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B. Let BMO^{-1} be the collection of distributions $f \in \mathscr{D}'(\mathbb{R}^n)$ with $f = \mathrm{div} g$ for some $g \in \mathrm{BMO}(\mathbb{R}^n; \mathbb{C}^n)$. The space BMO^{-1} can be embedded into $\mathscr{S}'(\mathbb{R}^n)$. It is isomorphic to the homogeneous Triebel–Lizorkin space $\dot{F}_{\infty,2}^{-1}$, with equivalent norms, see, e.g., [13, § 5.1]. Moreover, a well-known characterisation of BMO^{-1} is that

$$\|f\|_{\mathrm{BMO}^{-1}} \approx \left\| e^{t\Delta} f \right\|_{T^\infty}. \tag{11}$$

In particular, a tempered distribution f lies in BMO^{-1} if $(e^{t\Delta} f)(x)$ lies in T^∞. Remark that if f belongs to BMO^{-1}, as a function of $t \in [0,\infty)$, $e^{t\Delta} f$ is continuous in BMO^{-1} equipped with its weak star topology (or equipped with the topology inherited from that of $\mathscr{S}'(\mathbb{R}^n)$, by density).

   A natural question is whether all T^∞ functions solving the heat equation are of that form. We answer it in the affirmative.

Theorem 8. *Given a global distributional solution to the heat equation $u \in T^\infty$, there exists a unique $u_0 \in \mathrm{BMO}^{-1}$ so that $u(t) = e^{t\Delta} u_0$ for any $t > 0$.*

Proof. It suffices to verify the two conditions in Theorem 1. Indeed, it shows that there exists a unique $u_0 \in \mathscr{S}'(\mathbb{R}^n)$ so that $u(t) = e^{t\Delta} u_0$ for any $t > 0$. We then get $u_0 \in \mathrm{BMO}^{-1}$ by (11) since $(e^{t\Delta} u_0)(x) = u(t,x)$ belongs to T^∞.

   Let us verify the conditions. First, (i) readily follows as for $0 < a < b < \infty$,

$$\int_a^b \int_{B(0,R)} |u(t,y)|^2 \, dt dy = \int_a^b \int_{B(0,R)} |u(t,y)|^2 \left(\frac{1}{|B(y,b^{1/2})|} \int_{B(y,b^{1/2})} dx \right) dt dy$$

$$\leq \int_{B(0,R+b^{1/2})} \left(\frac{1}{|B(x,b^{1/2})|} \int_a^b \int_{B(x,b^{1/2})} |u(t,y)|^2 \, dt dy \right) dx$$

$$\leq \|u\|_{T^\infty}^2 \left| B(0,R+b^{1/2}) \right|.$$

   Next, we claim that there exists $M > 0$ so that for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\sup_{0 < t < 1/2} |\langle u(t), \varphi \rangle| \lesssim \mathscr{P}_M(\varphi) \|u\|_{T^\infty}, \tag{12}$$

where $\mathscr{P}_M$ is the semi-norm given by

$$\mathscr{P}_M(\varphi) := \sup_{|\alpha|+|\beta| \leq M} \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi(x) \right|.$$

For fixed $0 < t < 1/2$, standard considerations allow one to extend $u(t)$ to a tempered distribution so that (12) holds for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$, which proves (ii). As for the claim, fix $0 < t < 1/2 < t' < 1$ and let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Using the equation for u and integration by parts,

$$\left| \langle u(t'), \varphi \rangle - \langle u(t), \varphi \rangle \right| \leq \int_t^{t'} \int_{\mathbb{R}^n} |u(s,x)| \, |\Delta \varphi(x)| ds dx$$

$$= \int_t^{t'} \int_{B(0,1)} |u(s,x)| \left| \Delta \varphi(x) \right| ds dx + \sum_{k=1}^\infty \int_t^{t'} \int_{2^{k-1} \leq |x| < 2^k} |u(s,x)| \left| \Delta \varphi(x) \right| ds dx.$$

Denote by I_0 the first term and I_k the kth term in the summation. Cauchy-Schwarz inequality yields

$$I_0 \le |B(0,1)| \left(\sup_{|x|<1} \left| \Delta \varphi(x) \right| \right) \left(\frac{1}{|B(0,1)|} \int_0^1 \int_{B(0,1)} |u(s,x)|^2 \, ds \, dx \right)^{1/2}$$
$$\lesssim_n \mathscr{P}_2(\varphi) \|u\|_{T^\infty}.$$

Similarly, we also have that

$$I_k \lesssim_n 2^{kn} \left(\sup_{2^{k-1} \le |x| < 2^k} \left| \Delta \varphi(x) \right| \right) \left(\frac{1}{|B(0,2^k)|} \int_0^1 \int_{B(0,2^k)} |u(s,x)|^2 \, ds \, dx \right)^{1/2}$$
$$\lesssim_n 2^{-k} \left(\sup_{x \in \mathbb{R}^n} |x|^{n+1} |\Delta \varphi(x)| \right) \|u\|_{T^\infty} \le 2^{-k} \mathscr{P}_{n+3}(\varphi) \|u\|_{T^\infty}.$$

We thus obtain

$$\left| \langle u(t'), \varphi \rangle - \langle u(t), \varphi \rangle \right| \lesssim_n \mathscr{P}_{n+3}(\varphi) \|u\|_{T^\infty}.$$

Taking average in $t' \in (1/2, 1)$ implies

$$\left| \langle u(t), \varphi \rangle \right| \lesssim_n \int_{1/2}^1 \left| \langle u(t'), \varphi \rangle \right| \, dt' + \mathscr{P}_{n+3}(\varphi) \|u\|_{T^\infty}.$$

The same argument as above yields

$$\int_{1/2}^1 \left| \langle u(t'), \varphi \rangle \right| \, dt' \le \int_{1/2}^1 \int_{\mathbb{R}^n} \left| u(t', x) \right| |\varphi(x)| \, dt' \, dx \lesssim_n \mathscr{P}_{n+1}(\varphi) \|u\|_{T^\infty}.$$

This completes the proof of Theorem 8. $\qquad\square$

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