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Some lacunarity properties of partial quotients of real numbers

Quelques propriétés de lacunarité des quotients partiels de nombres réels

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Abstract. We consider lacunarity properties of sequence of partial quotients for real numbers in their continued fraction expansions. Hausdorff dimension of the sets of points with different lacunarity conditions on their partial quotients are calculated.

Résumé. Nous considérons des propriétés de lacunarité de la suite des quotients partiels du développement en fraction continue de nombres réels. Nous calculons la dimension de Hausdorff d'ensembles de points dont la suite des quotients partiels satisfait à différentes conditions de lacunarité.

Keywords. Hausdorff dimension, Continued fraction expansion.

Mots-clés. Dimension de Hausdorff, développement en fraction continue.

Mathematical subject classification (2010). 11K50, 28A78.

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1. Introduction

Continued fraction expansion is induced by the Gauss transformation $T : [0,1) \to [0,1)$ given by

$$T(0) := 0, \quad T(x) = \frac{1}{x} \bmod 1, \ x \in (0,1).$$

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Then every irrational number $x \in [0,1)$ can be uniquely expanded into an infinite continued fraction:

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots}} := [a_1(x), a_2(x), \ldots,],$$

where $a_1(x) = \lfloor 1/x \rfloor$ and $a_n(x) = a_1\left(T^{n-1}(x)\right)$ for $n \geq 2$ are called the partial quotients of x. The finite truncation

$$\frac{p_n(x)}{q_n(x)} = [a_1(x), \ldots, a_n(x)]$$

is called the nth convergent of x.

It is well-known that continued fraction expansion plays an important role in Diophantine approximation and dynamical systems:

- In Diophantine approximation, how well an irrational number can be approximated by rationals depends on the growth rate of the partial quotients. For example, the classic Jarník set [12] can be expressed as

$$\mathscr{K}(\psi) = \Big\{x \in [0,1) : a_{n+1}(x) \geq \psi(q_n(x)), \text{ for infinitely many } n \in \mathbb{N}\Big\}.$$

- In dynamical system, continued fraction is a classic dynamics with infinitely many branches [17].

As said above, the growth rate of the partial quotient is tightly related to the Diophantine properties of an irrational number. Many metric results have been achieved in this aspect, such as the Borel–Bernstein theorem that deals with Lebesgue measure theory on the growth rate of $\{a_n(x)\}_{n\geq 1}$. Hausdorff dimension of sets obeying some restrictions on the partial quotients have been well established in Good [6], Łcuzak [16], Wang & Wu [18], etc.

Very recently, it was found by Kleinbock and Wadleigh [14] that the Dirichlet improvable set is highly related to the growth rate of the product of two consecutive partial quotients in the following sense. Let $\psi\colon [t_0, \infty) \to \mathbb{R}_+$ be non-increasing with $t_0 > 1$ is fixed and $t\psi(t) < 1$ for all $t \geq t_0$. The ψ-Dirichlet improvavable set is defined as

$$\mathscr{D}(\psi) = \Big\{x \in [0,1) : \min_{1 \leq q < Q} \|qx\| < \psi(Q), \text{ for all } Q \gg 1\Big\}.$$

Then by taking $\Psi(q) = \frac{q\psi(q)}{1 - q\psi(q)}$, one has

$$G(\Psi) \subset [0,1) \setminus \mathscr{D}(\psi) \subset G\left(\frac{\Psi}{4}\right)$$

where

$$G(\Psi) = \Big\{x \in [0,1) : a_{n+1}(x) a_n(x) \geq \Psi(q_n(x)), \text{ for infinitely many } n \in \mathbb{N}\Big\}.$$

Since then, the metric theory relating the growth of two consecutive partial quotients are extensively studied in [1–3, 7–10, 15].

In this note, we take another turn by studying the relative growth rate of two consecutive partial quotients. All the above mentioned works are concerning the size of limsup sets, however, the sets we will consider below are of liminf nature. So the method used here is different from the above works (see [10] for a unified way of dealing with the Hausdorff dimension of the above mentioned works). These liminf sets concerns the points with partial quotients increasing very fast, so can be uniformly well approximated by their convergents. At first, we give some notations. Let $\{s_n\}_{n\geq 1}$ be a sequence of strictly increasing sequence of integers.

- call it a sub-lacunary sequence, if
$$\lim_{n\to\infty} s_{n+1}/s_n = 1;$$
- call it lacunary if $\exists c > 1$ such that
$$s_{n+1}/s_n \geq c, \text{ for all } n \gg 1;$$
  - log-lacunary if for some $t > 0$,
$$\frac{s_{n+1}}{s_n} \geq (\log n)^t, \text{ for all } n \gg 1;$$
  - polynomial-lacunary if for some $t > 0$,
$$\frac{s_{n+1}}{s_n} \geq n^t, \quad \text{for all } n \gg 1;$$
  - exponential-lacunary if for some $b > 1$,
$$\frac{s_{n+1}}{s_n} \geq b^n, \quad \text{for all } n \gg 1.$$

At first, we give an auxiliary set to be compliant with the set of points with strictly increasing partial quotients.
$$G = \Big\{x \in [0,1) : \{a_n(x)\} \text{ is a strictly increasing sequence}\Big\}. \tag{1}$$
Then we define
$$\mathscr{S}_G = \Big\{x \in G : \{a_n(x)\} \text{ is sub-lacunary}\Big\},$$
$$\mathscr{L}_G = \Big\{x \in G : \{a_n(x)\} \text{ is log-lacunary}\Big\},$$
$$\mathscr{P}_G = \Big\{x \in G : \{a_n(x)\} \text{ is polynomial-lacunary}\Big\},$$
$$\mathscr{E}_G = \Big\{x \in G : \{a_n(x)\} \text{ is exponential-lacunary}\Big\}.$$
In this note, we show that

Theorem 1. *By denoting* $\dim_{\mathrm{H}}$ *the Hausdorff dimension, we have*
$$\dim_{\mathrm{H}} \mathscr{S}_G = \dim_{\mathrm{H}} \mathscr{L}_G = \dim_{\mathrm{H}} \mathscr{P}_G = \dim_{\mathrm{H}} \mathscr{E}_G = 1/2.$$

2. Preliminaries

Recall that $p_n(x)/q_n(x)$ is the nth convergent of x. The numerator and denominator of $p_n(x)/q_n(x)$ can be determined recursively: for any $k \geq 1$
$$p_k(x) = a_k(x)p_{k-1}(x) + p_{k-2}(x), \quad q_k(x) = a_k(x)q_{k-1}(x) + q_{k-2}(x) \tag{2}$$
with the conventions $p_0 = 0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

For simplicity, we write
$$p_n(x) = p_n(a_1,\dots,a_n) = p_n, \quad q_n(x) = q_n(a_1,\dots,a_n) = q_n \tag{3}$$
when the partial quotients $a_1,\dots,a_n$ are clear.

For any positive integers $a_1,\dots,a_n$, define
$$I_n(a_1,\dots,a_n) := \{x \in [0,1) : a_1(x) = a_1,\dots,a_n(x) = a_n\}$$
and call it *a cylinder of order n*. We use $I_n(x)$ to denote the nth order cylinder containing x.

Proposition 2 (Khinchin [13]). *For any $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, p_k, q_k are defined recursively by* (2) *for* $0 \leq k \leq n$. *Then*

$$I_n(a_1, \ldots, a_n) = \begin{cases} \left[\dfrac{p_n}{q_n}, \dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}\right) & \text{if } n \text{ is even} \\ \left(\dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}, \dfrac{p_n}{q_n}\right] & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

Therefore, the length of a cylinder of order n is given by

$$|I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})}.$$

The next lemma relates a ball with the cylinders, basically following from Proposition 2 on the distribution of cylinders [11].

Proposition 3. *Let $x \in I_n(a_1, \ldots, a_n)$ with $a_n \geq 2$. Then*

$$B(x, |I_n(a_1, \ldots, a_n)|) \subset \bigcup_{i=-1}^{2} I_n(a_1, \ldots, a_n + i).$$

Next, we introduce the mass distribution principle which is a classic method in estimating the Hausdorff dimension of a set from below.

Proposition 4 ([4]). *Let E be a Borel set and μ be a measure with $\mu(E) > 0$. Suppose that for any $x \in E$,*

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq s \tag{5}$$

where $B(x, r)$ denotes an open ball centered at x and radius r, then $\dim_{\mathrm{H}} E \geq s$.

3. Proof of Theorem 1

Good [6] showed that the Hausdorff dimension of G defined in (1) is one-half, so by the simple inclusion

$$G \supset \mathscr{S}_G, \text{ and } G \supset \mathscr{L}_G \supset \mathscr{P}_G \supset \mathscr{E}_G,$$

it is sufficient to show that

$$\dim_{\mathrm{H}} \mathscr{S}_G \geq 1/2 \text{ and } \dim_{\mathrm{H}} \mathscr{E}_G \geq 1/2.$$

Lemma 5. *Let $\alpha > 1$. Define*

$$E_\alpha = \left\{x \in [0,1) : (2n-1)^\alpha \leq a_n(x) < (2n)^\alpha, \textit{ for all } n \geq 1\right\}.$$

Then

$$\dim_{\mathrm{H}} E_\alpha = \frac{\alpha - 1}{2\alpha}.$$

Proof.

(I) For the upper bound of $\dim_{\mathrm{H}} E_\alpha$, we consider a natural cover of E_α. It is clear that for any $N \geq 1$, the family of intervals

$$\bigcup_{a_1, \ldots, a_N : (2n-1)^\alpha \leq a_n(x) < (2n)^\alpha,\ 1 \leq n \leq N} I_N(a_1, \ldots, a_N)$$

covers E_α. Therefore, for any $s>0$, the s-Hausdorff measure of E_α can be estimated from above by

$$\begin{aligned}\mathcal{H}^s(E_\alpha) &\le \liminf_{N\to\infty} \sum_{a_1,\dots,a_N:(2n-1)^\alpha\le a_n(x)<(2n)^\alpha,1\le n\le N} |I_N(a_1,\dots,a_n)|^s \\ &\le \liminf_{N\to\infty} \sum_{a_1,\dots,a_N:(2n-1)^\alpha\le a_n<(2n)^\alpha,1\le n\le N} \prod_{n=1}^{N} \frac{1}{a_n^{2s}} \\ &= \liminf_{N\to\infty} \prod_{n=1}^{N} \sum_{(2n-1)^\alpha\le a_n<(2n)^\alpha} \frac{1}{a_n^{2s}}.\end{aligned}$$

For any $s>\frac{\alpha-1}{2\alpha}$, one can choose n_o such that for all $n\ge n_o$,

$$\sum_{(2n-1)^\alpha\le a_n<(2n)^\alpha} \frac{1}{a_n^{2s}} \le \frac{(2n)^\alpha-(2n-1)^\alpha}{(2n-1)^{2\alpha s}} \le \frac{2\alpha\cdot(2n)^{\alpha-1}}{(2n-1)^{2\alpha s}} < 1,$$

and thus it follows that

$$\mathcal{H}^s(E_\alpha) \le \prod_{n=1}^{n_o} \sum_{(2n-1)^\alpha\le a_n<(2n)^\alpha} \frac{1}{a_n^{2s}} < \infty.$$

As a result,

$$\dim_{\mathrm{H}} E_\alpha \le \frac{\alpha-1}{2\alpha}.$$

(II) For the lower bound of $\dim_{\mathrm{H}} E_\alpha$, notice that the set E_α has a nice Cantor structure. For each $n\ge 1$, let

$$\mathcal{E}_n = \Big\{ I_n(a_1,\dots,a_n) : (2k-1)^\alpha \le a_k < (2k)^\alpha, \ \text{ for all } 1\le k\le n \Big\}.$$

Then

$$E = \bigcap_{n=1}^{\infty} \bigcup_{I_n(a_1,\dots,a_n)\in\mathcal{E}_n} I_n(a_1,\dots,a_n),$$

and every element in $I_{n-1}(a_1,\dots,a_{n-1})$ in $\mathcal{E}_{n-1}$ contains exactly

$$c_1 n^{\alpha-1} \le D_n := (2n)^\alpha - (2n-1)^\alpha \le c_2 n^{\alpha-1}.$$

elements $I_n(a_1,\dots,a_n)$ in $\mathcal{E}_n$.

Then, we distribute a mass supported on E_α by setting

$$\mu(I_n(a_1,\dots,a_n)) = \frac{1}{D_1\cdots D_n},$$

for all $n\ge 1$ and $I_n(a_1,\dots,a_n)$ in $\mathcal{E}_n$.

For any $x\in E_\alpha$, for each $r>0$ small enough, let n be the integer such that

$$|I_{n+1}(x)| < r \le |I_n(x)|.$$

Then, by Proposition 3, the ball $B(x,r)$ can intersect at most three cylinders of order n, thus

$$\liminf_{r\to\infty} \frac{\log\mu(B(x,r))}{\log r} \ge \liminf_{n\to\infty} \frac{\log[3\cdot(D_1\cdots D_n)^{-1}]}{\log|I_{n+1}(x)|}.$$

By the recursive relation of $q_n(x)$, it follows that

$$q_{n+1}(x) \le \prod_{k=1}^{n+1} (a_k(x)+1) \le \prod_{k=1}^{n+1} (2k)^\alpha \le c^n \prod_{k=1}^{n+1} k^\alpha;$$

on the other hand,

$$\prod_{k=1}^{n} D_k \ge c_1^n \prod_{k=1}^{n} k^{\alpha-1}.$$

Thus it follows that

$$\liminf_{r\to 0}\frac{\log\mu((x,r))}{\log r}\geq\liminf_{n\to\infty}\frac{\log\prod_{k=1}^{n}k^{\alpha-1}}{\log\left(\prod_{k=1}^{n+1}k^{\alpha}\right)^{2}}=\frac{\alpha-1}{2\alpha}.$$

Finally by Proposition 4, one has

$$\dim_{\mathrm{H}}E_{\alpha}\geq\frac{\alpha-1}{2\alpha}.\qquad\square$$

By a result in [5] or the same line of the argument as above, except minor modifications on notation, one can show the following.

Lemma 6. *Let* $b>1$. *Define*

$$F_{b}=\left\{x\in[0,1):b^{1+2+\cdots+n}\leq a_{n}(x)<2\cdot b^{1+2+\cdots+n},\ \textit{for all}\ n\geq 1\right\}.$$

Then $\dim_{\mathrm{H}}F_{b}=1/2$.

Proof of Theorem 1. It is clear that for any $\alpha>1$,

$$E_{\alpha}\subset\mathscr{S}_{G},\ \text{and so,}\ \dim_{\mathrm{H}}\mathscr{S}_{G}\geq\lim_{\alpha\to\infty}\frac{\alpha-1}{2\alpha}=\frac{1}{2}.$$

It is also clear that for any $b>1$,

$$F_{b}\subset\mathscr{E}_{G},\ \text{and so,}\ \dim_{\mathrm{H}}\mathscr{E}_{G}\geq\frac{1}{2}.\qquad\square$$

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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