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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 775-778

Online since: 17 September 2024

https://doi.org/10.5802/crmath.595

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Research article / *Article de recherche*
Partial differential equations, Probability theory / *Equations aux dérivées partielles, Probabilités*

Granular media equation with double-well external landscape: limiting steady state

Équation des milieux granulaires avec potentiel externe à double puits : état stationnaire limite

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Abstract. In this paper, we give a simple condition on the initial state of the granular media equation which ensures that the limit as the time goes to infinity is the unique steady state with positive center of mass. To do so, we use functional inequalities, Laplace method and McKean–Vlasov diffusion (which corresponds to the probabilistic interpretation of the granular media equation).

Résumé. Dans ce papier, nous donnons une condition simple portant sur l'état initial de l'équation des milieux granulaires qui assure que la limite en temps long est l'unique probabilité invariante avec un centre de masse strictement positif. Pour ce faire, nous utilisons des inégalités fonctionnelles, la méthode de Laplace et la diffusion de McKean–Vlasov (qui correspond à l'interprétation probabiliste de l'équation des milieux granulaires).

2020 Mathematics Subject Classification. 35K55, 60J60, 60G10, 39B72.

Funding. This work is supported by the French ANR grant METANOLIN (ANR-19-CE40-0009).

Manuscript received 21 October 2023, accepted 29 November 2023.

In this work, we are interested in the long-time behavior of the following granular media equation in the one-dimensional setting:

$$\frac{\partial}{\partial t}\mu_t^\sigma(x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\mu_t^\sigma(x) + \frac{\partial}{\partial x}\left\{\mu_t^\sigma(x)\left(\nabla V(x) + \alpha\left(x - m_1^\sigma(t)\right)\right)\right\} \tag{1}$$

with $\alpha > 0$ and $m_1^\sigma(t) := \int_{\mathbb{R}} x\mu_t^\sigma(dx)$. Since $m_1^\sigma(t)$ depends on μ_t^σ, (1) is nonlinear. We assume the following on V:

- (**V-1**) *Polynomial function:* V is a polynomial function with $\deg(V) \geq 4$.
- (**V-2**) *Symmetry:* V is an even function.
- (**V-3**) *Double-well potential:* The equation $V'(x) = 0$ admits exactly three solutions: a, $-a$ and 0 with $a > 0$; $V''(a) > 0$ and $V''(0) < 0$. The bottoms of the wells are reached for $x = a$ and $x = -a$. Moreover, $V^{(2k)}(0) \geq 0$, for all $k \geq 2$.
- (**V-4**) $\lim_{x\to\pm\infty} V''(x) = +\infty$ and for any $x \geq a$, $V''(x) > 0$.

The nonconvexity of V is measured by the following constant: $\theta := \sup_{\mathbb{R}} -V''$. We also assume, eventually, that synchronization occurs that means: $\alpha > \theta$. The initial measure μ_0^σ is assumed to be absolutely continuous with respect to the Lebesgue measure with a density that we denote for simplicity by μ_0^σ. Moreover, $\int_{\mathbb{R}} x^{2k}\mu_0^\sigma(x)dx < \infty$ for any $k \in \mathbb{N}$ and $\int_{\mathbb{R}} \mu_0^\sigma(x)\log(\mu_0^\sigma(x))dx < +\infty$ that is the initial entropy is finite and so the same is true for the initial free-energy.

In this setting, it is well-known that if σ is small enough, then there are several steady states, see [4, 7]. We have proven that if μ^σ is a steady state with total mass equal to 1 of the granular media equation (1) then there exists $m \in \mathbb{R}$ such that $\mu^\sigma = \mu^{m,\sigma}$ where the measure $\mu^{m,\sigma}$ is defined as

$$\mu^{m,\sigma}(dx) := \frac{\exp\left[-\frac{2}{\sigma^2}\left(V(x)+\frac{\alpha}{2}x^2-\alpha m x\right)\right]}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\sigma^2}\left(V(y)+\frac{\alpha}{2}y^2-\alpha m y\right)\right]dy}\,dx.$$

Moreover, m is a zero of the following function:

$$\chi_\sigma(m) := \frac{\int_{\mathbb{R}} x\exp\left[-\frac{2}{\sigma^2}\left(V(x)+\frac{\alpha}{2}x^2-\alpha m x\right)\right]dx}{\int_{\mathbb{R}}\exp\left[-\frac{2}{\sigma^2}\left(V(x)+\frac{\alpha}{2}x^2-\alpha m x\right)\right]dx} - m.$$

In [7], we have shown that if $\sigma < \sigma_c(\alpha)$, then there exists $m(\sigma) > 0$ such that χ_σ is positive on the interval $(0; m(\sigma))$ and negative on $(m(\sigma); +\infty)$. Thus, due to the symmetry of V, there exist exactly three steady states : ν_+^σ with positive expectation, ν_-^σ with negative expectation and ν_0^σ with zero expectation. We point out that we can extend our work to non symmetrical case provided that χ_σ is positive on the interval $(0; m(\sigma))$. Despite this non-uniqueness, the convergence towards one of the invariant probability measures has been proven in [6]. Nevertheless, very few is known about the basins of attraction. Here, we exhibit a simple condition ensuring that μ_t^σ converges towards ν_+^σ.

Theorem 1. *We assume there exists $\delta \in (0; m(\sigma))$ such that $\int_{\mathbb{R}} x\mu_0^\sigma(dx) > m(\sigma) - \delta$ and*

$$\chi_\sigma(m(\sigma)-\delta) > \begin{cases} \mathbb{W}_2\left(\mu_0^\sigma; \mu^{m(\sigma)-\delta,\sigma}\right) & \text{if } \alpha > \theta \\ \mathrm{L}_2\left(\mu_0^\sigma; \mu^{m(\sigma)-\delta,\sigma}\right) & \text{if } \alpha \leq \theta \end{cases}, \tag{2}$$

where $\mathbb{W}_2$ stands for the quadratic Wasserstein distance. Then, μ_t^σ weakly converges towards ν_+^σ as t goes to infinity.

Proof. We introduce the self-stabilizing diffusion:

$$X_t^\sigma = X_0 + \sigma B_t - \int_0^t \nabla V(X_s^\sigma)ds - \alpha\int_0^t \left(X_s^\sigma - \mathbb{E}\left[X_s^\sigma\right]\right)ds. \tag{3}$$

Here, X_0 is a random variable, independent from the Brownian motion B and we assume that X_0 follows the law μ_0^σ. This kind of processes has been introduced in the seminal work [5]. We do not discuss the wellposedness of Equation (3). About this, we refer to [3]. We point out that $\mathscr{L}(X_t^\sigma) = \mu_t^\sigma$ for any $t \geq 0$. We can rewrite the equation in this way: $X_t^\sigma = X_0 + \sigma B_t - \int_0^t \nabla V(X_s^\sigma)ds - \alpha\int_0^t (X_s^\sigma - m_1^\sigma(s))ds$. We introduce the following diffusion

$$Y_t^\sigma = X_0 + \sigma B_t - \int_0^t \nabla V\left(Y_s^\sigma\right)ds - \alpha\int_0^t \left(Y_s^\sigma - (m(\sigma)-\delta)\right)ds. \tag{4}$$

We consider the *deterministic* time $T_0^\sigma := \inf\{t \geq 0 \,:\, \mathbb{E}(X_t^\sigma) \leq m(\sigma) - \delta\}$ where X^σ is defined in Equation (3). Then, for any $t \in [0; T_0^\sigma)$, we have $X_t^\sigma \geq Y_t^\sigma$ so $\mathbb{E}(X_t^\sigma) \geq \mathbb{E}(Y_t^\sigma)$. We remark $\mathbb{E}(Y_t^\sigma) = \mathbb{E}(Y_\infty^\sigma) + \mathbb{E}(Y_t^\sigma - Y_\infty^\sigma)$ where Y_∞^σ follows the law $\mu^{m(\sigma)-\delta,\sigma}$, which is the unique invariant probability of Diffusion (4).

On the one hand, we assume that $\alpha > \theta$. Then, $x \mapsto V(x) + \frac{\alpha}{2}(x - (m(\sigma) - \delta))^2$ is uniformly convex. Also, the unique invariant probability measure of Equation (4) is the measure $\mu^{m(\sigma)-\delta,\sigma}$. Hence, by applying [2, Theorem 2.1.], we deduce:

$$\mathbb{W}_2\left(\mathscr{L}\left(Y_t^\sigma\right);\mu^{m(\sigma)-\delta,\sigma}\right) \le e^{-(\alpha-\theta)t}\mathbb{W}_2\left(\mathscr{L}\left(X_0\right);\mu^{m(\sigma)-\delta,\sigma}\right).$$

With $\widetilde{Y_\infty^\sigma}$ and $\widetilde{Y_t^\sigma}$ realizing the optimal coupling in $\mathbb{W}_2$, we have:

$$\begin{aligned}\mathbb{E}\left(Y_t^\sigma - Y_\infty^\sigma\right) =& \mathbb{E}\left(\widetilde{Y_t^\sigma} - \widetilde{Y_\infty^\sigma}\right) \ge -\mathbb{E}\left(\left|\widetilde{Y_t^\sigma} - \widetilde{Y_\infty^\sigma}\right|\right) \ge -\mathbb{W}_2\left(\mathscr{L}\left(Y_t^\sigma\right);\mu^{m(\sigma)-\delta,\sigma}\right)\\ &\ge -e^{-(\alpha-\theta)t}\mathbb{W}_2\left(\mathscr{L}\left(X_0\right);\mu^{m(\sigma)-\delta,\sigma}\right) \ge -\mathbb{W}_2\left(\mathscr{L}\left(X_0\right);\mu^{m(\sigma)-\delta,\sigma}\right).\end{aligned}$$

Hence, for any $t \le T_0^\sigma$, we have $\mathbb{E}(Y_t^\sigma) \ge \int_{\mathbb{R}} x\mu^{m(\sigma)-\delta,\sigma}(dx) - \mathbb{W}_2\left(\mathscr{L}\left(X_0\right);\mu^{m(\sigma)-\delta,\sigma}\right)$. We deduce $\mathbb{E}(Y_t^\sigma) \ge (m(\sigma)-\delta) + \chi_\sigma\left(m(\sigma)-\delta\right) - \mathbb{W}_2\left(\mathscr{L}\left(X_0\right);\mu^{m(\sigma)-\delta,\sigma}\right)$. Consequently, for any $t \le T_0^\sigma$, we have

$$\mathbb{E}(X_t^\sigma) \ge (m(\sigma)-\delta) + \chi_\sigma\left(m(\sigma)-\delta\right) - \mathbb{W}_2\left(\mathscr{L}\left(X_0\right);\mu^{m(\sigma)-\delta,\sigma}\right) > m(\sigma)-\delta.$$

Hence, $T_0^\sigma = +\infty$ so that for any $t \ge 0$: $\mathbb{E}(X_t^\sigma) \ge m(\sigma) - \delta$.

By applying main theorem in [6], we deduce that $\mathscr{L}(X_t^\sigma)$ weakly converges towards one of the steady state: ν_-^σ, ν_0^σ or ν_+^σ.

Since $\int x\nu_-^\sigma(dx) < 0 = \int x\nu_0^\sigma(dx) < \int x\mathscr{L}(X_t^\sigma)(dx)$ for any $t \ge 0$, we deduce that $\mathscr{L}(X_t^\sigma)$ does not converge towards either ν_-^σ nor ν_0^σ. Consequently, $\mathscr{L}(X_t^\sigma)$ weakly converges towards ν_+^σ as t goes to infinity.

On the other hand, if $\alpha \le \theta$, we can use Poincaré inequality from [1, Corollary 1.6.] since $V''(x) + \alpha > 0$ for sufficiently large x. We deduce the existence of $C(\sigma) > 0$ such that

$$\mathrm{L}_2\left(\mathscr{L}\left(Y_t^\sigma\right);\mu^{m(\sigma)-\delta,\sigma}\right) \le e^{-C(\sigma)t}\mathrm{L}_2\left(\mathscr{L}\left(X_0\right);\mu^{m(\sigma)-\delta,\sigma}\right)$$

so that

$$\mathbb{W}_2\left(\mathscr{L}\left(Y_t^\sigma\right);\mu^{m(\sigma)-\delta,\sigma}\right) \le e^{-C(\sigma)t}\mathrm{L}_2\left(\mathscr{L}\left(X_0\right);\mu^{m(\sigma)-\delta,\sigma}\right).$$

We then complete the proof as previously with

$$\mathrm{L}_2\left(\mathscr{L}(X_0);\mu^{m(\sigma)-\delta,\sigma}\right) \quad \text{instead of } \mathbb{W}_2\left(\mathscr{L}(X_0);\mu^{m(\sigma)-\delta,\sigma}\right).$$ □

As mentioned above, $\chi_\sigma(m) \le 0$ for any $m \ge m(\sigma)$ so we immediately deduce that the theorem can not be applied if $\int_{\mathbb{R}} x\mu_0^\sigma(dx) \ge m(\sigma)$. We have an immediate corollary:

Corollary 2. *We assume that there exists $m \in (0;m(\sigma))$ such that $\mu_0^\sigma = \mu^{m,\sigma}$. Then, μ_t^σ weakly converges towards ν_+^σ as t goes to infinity.*

Proof. Hypothesis (2) is satisfied since for any $m \in (0;m(\sigma))$, there exists $\delta \in (0;m(\sigma))$ satisfying $m = m(\sigma) - \delta$. Then, we just remark:

$$\chi_\sigma(m(\sigma)-\delta) > 0 = \mathbb{W}_2\left(\mu^{m(\sigma)-\delta,\sigma};\mu^{m(\sigma)-\delta,\sigma}\right) = \mathrm{L}_2\left(\mu^{m(\sigma)-\delta,\sigma};\mu^{m(\sigma)-\delta,\sigma}\right).$$

First condition is also satisfied due to $\int_{\mathbb{R}} x\mu^{m(\sigma)-\delta,\sigma}(dx) = m(\sigma) - \delta + \chi_\sigma(m(\sigma)-\delta) > m(\sigma) - \delta$. Thus, we may apply Theorem 1. □

We stress that we only used the decrease of the Wasserstein distance in the proof of the theorem. Hence, it is not a priori required that the decay is exponential.

We point out that we have a similar result with the other half-line: if $\mu_0^\sigma = \mu^{-m(\sigma)+\delta,\sigma}$. Then, μ_t^σ converges weakly towards ν_-^σ as t goes to infinity.

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