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Oscillatory integrals for Mittag-Leffler functions with two variables

Intégrales oscillatoires pour les fonctions de Mittag-Leffler à deux variables

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Abstract. In this paper we consider the problem of estimation of oscillatory integrals with Mittag-Leffler functions in two variables. The generalisation is that we replace the exponential function with the Mittag-Leffler-type function, to study oscillatory type integrals.

Résumé. Dans cet article, nous considérons le problème de l'estimation des intégrales oscillatoires avec les fonctions de Mittag-Leffler à deux variables. La généralisation est que l'on remplace la fonction exponentielle par la fonction de type Mittag-Leffler, pour étudier les intégrales de type oscillatoire.

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1. Introduction

The function $E_\alpha(z)$ is named after the Swedish mathematican Gösta Magnus Mittag-Leffler (1846-1927) who defined it by a power series

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\ \alpha\in\mathbb{C}, Re(\alpha)>0, \tag{1}$$

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and studied its properties in 1902-1905 in several subsequent notes [19–22] in connection with his summation method for divergent series.

A classical generalization of the Mittag-Leffler function, namely the two-parametric Mittag-Leffler function is

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\alpha k+\beta\right)},\ \alpha,\beta\in\mathbb{C},Re(\alpha)>0, \tag{2}$$

which was deeply investigated independently by Humbert and Agarval in 1953 [1, 11, 12] and by Dzherbashyan in 1954 [5–7] as well as in [8].

It has the property that

$$E_{1,1}(x)=e^x,\text{and we can refer to [24] for other properties.} \tag{3}$$

In harmonic analysis one of the most important estimates for oscillatory integral is van der Corput lemma [4, 25, 26, 28]. Estimates for oscillatory integrals with polynomial phases can be found, for instance, in papers [2, 13, 17, 30–33]. In the current paper we replace the exponential function with the Mittag-Leffler-type function and study oscillatory type integrals (6). In the papers [28] and [29] analogues of the van der Corput lemmas involving Mittag-Leffler functions for one dimensional integrals have been considered. We extend results of [28] and [29] for two-dimensional integrals with phase having some simple singularities. Analogous problem on estimates for Mittag-Leffler functions with the smooth phase functions of two variables having simple singularities was considered in [27] and [34].

2. Preliminaries

Definition 1. *An oscillatory integral with phase f and amplitude a is an integral of the form*

$$J(\lambda,f,a)=\int_{\mathbb{R}^n}a(x)e^{i\lambda f(x)}dx, \tag{4}$$

where $a\in C_0^{\infty}(\mathbb{R}^n)$ and $\lambda\in\mathbb{R}$.

If the support of a lies in a sufficiently small neighborhood of the origin and f is an analytic function at $x=0$, then for $\lambda\to\infty$ the following asymptotic expansion holds ([18]):

$$J(\lambda,f,a)\approx e^{i\lambda f(0)}\sum_{s}\sum_{k=0}^{n-1}b_{s,k}(a)\lambda^s(\ln\lambda)^k, \tag{5}$$

where s belongs to a finite number of arithmetic progressions, independent of a, composed of negative rational numbers, $b_{s,k}$ is a distribution with support in the critical set $\{x:\nabla f(x)=0\}$.

Inspired by the terminology from [3], we refer to the maximal value of s, denoting it by α in this case, as the *growth index* of f, or the oscillation index at the origin, and the corresponding value of k is referred to as the *multiplicity*.

More precisely, the *multiplicity* of the oscillation index of an analytic phase at a critical point is the maximal number k possessing the property: for any neighbourhood of the critical point there is an amplitude with support in this neighbourhood for which in the asymptotic series (5) the coefficient $b_{s,k}(a)$ is not equal to zero. The multiplicity of the oscillation index will be denoted by m (see [3]).

Let f be a smooth real-valued function defined on a neighborhood of the origin in $\mathbb{R}^2$ with $f(0,0)=0$, $\nabla f(0,0)=0$, and consider the associated Taylor series

$$f(x_1,x_2)\sim\sum_{j,k=0}^{\infty}c_{jk}x_1^jx_2^k$$

of f centered at the origin. The set

$$\Im(f) := \left\{(j,k) \in \mathbb{Z}_+^2 : c_{jk} = \frac{1}{j!k!}\partial_{x_1}^j \partial_{x_2}^k f(0,0) \neq 0\right\}$$

is called the Taylor support of f at $(0,0)$. We shall always assume that

$$\Im(f) \neq \emptyset,$$

i.e., that the function f is of finite type at the origin. If f is real analytic, so that the Taylor series converges to f near the origin, this just means that $f \neq 0$. The *Newton polyhedron* $\aleph(f)$ of f at the origin is defined to be the convex hull of the union of all the quadrants $(j,k)+\mathbb{R}_+^2$, with $(j,k) \in \Im(f)$. The associated *Newton diagram* $\aleph_d(f)$ in the sense of Varchenko [35] is the union of all compact faces of the Newton polyhedron; here, by a *face*, we mean an edge or a vertex.

We shall use coordinates (t_1, t_2) for points in the plane containing the Newton polyhedron, in order to distinguish this plane from the (x_1, x_2) - plane.

The *distance* $d = d(f)$ between the Newton polyhedron and the origin in the sense of Varchenko is given by the coordinate d of the point (d,d) at which the bisectrix $t_1 = t_2$ intersects the boundary of the Newton polyhedron, where $d \geq 1$.

The *principal face* $\pi(f)$ of the Newton polyhedron of f is the face of minimal dimension containing the point (d,d). Deviating from the notation in [35], we shall call the series

$$f_p(x_1,x_2) := \sum_{j,k\in\pi(f)} c_{jk}x_1^j x_2^k$$

the *principal part* of f. In the case that $\pi(f)$ is compact, f_π is a mixed homogeneous polynomial; otherwise, we shall consider f_π as a formal power series.

Note that the distance between the Newton polyhedron and the origin depends on the chosen local coordinate system in which f is expressed. By a *local analytic (respectively smooth) coordinate system at the origin* we shall mean an analytic (respectively smooth) coordinate system defined near the origin which preserves 0. If we work in the category of smooth functions f, we shall always consider smooth coordinate systems, and if f is analytic, then one usually restricts oneself to analytic coordinate systems (even though this will not really be necessary for the questions we are going to study, as we will see). The *height* of the analytic (respectively smooth) function f is defined by

$$h := h(f) := sup\{d_x\},$$

where the supremum is taken over all local analytic (respectively smooth) coordinate systems x at the origin, and where d_x is the distance between the Newton polyhedron and the origin in the coordinates x, also $h \geq 1$.

A given coordinate system x is said to be adapted to f if $h(f) = d_x$.

Let π be the principal face. We assume that π is a point or a compact edge, then f_π is a weighted homogeneous polynomial. Denote by ν the maximal order of roots of f_π on the unit circle at the origin, so

$$\nu := \max_{S^1} ord(f_\pi).$$

If there exists a coordinate system x such that $\nu = d_x$ then we set $m = 1$. It can be shown that in this case x is adapted to f (see [16]). Otherwise we take $m = 0$. Following A. N. Varchenko we call m the *multiplicity of the Newton polyhedron.* One may connect the multiplicity of the Newton polyhedron and the asymptotic expansions for the oscillatory integrals as

$$J(\lambda, f, a) = O\left(\lambda^{-\frac{1}{h}}(\ln\lambda)^m\right), \text{ as } \lambda \to +\infty.$$

In the classical paper by A. N. Varchenko [35], he obtained the sharp estimates for oscillatory integrals in terms of the height. Also in the paper [14] the height was used to get the sharp bound for maximal operators associated to smooth surfaces in $\mathbb{R}^3$. It turns out that analogous quantities can be used for oscillatory integrals with the Mittag-Leffler function.

We consider the following integral with phase f and amplitude ψ, of the form

$$I_{\alpha,\beta} = \int_U E_{\alpha,\beta}\left(i\lambda f(x)\right)\psi(x)dx, \tag{6}$$

where $0<\alpha<1$, $\beta>0$, U is a sufficiently small neighborhood of the origin. We are interested in particular in the behavior of $I_{\alpha,\beta}$ when λ is large, as for small λ the integral is just bounded. In particular if $\alpha=1$ and $\beta=1$ we have oscillatory integral (4).

The main result of the work is the following.

Theorem 2. *Let f be a smooth finite type function of two variables defined in a sufficiently small neighborhood of the origin and let $\psi\in C_0^\infty(U)$.*

Let h be the height of the function f, and let $m=0,1$ be the multiplicity of its Newton polyhedron. If $\frac{1}{2}<\alpha<1$, $\beta>0$, $h>1$, and $\lambda\gg 1$ then we have the estimate

$$\left|\int_U E_{\alpha,\beta}\big(i\lambda f(x_1,x_2)\big)\psi(x)dx\right| \le \frac{C|\ln\lambda|^m\|\psi\|_{L^\infty(\overline{U})}}{\lambda^{\frac{1}{h}}}. \tag{7}$$

If $\frac{1}{2}<\alpha<1$, $\beta>0$, $h=1$ and $\lambda\gg 1$, then we have following estimate

$$\left|\int_U E_{\alpha,\beta}\big(i\lambda f(x_1,x_2)\big)\psi(x)dx\right| \le \frac{C|\ln\lambda|^2\|\psi\|_{L^\infty(\overline{U})}}{\lambda}, \tag{8}$$

where the constants C are independent of the phase, amplitude and λ.

Remark 3. Note that the inequalities (7) and (8) do not hold for the case $\alpha=\beta=1$ of the classical oscillatory integrals, because we have estimates with the L^∞ norm of the amplitude function.

3. Auxiliary statements

We first recall some useful properties.

Proposition 4. *If $0<\alpha<2,\beta$ is an arbitrary real number, μ is such that $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$, then there is $C>0$, such that we have*

$$\left|E_{\alpha,\beta}(z)\right| \le \frac{C}{1+|z|},\ z\in\mathbb{C},\ \mu\le\left|\arg(z)\right|\le\pi. \tag{9}$$

See [7, 8, 24].

Proposition 5. *Let Ω be an open, bounded subset of $\mathbb{R}^2$, and let $f:\Omega\to\mathbb{R}$ be a measurable function such that for all $\lambda\gg 1$ and for some positive $\delta\neq 1$, we have*

$$\left|\int_\Omega e^{i\lambda f(x)}dx\right| \le C|\lambda|^{-\delta}|\ln\lambda|^m, \tag{10}$$

with $m\ge 0$. Then, we have

$$\left|x\in\Omega:\left|f(x)\right|\le\varepsilon\right| \le C_\delta\varepsilon^\delta|\ln\varepsilon|^m,\ for\ \delta<1,$$
$$for\ 0<\varepsilon\ll 1,\ and\ for\ \delta>1, |x\in\Omega:\left|f(x)\right|\le\varepsilon| \le C_\delta\varepsilon,$$
$$for\ \delta=1,\ \left|x\in\Omega:|f(x)|\le\varepsilon\right| \le C_\delta\varepsilon|\ln\varepsilon|^{m+1},$$

where C_δ depends only on δ, $|A|$ means the Lebesgue measure of a set A. See [9].

Proof. For the convenience of the reader we give an independent proof of Proposition 5. We consider an even non-negative smooth function

$$\omega(x) = \begin{cases} 1, & \text{when } |x| \le 1, \\ 0, & \text{when } |x| \ge 2. \end{cases}$$

For the characteristic function of Ω with $\overline{\Omega} \subset U$, the following inequality holds true

$$|x \in \Omega : \left|f(x)\right| \le \varepsilon| = \int_\Omega \chi_{[0,1]}\left(\frac{|f(x)|}{\varepsilon}\right) dx \le \int_\Omega \omega\left(\frac{f(x)}{\varepsilon}\right) dx.$$

Now we will use the Fourier inversion formula, and rewrite the last integral as

$$\int_\Omega \omega\left(\frac{f(x)}{\varepsilon}\right) dx = \frac{1}{2\pi}\int_\Omega \int_{-\infty}^{\infty} \check{\omega}(\xi) e^{i\xi\frac{f(x)}{\varepsilon}} d\xi dx.$$

As $\check{\omega}(\xi)$ is a Schwartz function, we can use Fubini theorem and change the order of integration. So we have

$$\int_\Omega \int_{-\infty}^{\infty} \check{\omega}(\xi) e^{i\xi\frac{f(x)}{\varepsilon}} d\xi dx = \int_{-\infty}^{\infty} \check{\omega}(\xi) \int_\Omega e^{i\xi\frac{f(x)}{\varepsilon}} dx d\xi.$$

We use inequality (10) for the inner integral and get

$$\left|\int_\Omega e^{i\xi\frac{f(x)}{\varepsilon}} dx\right| \le \frac{C\left|\ln\left(2+\frac{\xi}{\varepsilon}\right)\right|^m}{\left(1+|\frac{\xi}{\varepsilon}|\right)^\delta}.$$

As $\check{\omega}(\xi)$ is a Schwartz function, we also have

$$|\check{\omega}(\xi)| \le \frac{C}{1+|\xi|}.$$

So

$$\left|\int_{-\infty}^{\infty} \frac{C\check{\omega}(\xi)\left|\ln\left(2+\frac{\xi}{\varepsilon}\right)\right|^m}{\left(2+\left|\frac{\xi}{\varepsilon}\right|\right)^\delta} d\xi\right| \lesssim \int_0^\infty \frac{2C\left|\ln\left(\frac{\xi}{\varepsilon}\right)\right|^m}{(1+|\xi|)\left(2+\left|\frac{\xi}{\varepsilon}\right|\right)^\delta} d\xi.$$

Now we change the variable as $\xi = \eta\varepsilon$, and we get

$$\int_0^\infty \frac{\left|\ln\left(\frac{\xi}{\varepsilon}\right)\right|^m}{(1+|\xi|)\left(2+\left|\frac{\xi}{\varepsilon}\right|\right)^\delta} d\xi = \int_0^\infty \frac{\varepsilon|\ln\eta|^m}{(1+|\varepsilon\eta|)(2+|\eta|)^\delta} d\eta.$$

Now we estimate the last integral for different values of δ.

If $\delta < 1$ then we have

$$\int_0^\infty \frac{\varepsilon|\ln\eta|^m}{\left(1+|\varepsilon\eta|\right)\left(2+|\eta|\right)^\delta} d\eta \le C\varepsilon\int_0^{\frac{1}{\varepsilon}} \frac{|\ln\eta|^m d\eta}{(2+\eta)^\delta} + C\varepsilon\int_{\frac{1}{\varepsilon}}^\infty \frac{|\ln\eta|^m d\eta}{\varepsilon\eta^{\delta+1}}.$$

We represent $\frac{1}{(2+\eta)^\delta} = \frac{1}{\eta^\delta(1+\frac{2}{\eta})^\delta} = \frac{1}{\eta^\delta} + O(\frac{1}{\eta^{\delta+1}})$. So

$$C\varepsilon\int_0^{\frac{1}{\varepsilon}} \frac{|\ln\eta|^m d\eta}{(2+\eta)^\delta} = \varepsilon\int_0^2 \frac{|\ln\eta|^m d\eta}{(2+\eta)^\delta} + \varepsilon\int_2^{\frac{1}{\varepsilon}} \frac{|\ln\eta|^m d\eta}{(2+\eta)^\delta}.$$

Integrating by parts we obtain

$$\varepsilon\int_2^{\frac{1}{\varepsilon}} \frac{|\ln\eta|^m d\eta}{(2+\eta)^\delta} \le \varepsilon\int_2^{\frac{1}{\varepsilon}} \frac{|\ln\eta|^m d\eta}{\eta^\delta} \le C\varepsilon^\delta|\ln\varepsilon|^m.$$

As $\delta < 1$, the integrals $\int_0^2 \frac{|\ln\eta|^m d\eta}{(2+\eta)^\delta}$ and $\int_{\frac{1}{\varepsilon}}^\infty \frac{|\ln\eta|^m d\eta}{\varepsilon\eta^{\delta+1}}$ convergence.

If $\delta > 1$ then we trivially obtain

$$\left|\int_0^\infty \frac{C\varepsilon|\ln\eta|^m}{\left(1+|\varepsilon\eta|\right)\left(2+|\eta|\right)^\delta}d\eta\right| \le C\varepsilon.$$

If $\delta = 1$ then assuming $0 < \varepsilon < \frac{1}{2}$ we get $|\varepsilon\eta| < 1$ (for $|\eta| < 2$), then write the integral as the sum of three integrals and obtain

$$\left|\int_0^\infty \frac{C\varepsilon|\ln\eta|^m}{\left(1+|\varepsilon\eta|\right)\left(1+|\eta|\right)}d\eta\right| \le \left|\int_0^2 C\varepsilon\left|\ln\eta\right|^m d\eta\right| + \left|\int_2^{\frac{1}{\varepsilon}} \frac{C\varepsilon\left|\ln\eta\right|^m}{\eta}d\eta\right| + \left|\int_{\frac{1}{\varepsilon}}^\infty \frac{C\varepsilon\left|\ln\eta\right|^m}{\eta}d\eta\right|.$$

Then we have

$$\left|\int_0^2 C\varepsilon\left|\ln\eta\right|^m d\eta\right| \le C\varepsilon,$$

and we get with simple calculating that

$$\left|\int_2^{\frac{1}{\varepsilon}} \frac{C\varepsilon\left|\ln\eta\right|^m}{\eta}d\eta\right| \le C\varepsilon|\ln\varepsilon|^{m+1}.$$

We use the formula of integrating by parts several times, to get

$$\left|\int_{\frac{1}{\varepsilon}}^\infty \frac{C\varepsilon|\ln\eta|^m}{\eta}d\eta\right| \le C\varepsilon|\ln\varepsilon|^m,$$

completing the proof of Proposition 5. □

From Proposition 5 we get the following corollaries.

Corollary 6. *Let $f(x_1,x_2)$ be a smooth function with $f(0,0)=0$, $\nabla f(0,0)=0$, and h be the height of the function $f(x_1,x_2)$, and let $m = 0,1$ be the multiplicity of its Newton polyhedron. Let also*

$$a(x) = \begin{cases} 1, & when\,|x| \le \sigma, \\ 0, & when\,|x| \ge 2\sigma, \end{cases} \quad \sigma > 0,$$

and $a(x) \ge 0$ with $a \in C_0^\infty(\mathbb{R}^2)$. If for all real $\lambda \gg 1$ and for any positive $\delta \ne 1$, the following inequality holds

$$\left|\int_{\mathbb{R}^2} e^{i\lambda f(x)} a(x)dx\right| \le C|\lambda|^{-\delta}|\ln\lambda|^m, \tag{11}$$

then we have

$$\left||x| \le \sigma : \left|f(x)\right| \le \varepsilon\right| \le C\varepsilon^\delta|\ln\varepsilon|^m,$$

where $m \ge 0$. See [10, 15, 16, 23].

Corollary 7. *Let $f(x_1,x_2)$ be a smooth function with $f(0,0)=0$, $\nabla f(0,0)=0$, and let $\overline{\Omega}$ be a sufficiently small compact set around the origin. Let also h be the height of the function $f(x_1,x_2)$, and let $m = 0,1$ be the multiplicity of its Newton polyhedron. Then for all $0 < \varepsilon \ll 1$ we have*

$$|x \in \Omega : |f(x)| \le \varepsilon| \le C\varepsilon^{\frac{1}{h}}|\ln\varepsilon|^m,$$

where h is the height of f and m is its multiplicity [10].

4. Proof of the main result

Proof of Theorem 2. As for $\lambda < 2$ the integral (6) is just bounded, we consider the case $\lambda \geq 2$. Without loss of generality, we can consider the integral over U. Using inequality (9), we have

$$\left|E_{\alpha,\beta}\left(i\lambda f(x)\right)\right| \leq \frac{C}{1+\lambda\left|f(x)\right|}. \tag{12}$$

We then use (12) for the integral (6), and get that

$$\left|I_{\alpha,\beta}\right| \leq \left|\int_U E_{\alpha,\beta}\left(i\lambda f(x)\right)\psi(x)dx\right| \leq C\int_U \frac{|\psi(x)|dx}{1+\lambda\left|f(x)\right|}. \tag{13}$$

Now we represent the integral $I_{\alpha,\beta}$ over the union of sets $\Omega_1 := \Omega \cap \{\lambda|f(x_1,x_2)| < M\}$ and $\Omega_2 := \Omega \cap \{\lambda|f(x_1,x_2)| \geq M\}$ respectively, where M is a positive real number.

We estimate the integral $I_{\alpha,\beta}$ over the sets Ω_1 and Ω_2, respectively,

$$\left|I_{\alpha,\beta}\right| \leq C\int_U \frac{|\psi(x)|dx}{1+\lambda\left|f(x)\right|} = J_1 + J_2 := C\int_{\Omega_1} \frac{|\psi(x)|dx}{1+\lambda\left|f(x)\right|} + C\int_{\Omega_2} \frac{|\psi(x)|dx}{1+\lambda\left|f(x)\right|}.$$

First we estimate the integral over the set Ω_1. Using Corollary 7 we obtain

$$|J_1| = C\int_{\Omega_1} \frac{|\psi(x)|dx}{1+\lambda\left|f(x)\right|} \leq \frac{C|\ln\lambda|^m \|\psi\|_{L^\infty(\overline{\Omega_1})}}{\lambda^{\frac{1}{h}}}.$$

Lemma 8. *Let $f \in C^\infty$ and h be the height of the function f, and let $m = 0,1$ be the multiplicity of its Newton polyhedron. For any smooth function $a = a(x,y)$ with sufficiently small support and for $h > 1$ the following inequality holds*

$$I := \int_{\{|f(x,y)| \geq \frac{M}{\lambda}\}} \frac{a(x,y)}{1+\lambda\left|f(x,y)\right|}dxdy \leq \frac{C|\ln\lambda|^m \|a\|_{L^\infty(\overline{U})}}{\lambda^{\frac{1}{h}}}, \tag{14}$$

where supp$\{a(x,y)\} = U$.

Proof. Let $h > 1$. Consider the sets

$$A_k = \left\{x \in U : \frac{2^k}{\lambda} \leq \left|f(x)\right| \leq \frac{2^{k+1}}{\lambda}\right\}.$$

For the measure of a set of smaller values we use Corollary 7, and we have

$$\mu\left(\left|f(x)\right| \leq \frac{2^{k+1}}{\lambda}, x \in U\right) \leq C\left(\frac{2^{k+1}}{\lambda}\right)^{\frac{1}{h}}\left(\ln\left|\frac{\lambda}{2^{k+1}}\right|\right)^m.$$

Let

$$I_k := \int_{A_k} \frac{a(x,y)}{1+\lambda\left|f(x,y)\right|}dxdy.$$

For the integral

$$\sum_{2^k \leq \lambda|f(x)| \leq 2^{k+1}} I_k = \int_{\Omega_2} \frac{a(x,y)}{1+\lambda\left|f(x,y)\right|}dxdy,$$

we find the following estimate:

$$|I_k| = \left|\int_{A_k} \frac{a(x,y)}{1+\lambda\left|f(x,y)\right|}dxdy\right| \leq C\|a\|_{L^\infty\left(\overline{U}\right)}\left(\frac{2^{k+1}}{\lambda}\right)^{\frac{1}{h}}\left|\ln\frac{2^{k+1}}{\lambda}\right|^m 2^{-k}.$$

From here we find the sum of I_k and, by estimating the integral I, we get

$$I \leq \|a\|_{L^\infty\left(\overline{U}\right)}\sum_{k=1}^\infty I_k \leq \|a\|_{L^\infty\left(\overline{U}\right)}\sum_{k=1}^\infty\left(\frac{2^{k+1}}{\lambda}\right)^{\frac{1}{h}}\left|\ln\frac{2^{k+1}}{\lambda}\right|^m 2^{-k} \leq \|a\|_{L^\infty\left(\overline{U}\right)}\frac{|\ln\lambda|^m}{\lambda^{\frac{1}{h}}}\sum_{k=1}^\infty 2^{\frac{k+1}{h}-k}k^m.$$

As $h > 1$, the last series is convergent, proving the lemma. □

Remark 9. Consider the case $h = 1$. The smooth function has non-degenerate critical point at the origin if and only if $h = 1$. As $f(x,y)$ is a smooth function with $\nabla f(0,0) = 0$, using Morse lemma we have $f \sim x^2 \pm y^2$. So in this case we estimate two sets $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1 := \{(x,y) : \lambda|x^2 \pm y^2| \le M, |x| \le 1, |y| \le 1\}$ and $\Delta_2 := \{(x,y) : \lambda|x^2 \pm y^2| > M, |x| \le 1, |y| \le 1\}$. First we consider the integral over the set Δ_1. Then we have

$$\left|\int_{\Delta_1} \frac{a(x,y)}{1+\lambda|x^2 \pm y^2|} dxdy\right| \le C\|a\|_{L^\infty(\Delta_1)} \left|\int_{\Delta_1} dxdy\right|.$$

Now we estimate the last integral as

$$\left|\int_{\lambda|x^2+y^2|\le M} dxdy\right| \le \frac{C}{\lambda}.$$

Then we estimate the measure of the set $\{|x^2 - y^2| \le \varepsilon M\}$, where $\varepsilon = \frac{1}{\lambda}$. We have, for simplicity putting $M = 1$,

$$\begin{aligned}
&\left|\int_{|x^2-y^2|\le\varepsilon M} dxdy\right| \\
&\le C\left|\int_{\sqrt{\varepsilon}}^{\sqrt{1-\varepsilon}} dy \int_{\sqrt{y^2-\varepsilon}}^{\sqrt{y^2+\varepsilon}} dx\right| = \left|\int_{\sqrt{\varepsilon}}^{\sqrt{1-\varepsilon}} \left(\sqrt{y^2+\varepsilon} - \sqrt{y^2-\varepsilon}\right) dy\right| \\
&= \left(\frac{y}{2}\sqrt{y^2+\varepsilon} + \frac{\varepsilon}{2}\ln\left|y + \sqrt{y^2+\varepsilon}\right|\right)\Big|_{\sqrt{\varepsilon}}^{\sqrt{1-\varepsilon}} - \left(\frac{y}{2}\sqrt{y^2-\varepsilon} - \frac{\varepsilon}{2}\ln\left|y + \sqrt{y^2-\varepsilon}\right|\right)\Big|_{\sqrt{\varepsilon}}^{\sqrt{1-\varepsilon}} \\
&= \Bigg|\frac{\sqrt{1-\varepsilon}}{2} + \frac{\varepsilon}{2}\ln\frac{\sqrt{1-\varepsilon}+1}{\sqrt{\varepsilon}} - \frac{\sqrt{2}}{2}\varepsilon - \frac{\varepsilon}{2}\ln\left|\sqrt{\varepsilon}(1+\sqrt{2})\right| - \\
&\quad - \left(\frac{\sqrt{(1-\varepsilon)(1-2\varepsilon)}}{2} - \frac{\varepsilon}{2}\ln\left|\sqrt{1-\varepsilon} + \sqrt{1-2\varepsilon}\right| + \frac{\varepsilon}{2}\ln\sqrt{\varepsilon}\right)\Bigg| \le C\varepsilon\ln\varepsilon.
\end{aligned}$$

Now we consider the integral over the set Δ_2. In this case we change the variables to polar coordinate system and with easy calculating we get

$$\left|\int_{\{\lambda|x^2+y^2|\ge M\}} \frac{a(x,y)}{1+\lambda\left|x^2+y^2\right|} dxdy\right| \le \frac{C|\ln\lambda|\|a\|_{L^\infty(\Delta_2)}}{\lambda} \tag{15}$$

and

$$\left|\int_{\{\lambda|x^2-y^2|\ge M\}} \frac{a(x,y)}{1+\lambda\left|x^2-y^2\right|} dxdy\right| \le \frac{C|\ln\lambda|^2\|a\|_{L^\infty(\Delta_2)}}{\lambda}. \tag{16}$$

Now we continue the proof of Theorem 2. Let $h > 1$. We use Proposition 5 for the integral J_1, to get

$$|J_1| \le \frac{C|\ln\lambda|^m\|a\|_{L^\infty(\overline{U})}}{\lambda^{\frac{1}{h}}}.$$

Let consider the integral J_2. If $h > 1$, then using Lemma 8 we get

$$|J_2| \le \frac{C|\ln\lambda|\|a\|_{L^\infty(\overline{U})}}{\lambda^{\frac{1}{h}}}.$$

If $h = 1$, using the Remark 9 we get the inequality (8). The proof of Theorem 2. is complete. $\square$

The proof of Theorem 2 shows that if $h = 1$, we can get a more precise result.

Proposition 10. *If $h = 1$ and f has an extremal point at the point (0,0) (then f is diffeomorhic equivalent to $x_1^2 + x_2^2$ or $-x_1^2 - x_2^2$), then we have*

$$|I_{\alpha,\beta}| \le \frac{C|\ln\lambda|\|\psi\|_{L^\infty(\overline{U})}}{\lambda},$$

for all $\lambda \ge 2$.

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