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Equality in the Miyaoka–Yau inequality and uniformization of non-positively curved klt pairs

Cas d'égalité de l'inégalité de Miyaoka–Yau et uniformisation des paires klt à courbure négative

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In memory of Jean-Pierre Demailly

Abstract. Let (X,Δ) be a compact Kähler klt pair, where $K_X+\Delta$ is ample or numerically trivial, and Δ has standard coefficients. We show that if equality holds in the orbifold Miyaoka–Yau inequality for (X,Δ), then its orbifold universal cover is either the unit ball (ample case) or the affine space (numerically trivial case).

Résumé. Soit (X,Δ) une paire klt compacte kählérienne pour laquelle $K_X+\Delta$ est ample ou numériquement trivial, et Δ à coefficients standard. Nous démontrons que, si l'inégalité de Miyaoka–Yau orbifold pour (X,Δ) est une égalité, alors le revêtement universel orbifold de la paire est soit la boule (cas ample), soit l'espace affine (cas numériquement trivial).

Keywords. Miyaoka–Yau inequality, orbifold uniformization, klt pairs.

Mots-clés. inégalité de Miyaoka–Yau, uniformisation orbifold, paires klt.

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1. Introduction

Let X be an n-dimensional compact Kähler manifold and let us assume that either

(I) K_X is ample (and X is thus projective), or
(II) K_X is numerically trivial (equivalently, $c_1(X) = 0$ in $H^2(X,\mathbb{R})$).

As a consequence of the existence of a Kähler–Einstein metric ω_{KE} on X (proved by Aubin [4] and Yau [43]), the Chern classes of X satisfy the *Miyaoka–Yau inequality*

$$\left(2(n+1)\,c_2(X) - n\,c_1^2(X)\right)\cdot \alpha^{n-2} \geq 0. \tag{MY}$$

where in case (I), we set $\alpha = [K_X]$, while in case (II), α can be an arbitrary Kähler class. Furthermore, in case of equality, the universal cover $\pi\colon \widetilde{X} \to X$ is (biholomorphic to)

(I) the n-dimensional unit ball $\mathbb{B}^n = \left\{(z_1,\dots,z_n)\in\mathbb{C}^n \;\middle|\; |z_1|^2+\cdots+|z_n|^2<1\right\}$,
(II) the n-dimensional affine space $\mathbb{C}^n$.

We can reformulate the above conclusion by saying that

(I) $X = \mathbb{B}^n/\Gamma$ with $\Gamma \subset \mathrm{PU}(1,n) = \mathrm{Aut}(\mathbb{B}^n)$,
(II) $X = \mathbb{C}^n/\Gamma$ with $\Gamma \subset \mathbb{C}^n \rtimes \mathrm{U}(n) = \mathrm{Aut}(\mathbb{C}^n, \pi^*\omega_{KE})$,

where in both cases, the action of Γ on $\widetilde{X}$ is *fixed point-free*. Not surprisingly, there is a beautiful exposition of this circle of ideas by Jean-Pierre Demailly [18].

It seems natural to investigate the general case of quotients by cocompact lattices $\Gamma \subset \mathrm{Aut}(\widetilde{X})$ (with $\widetilde{X} = \mathbb{B}^n$ or $\mathbb{C}^n$ endowed with the Bergman metric or the flat metric, respectively), the action being of course assumed to be properly discontinuous. The corresponding quotients are then naturally endowed with an orbifold structure that can be encoded in the datum of a $\mathbb{Q}$-divisor with standard coefficients (see Setup 1 below). To sum up, it is natural to consider pairs (X,Δ) when dealing with these quotients.

The question of uniformizing *spaces* (as opposed to pairs) in the cases (I) and (II) has been considered in the framework of klt singularities. To quote a few relevant papers: [15, 24, 27, 28, 29, 30, 38]. This article grew out of an attempt to understand the general situation with an orbifold structure in codimension one.

Unfortunately, the parallels between cases (I) and (II) cannot be pursued throughout this introductory section since the difficulties (when dealing with the inequality (MY) in the singular setting) are not of the same nature. The following three facts illustrate this point:

- In case (I), the variety X is necessarily projective, but the codimension one part of the orbifold structure cannot be easily eliminated. Therefore we have to use orbifold techniques in the proof.
- In case (II), we also need to consider (non-algebraic) compact Kähler spaces, but we can get rid of the codimension one part of the orbifold structure via a cyclic covering (see Proposition 12). This enables us to assume that $\Delta = 0$ for most of the argument.
- In case (I), the Bergman metric is invariant under the full automorphism group of $\mathbb{B}^n$, but this is not true of the flat metric in case (II). Therefore (2) below does not have an analog in Corollary 7, although a conjecture due to Iitaka [32] (or rather an orbifold version thereof) predicts that this should in fact be true.

Due to this break in symmetry, we split the discussion according to the sign of the canonical bundle.

The canonically polarized case

Let us recall the singular version of the inequality (MY) as proven by the third-named author together with B. Taji [31]. When dealing with case (I), we work in the following setting:

Setup 1. Let (X, Δ) be an n-dimensional klt pair, where X is a projective variety and Δ has standard coefficients, i.e. $\Delta = \sum_{i \in I} \big(1 - \frac{1}{m_i}\big) \Delta_i$ with integers $m_i \geq 2$ and the Δ_i irreducible and pairwise distinct.

Theorem 2 (⊂ [31, Thm. B]). *Let (X, Δ) be as in Setup 1, and assume that $K_X + \Delta$ is big and nef. Assume additionally that every irreducible component Δ_i of Δ is $\mathbb{Q}$-Cartier. Then the following inequality holds:*

$$\big(2(n+1)\,\widetilde{c}_2(X, \Delta) - n\,\widetilde{c}_1^{\,2}(X, \Delta)\big) \cdot [K_X + \Delta]^{n-2} \geq 0. \tag{2}$$

Here, $\widetilde{c}_2(X, \Delta)$ and $\widetilde{c}_1^{\,2}(X, \Delta)$ denote the appropriate orbifold Chern classes of the pair (X, Δ), as defined e.g. in [31, Notation 3.7]. □

Remark. In the above theorem, the assumption that the Δ_i be $\mathbb{Q}$-Cartier is not necessary, and establishing this is one of the (minor) contributions of this paper, cf. Theorem 36. While this may seem like an innocuous technical issue at first sight, eliminating the $\mathbb{Q}$-Cartier assumption will become crucial below when deducing Corollary 4 from Theorem A, see Remark 38.

As in the smooth case, it is interesting to characterize geometrically those pairs that achieve equality in (2). In the case where $\Delta = 0$, this has been achieved in [29, Thm. 1.2] and [30, Thm. 1.5]: equality holds if and only if there is a finite quasi-étale Galois cover $Y \to X$ such that the universal cover of Y is the unit ball. An expectation concerning the general case was formulated in [29, §10.2]. Our first main result confirms this expectation.

Theorem A (Uniformization of canonical models). *Let (X, Δ) be as in Setup 1. Assume that $K_X + \Delta$ is ample and that equality holds in* (2). *Then the orbifold universal cover $\pi \colon \widetilde{X}_\Delta \to X$ of (X, Δ) is the unit ball (cf. Definition 24). More precisely, $(\widetilde{X}_\Delta, \widetilde{\Delta}) \cong (\mathbb{B}^n, \emptyset)$.*

In fact, a suitable converse of the above theorem also holds, and we obtain the following corollary.

Corollary 3 (Characterization of ball quotients). *Let (X, Δ) be as in Setup 1. The following are equivalent:*

(1) *$K_X + \Delta$ is ample, and equality holds in* (2).
(2) *The orbifold universal cover of (X, Δ) is the unit ball $\mathbb{B}^n$.*
(3) *(X, Δ) admits a finite orbi-étale Galois cover $f \colon Y \to X$ (cf. Definition 8), where Y is a projective manifold whose universal cover is the unit ball.*

In the spirit of [30, Thm. 1.5], we can also prove the following uniformization statement for minimal pairs of log general type.

Corollary 4 (Uniformization of minimal models). *Let (X, Δ) be as in Setup 1. Assume that $K_X + \Delta$ is big and nef and that equality holds in* (2). *Then the canonical model $(X, \Delta)_{\mathrm{can}} =: (X_{\mathrm{can}}, \Delta_{\mathrm{can}})$ of the pair (X, Δ) is a ball quotient in the sense of Theorem A.*

The flat case

As mentioned earlier, Kähler quotients of $\mathbb{C}^n$ by cocompact groups of isometries are in general not projective, so we have to consider the following framework.

Setup 5. Let (X, Δ) be an n-dimensional klt pair, where X is a compact Kähler space and Δ has standard coefficients, i.e. $\Delta = \sum_{i \in I} \big(1 - \frac{1}{m_i}\big) \Delta_i$ with integers $m_i \geq 2$ and the Δ_i irreducible and pairwise distinct.

In this more general Kähler setting, the methods of [31] cannot be used to prove a singular analogue of the Miyaoka–Yau inequality. Instead, we rely on the Decomposition Theorem from [5] to deduce the following singular version of the inequality (MY) in case (II).

Theorem 6 (Singular Miyaoka–Yau inequality). *Let (X,Δ) be as in Setup 5 and assume that* $c_1(K_X+\Delta)=0\in \mathrm{H}^2(X,\mathbb{R})$. *Let $\alpha\in \mathrm{H}^2(X,\mathbb{R})$ be any Kähler class. We then have:*

$$\widetilde{c}_2(X,\Delta)\cdot\alpha^{n-2}\geq 0. \tag{3}$$

As before, we are particularly interested in what happens if equality is achieved.

Theorem B (Uniformization in the flat case). *Let (X,Δ) be as in Setup 5. Assume that* $c_1(K_X+\Delta)=0\in \mathrm{H}^2(X,\mathbb{R})$ *and that equality holds in* (3) *for* some *Kähler class α. Then the orbifold universal cover $\pi\colon \widetilde{X}_\Delta\to X$ of (X,Δ) is the affine space (cf. Definition 24). More precisely, $(\widetilde{X}_\Delta,\widetilde{\Delta})\cong(\mathbb{C}^n,\emptyset)$.*

As above, we can formulate a converse and get the following corollary.

Corollary 7 (Characterization of torus quotients). *Let (X,Δ) be as in Setup 5. The following are equivalent:*

(1) $c_1(K_X+\Delta)=0\in \mathrm{H}^2(X,\mathbb{R})$, *and equality holds in* (3) *for some Kähler class α.*
(2) *(X,Δ) admits a finite orbi-étale Galois cover $f\colon T\to X$ (cf. Definition 8), where T is a complex torus.*

The previous statements are thus generalizations of [38, Thm. 1.2] (itself elaborating on [27, Thm. 1.17]). The generalization is threefold:

- Here X is a compact Kähler space, not necessarily projective.
- The class α is transcendental, a priori not an ample class.
- Ramification is allowed in codimension one; i.e. we work with klt pairs rather than klt spaces.

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2. Generalities on orbifolds

In this section, we consider Kawamata log terminal (klt) pairs (X,Δ) consisting of a normal algebraic variety or complex space X of dimension n and a $\mathbb{Q}$-divisor $\Delta=\sum_{i\in I}\left(1-\frac{1}{m_i}\right)\Delta_i$ on X, with $m_i\geq 2$.

2.1. *Orbi-structures and orbi-sheaves*

Most of the definitions and basic properties given below can be found in e.g. [31, §2] in the slightly more general setting of *dlt* pairs with standard coefficients, at least if X is algebraic. Working exclusively with klt pairs will simplify the exposition.

Definition 8 (Adapted morphisms). *Let $f\colon Y\to X$ be a finite surjective Galois morphism from a normal variety or complex space Y. One says that f is:*

- adapted *to (X,Δ) if for all $i\in I$, there exists $a_i\in\mathbb{Z}^{\geq 1}$ and a reduced divisor Δ_i' on Y such that $f^*\Delta_i=a_i m_i\Delta_i'$,*
- strictly adapted *to (X,Δ) if it is adapted and if $a_i=1$ for all $i\in I$,*
- orbi-étale *if it is strictly adapted and the divisorial component of the branch locus of f is contained in* $\mathrm{supp}(\Delta)$. *Equivalently, if f is étale over $X_{\mathrm{reg}}\setminus\mathrm{supp}(\Delta)$.*

Remark. If X is compact, then a map $f\colon Y \to X$ as above is orbi-étale if and only if $K_Y = f^*(K_X+\Delta)$.

Definition 9 (Orbi-structures). *An* orbi-structure *for the pair* (X,Δ) *consists of a compatible collection of triples* $\mathscr{C} = \{(U_\alpha, f_\alpha, X_\alpha)\}_{\alpha\in J}$*, where* $(U_\alpha)_{\alpha\in J}$ *is a covering of* X *by étale-open subsets, and for each* $\alpha \in J$*,* $f_\alpha\colon X_\alpha \to U_\alpha$ *is an adapted morphism from a normal complex space* X_α *with respect to the pair structure on* U_α *induced by* (X,Δ)*. The compatibility condition means that for all* $\alpha,\beta\in J$*, the projection map* $g_{\alpha\beta}\colon X_{\alpha\beta}\to X_\alpha$ *is quasi-étale, where* $X_{\alpha\beta}$ *is the normalization of* $X_\alpha\times_X X_\beta$*.*

An orbi-structure $\mathscr{C} = \{(U_\alpha, f_\alpha, X_\alpha)\}_{\alpha\in J}$ *is called* strict *(resp.* orbi-étale*) if for each* $\alpha\in J$*, the morphism* f_α *is strictly adapted (resp. orbi-étale). It is called* smooth *if for each* $\alpha\in J$*, the variety* X_α *is smooth. In this case, the maps* $g_{\alpha\beta}$ *are étale by purity of branch locus.*

Definition 10 (Quotient singularities). *A pair* (X,Δ) *is said to have* quotient singularities *if locally analytically on* X*, there exists an orbi-étale morphism* $f\colon Y\to X$*, where* Y *is smooth. The maximal open subset of* X *where this condition is satisfied will also be referred to as the* orbifold locus *of* (X,Δ) *and will be denoted by* $X^\circ \subset X$ *or* $X^{\mathrm{orb}}\subset X$*.*

Remark. With the above terminology, a pair (X,Δ) admits a smooth orbi-étale orbi-structure if and only if it has quotient singularities. This is because the compatibility condition is automatically satisfied.

The following technical result will be useful in the sequel: a pair with quotient singularities whose underlying space is compact Kähler is a Kähler orbifold. The log smooth case had been already observed in [14, Prop. 2.1]. Slightly more generally, we have the following.

Lemma 11 (Existence of orbifold Kähler metrics). *Let* (Z,Δ) *be a pair with quotient singularities and such that* Z *is a Kähler space. Then for any relatively compact open subset* $X\Subset Z$*, there exists an orbifold Kähler metric* ω *adapted to* $(X,\Delta|_X)$ *in the sense that* ω *is a Kähler metric on* $X_{\mathrm{reg}}\setminus\operatorname{supp}\Delta$ *which pulls back to a smooth Kähler metric on the smooth local covers.*

Proof. One can find an open neighborhood X' of $\overline{X}\subset Z$ admitting a finite covering $X' = \bigcup_{\alpha\in I} X'_\alpha$ such that there exist smooth orbi-étale covers $p_\alpha\colon Y'_\alpha\to X'_\alpha$. We set $X_\alpha := X'_\alpha\cap X$ and $Y_\alpha := p_\alpha^{-1}(X_\alpha)$. We pick a Kähler metric ω_Z on Z, as well as potentials ϕ_α on X'_α such that $dd^c p_\alpha^*\phi_\alpha$ is a Kähler metric on Y'_α; the functions ϕ_α are solely continuous on X_α but $p_\alpha^*\phi_\alpha$ is smooth on Y'_α. We can assume that $|\phi_\alpha|\le 1$ on X_α. Finally, let $(\chi_\alpha)_{\alpha\in I}$ be some partition of unity subordinate to the covering $(X_\alpha)_{\alpha\in I}$ and set $\phi := \sum \chi_\alpha\phi_\alpha$. We set $N := |I|$ and pick a constant $C>0$ such that

$$\|dd^c\chi_\alpha\|^2_{\omega_Z} + \|d\chi_\alpha\|^2_{\omega_Z}\le C, \tag{4}$$

holds for any $\alpha\in I$ and we claim that the current

$$\omega := M\omega_Z + dd^c\phi$$

is an orbifold Kähler metric on X for $M\gg 1$. Clearly, ω is smooth as an orbifold differential form, as one can see directly by using the compatibility of the covers. Let $x\in X$ and let $J := \{\alpha\in I, x\in X_\alpha\} = \{\alpha_1,\dots,\alpha_s\}$. We set $X_J := \bigcap_{\alpha\in J} X_\alpha$ and choose a connected component Y_J of the normalization of $p_{\alpha_1}^{-1}(X_J)\times_{X_J}\cdots\times_{X_J} p_{\alpha_s}^{-1}(X_J)$. The space Y_J is a smooth manifold endowed with an orbi-étale map $p_J\colon Y_J\to X_J$ induced by the p_{α_i}, $i=1,\dots,s$.

We have $1 = \sum_{\alpha\in I}\chi_\alpha(x) = \sum_{\alpha\in J}\chi_\alpha(x)$, hence there exists $\beta\in J$ such that $\chi_\beta(x)\ge\frac{1}{N}$. Since $p_J^*(dd^c\phi_\beta|_{X_J})$ is a Kähler metric on Y_J (which extends slightly beyond), we infer that there exists $\delta>0$ such that

$$\forall\,\alpha\in J,\quad dd^c\phi_\beta\ge\delta d\phi_\alpha\wedge d^c\phi_\alpha\quad\text{on } X_J.$$

Next, we have the following inequality for any $\varepsilon>0$:

$$\pm(d\phi_\alpha\wedge d^c\chi_\alpha + d\chi_\alpha\wedge d^c\phi_\alpha)\le\varepsilon d\phi_\alpha\wedge d^c\phi_\alpha + \varepsilon^{-1}d\chi_\alpha\wedge d^c\chi_\alpha.$$

Combining the above inequality with (4), we get for any $\varepsilon > 0$:

$$\begin{aligned}\omega &= M\omega_Z + \sum_{\alpha\in I}\chi_\alpha \mathrm{dd}^c\phi_\alpha + \sum_{\alpha\in I}\phi_\alpha \mathrm{dd}^c\chi_\alpha + \sum_{\alpha\in I}(\mathrm{d}\phi_\alpha \wedge \mathrm{d}^c\chi_\alpha + \mathrm{d}\chi_\alpha \wedge \mathrm{d}^c\phi_\alpha)\\ &\geq (M - NC(1+\varepsilon^{-1}))\omega_Z + \chi_\beta \mathrm{dd}^c\phi_\beta - \varepsilon\sum_{\alpha\in I}\mathrm{d}\phi_\alpha\wedge \mathrm{d}^c\phi_\alpha\end{aligned}$$

which yields, at the point x:

$$\omega \geq (M - NC(1+\varepsilon^{-1}))\omega_Z + \left(\frac{1}{N} - \frac{N\varepsilon}{\delta}\right)\mathrm{dd}^c\phi_\beta.$$

Therefore, if we choose $\varepsilon := \frac{\delta}{2N^2}$ and $M = 2NC(1+\varepsilon^{-1})$, then ω is an orbifold Kähler metric near x. Since x is arbitrary and the constants N, C, δ are uniform, the lemma is now proved. □

2.2. *Covering constructions*

In what follows, we present some variations on the well-known cyclic covering theme. The first one, Proposition 12, is a consequence of [42, Ex. 2.4.1] when X is quasi-projective so that K_X is well-defined as a (class of) Weil divisor, but one needs to argue slightly differently in the complex analytic case. The second one, Proposition 13, improves upon previous results such as [33, Prop. 2.9], [31, Ex. 2.11] and [17, Prop. 2.38]. The main observation is that given a pair (X, Δ), it is (for our purposes) unnecessary to assume that the components of Δ are $\mathbb{Q}$-Cartier as long as $K_X + \Delta$ is. As explained in Remark 38, this is crucial for proving Corollary 4.

Proposition 12 (Existence of orbi-étale covers). *Let (X, Δ) be a (not necessarily klt) pair with standard coefficients, where X is a normal complex space. Assume that there is a reflexive rank 1 sheaf $\mathscr{L}$ and an integer $N \geq 1$ such that $N\Delta$ is a $\mathbb{Z}$-divisor and*

$$\mathscr{O}_X(N\Delta) \cong \mathscr{L}^{[N]}.$$

Then there exists an orbi-étale morphism $f\colon Y \to X$. In particular:

If (X, Δ) is klt and there is an integer $N \geq 1$ such that $N\Delta$ is a $\mathbb{Z}$-divisor and $\omega_X^{[N]}(N\Delta) \cong \mathscr{O}_X$, then we can find an orbi-étale morphism $f\colon Y \to X$ such that $\omega_Y \cong \mathscr{O}_Y$ and Y has canonical singularities.

Proof. Let $\sigma \in \mathrm{H}^0\big(X, \mathscr{L}^{[N]}\big)$ be such that $\mathrm{div}(\sigma) = N\Delta$, and let us consider the cyclic covering $g\colon Z \to X$ induced by σ, cf. e.g. [36, Def. 2.52]. In the analytic setting, we can construct f in the following way. On $X_{\mathrm{reg}} \setminus \mathrm{supp}(\Delta)$, $\mathscr{L}|_{X_{\mathrm{reg}}\setminus\mathrm{supp}(\Delta)}$ is torsion and it gives rise to an étale cover $g^\circ\colon Z^\circ \to X_{\mathrm{reg}} \setminus \mathrm{supp}(\Delta)$ (the N^{th}-root of $\sigma|_{X_{\mathrm{reg}}\setminus\mathrm{supp}(\Delta)}$) that is moreover a Galois cover with cyclic Galois group. According to [19, Thm. 3.4], the map g° can be extended to a finite cover $f\colon Z \to X$ with the same Galois group.

We claim that g ramifies exactly at order m_i along Δ_i. It is enough to check the claim at a general point of Δ_i. Therefore, there is no loss of generality assuming that $(X, \Delta) = (U, (1-\frac{1}{m})D)$ where $U \subset \mathbb{C}^n$ ($n = \dim(X)$) is a ball, $D = (z_1 = 0) \cap U$, and that $\sigma|_U = z_1^{N(1-\frac{1}{m})}\sigma_{\mathscr{L},U}^{\otimes N}$ with $\sigma_{\mathscr{L},U}$ a trivializing section of $\mathscr{L}$ over U.

Write $N = km$, and let $V := \{(t,z) \in \mathbb{C}\times\mathbb{C}^n \,|\, t^N = z_1^{k(m-1)}\} \subset \mathbb{C}\times\mathbb{C}^n$ and let $\nu\colon V^\nu \to V$ be its normalization. One can actually write down exactly what V^ν is. Indeed, let ζ be a primitive k-th root of unity, and set $V_p := \{(t,z) \,|\, t^m = \zeta^p z_1^{m-1}\} \subset \mathbb{C}\times\mathbb{C}^n$ for $p = 0, \ldots, k-1$. We have a decomposition $V = \bigcup_p V_p$ into irreducible components, and the normalization $\nu_p\colon V_p^\nu \to V_p$ is the affine space $V_p^\nu \cong \mathbb{C} \times \mathbb{C}^{n-1}$ with map $\nu_p(u,w) = (\xi u^{m-1}, u^m, w)$ where ξ is an m-th root of ζ^p. Now, set $V^\nu := \bigsqcup_p V_p^\nu$ and define $\nu\colon V^\nu \to V$ by $\nu|_{V_p^\nu} := \nu_p$. We have a diagram

$$\begin{tikzcd} V^\nu \arrow[r, "\nu"] \arrow[rr, bend left, dotted, "j"] & V \arrow[d, "\mathrm{pr}_{\mathbb{C}^n}"'] & Z \arrow[d, "g"] \\ & U \arrow[r, hook] & X \end{tikzcd}$$

where j is obtained by the universal property of normalization. In particular, j is finite and generically 1-to-1 between normal varieties, hence it is an open embedding. Moreover, if $(u,w) \in V_p^\nu$, we have $\mathrm{pr}_{\mathbb{C}^n} \circ \nu(u,w) = (u^m, w)$, hence the latter map ramifies at order m along D. It follows that g ramifies at order m along D.

Finally, one picks one irreducible component Y of Z and sets $f := g|_Y$. It yields the expected cover, which is Galois with group $G < \mathbb{Z}/n\mathbb{Z} \cong \mathrm{Gal}(Z \to X)$ defined as the stabilizer of Y.

As for the last part of the proposition, we can apply the above construction to $\mathscr{L} = \omega_X^{[-1]} := \omega_X^\vee$. This provides us with an orbi-étale morphism $f\colon Y \to X$. In particular, Y is klt and the computations made above show that $f^*(K_X + \Delta)$ is trivial over $X_{\mathrm{reg}} \setminus \Delta_{\mathrm{sg}}$. So we get that ω_Y is trivial as well and finally that Y has only canonical singularities. □

Proposition 13 (Existence of strictly adapted covers). *Let (X,Δ) be a projective pair with standard coefficients such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (but not necessarily klt). Then there exists a very ample divisor L on X such that for general $H \in |L|$, there exists a cyclic Galois cover $f\colon Y \to X$ with the following properties:*

(1) *The morphism f is orbi-étale for $\big(X, \Delta + (1-\frac{1}{N})H\big)$, where $N := \deg(f)$.*
(2) *The morphism f is strictly adapted for (X,Δ).*
(3) *If (X,Δ) is klt, then so are the pairs $\big(X, \Delta + (1-\frac{1}{N})H\big)$ and $(Y, \emptyset)$.*

Proof. Pick, once and for all, a representative K of K_X, that is, an integral (but not necessarily effective) Weil divisor K on X such that $K_X \sim K$. Choose a very ample divisor A on X and a positive integer N such that

$$L := N \cdot \big(A - (K + \Delta)\big)$$

is integral and very ample, and pick a general element $H \in |L|$. Consider the principal divisor

$$D := H - L = H + N \cdot (K + \Delta - A) \sim 0.$$

Let $f\colon Y \to X$ be the degree N cyclic cover associated to D, as in [42, §2.3]. (To be more precise, Y is an arbitrary irreducible component of the normalization of that cover.) We need to check properties (1)–(3).

By construction, the branch locus of f is contained in $\mathrm{supp}(D)$. Recall from [42] that writing $D = \sum_i d_i D_i$, the ramification order of f along each component of $f^{-1}(D_i)$ is given by $N/\mathrm{hcf}(d_i, N)$. Since K, A and H are $\mathbb{Z}$-divisors, where H is even reduced, this implies (1). Property (2) is an immediate consequence.

For (3), it is enough to show the first claim thanks to (1) and [36, Prop. 5.20]. To check the claim, we take a log resolution $\pi\colon \widetilde{X} \to X$ of (X,Δ) and write

$$K_{\widetilde{X}} + \Delta' = \pi^*(K_X + \Delta) + \sum a_i E_i$$

as usual, where Δ' is the strict transform of Δ. Since H is a general element of $|L|$, and $\pi^*|L|$ is basepoint-free, one can assume that $\pi^* H = \pi_*^{-1} H$ is smooth and intersects each stratum of the exceptional divisor of π and of Δ' smoothly. In particular, π is also a log resolution for the pair $\big(X, \Delta + (1-\frac{1}{N})H\big)$. Now, the identity

$$K_{\widetilde{X}} + \Delta' + \left(1 - \frac{1}{N}\right)\pi_*^{-1} H = \pi^*\left(K_X + \Delta + \left(1 - \frac{1}{N}\right) H\right) + \sum a_i E_i$$

shows that $\big(X, \Delta + (1-\frac{1}{N})H\big)$ is klt. $\square$

Remark. More generally, it can be observed that a pair (X, Δ) (with X a normal analytic space) admits strictly adapted covers if there exists a Cartier divisor D on X having no component in common with Δ and such that $m(K_X + \Delta) \sim D$ for some (sufficiently divisibe) integer $m \geq 1$. We can indeed apply Proposition 12 to the pair $(X \setminus D, \Delta|_{X\setminus D})$ and get an orbi-étale cover $Y^\circ \to X \setminus D$. Its completion over X is then adapted with respect to Δ and the extra-ramification is supported over the components of D.

The following result seems to have been known to experts for a long time. A proof of it was written down in [26] in the case where $\Delta = 0$, and the general case follows almost immediately from Proposition 12 as we will explain.

Lemma 14 (Klt pairs have quotient singularities in codimension two). *Let (X, Δ) be a klt pair with standard coefficients. Then there is a Zariski closed subset $Z \subset X_{\mathrm{sg}} \cup \operatorname{supp}\Delta$ with $\operatorname{codim}_X(Z) \geq 3$ such that for $X^\circ := X \setminus Z$, the pair $(X^\circ, \Delta|_{X^\circ})$ admits a smooth orbi-étale orbi-structure $\mathscr{C}^\circ$.*

Proof. Since $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor, we can cover X by (affine or Stein) open subsets $U_\beta \subset X$, $\beta \in I$, such that $(K_X + \Delta)|_{U_\beta} \sim_{\mathbb{Q}} 0$. By Proposition 12, we can find a finite cyclic cover $g_\beta\colon U'_\beta \to U_\beta$ that branches exactly over the $\Delta_i|_{U_\beta}$ with multiplicity m_i. Moreover, U'_β has klt singularities, since $K_{U'_\beta} = g_\beta^*(K_{U_\beta} + \Delta|_{U_\beta})$. We can now use [26, Prop. 9.3] or [24, Lem. 5.8] to find a *smooth* orbi-étale orbi-structure $\{U'_{\beta\gamma}, f_{\beta\gamma}, X'_{\beta\gamma}\}_{\gamma \in J}$ on $U'_\beta \setminus Z_\beta$, for some closed subset $Z_\beta \subset U'_\beta$ of codimension at least three. Set $U_{\beta\gamma} = g_\beta(U'_{\beta\gamma})$, so that $\bigcup_\beta U_{\beta\gamma} \subset U_\beta$ is an open subset whose complement is of codimension at least three. In summary, we get the following diagram:

$$\begin{tikzcd}
X'_{\beta\gamma} \arrow[r, "f_{\beta\gamma}"] \arrow[rr, bend left, "h_{\beta\gamma}"] & U'_{\beta\gamma} \arrow[r, "g_\beta"] \arrow[d, hook] & U_{\beta\gamma} \arrow[d, hook] & \\
& U'_\beta \arrow[r, "g_\beta"] & U_\beta \arrow[r, hook] & X
\end{tikzcd} \tag{5}$$

Now $\big\{U_{\beta\gamma}, h_{\beta\gamma}, X'_{\beta\gamma}\big\}_{(\beta,\gamma)\in I\times J}$ is the sought-after smooth orbi-étale orbi-structure on $(X^\circ, \Delta|_{X^\circ})$, where the open subset $X^\circ := \bigcup_{(\beta,\gamma)\in I\times J} U_{\beta\gamma}$ has complement of codimension at least three. $\square$

Remark 15. In particular, a klt surface pair with standard coefficients admits a smooth orbi-étale orbi-structure, hence it has quotient singularities in the sense of Definition 10. This is of course well-known and follows from the cyclic cover construction recalled above and [36, Prop. 4.18].

Definition 16 (Orbi-sheaves). *An* orbi-sheaf *with respect to an orbi-structure $\mathscr{C} = \big\{(U_\alpha, f_\alpha, X_\alpha)\big\}_{\alpha\in J}$ on (X, Δ) is the datum of a collection $(\mathscr{E}_\alpha)_{\alpha\in J}$ of coherent sheaves on each X_α, together with isomorphisms $g_{\alpha\beta}^*\mathscr{E}_\alpha \cong g_{\beta\alpha}^*\mathscr{E}_\beta$ of $\mathscr{O}_{X_{\alpha\beta}}$-modules satisfying the natural compatibility conditions on triple overlaps.*

All the usual notions for sheaves (locally free, reflexive, subsheaves, morphisms etc.) can be carried over to this setting in the obvious way, cf. [31, §2.7]. Ditto for Higgs fields and Higgs sheaves, cf. [31, Def. 2.24].

Recall the following definition from [17, §3]:

Definition 17 (Adapted differentials). *Let $\gamma\colon Y \to X$ be a strictly adapted morphism for (X,Δ). Let $X^\circ \subset X$ and $\iota\colon Y^\circ \hookrightarrow Y$ be the maximal open subsets where γ is* good *in the sense of [17, Def. 3.5]. The sheaf of* adapted reflexive differentials *is defined as*

$$\Omega^{[1]}_{(X,\Delta,\gamma)} := \iota_*\left[\left(\operatorname{im}\left(\gamma^*\Omega^1_{X^\circ} \to \Omega^1_{Y^\circ}\right) \otimes \mathscr{O}_{Y^\circ}(\gamma^*\Delta)\right) \cap \Omega^1_{Y^\circ}\right].$$

Lemma 18. *The following properties hold:*

(1) *The sheaf $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is a coherent reflexive subsheaf of $\Omega^{[1]}_Y$.*
(2) *If γ is orbi-étale for (X,Δ), then $\Omega^{[1]}_{(X,\Delta,\gamma)} = \Omega^{[1]}_Y$.*
(3) *Let $\gamma_2\colon Z \to Y$ be quasi-étale, where Z is normal. Then $\delta := \gamma\circ\gamma_2\colon Z \to X$ is strictly adapted for (X,Δ), and $\Omega^{[1]}_{(X,\Delta,\delta)} = \gamma_2^{[*]}\Omega^{[1]}_{(X,\Delta,\gamma)}$.* □

Definition 19 (Orbifold cotangent sheaf, cf. [31, Def. 2.23]). *Consider on (X,Δ) any strictly adapted orbi-structure $\mathscr{C} = \{(U_\alpha, f_\alpha, X_\alpha)\}_{\alpha\in J}$. Then the sheaves*

$$\left(\Omega^{[1]}_{(X,\Delta,f_\alpha)}\right)_{\alpha\in J}$$

induce a reflexive orbi-sheaf called the orbifold cotangent sheaf, *or* sheaf of reflexive differential forms, *which we denote by $\Omega^{[1]}_{\mathscr{C}}$. If the orbi-structure $\mathscr{C}$ is smooth and orbi-étale, then $\Omega^{[1]}_{\mathscr{C}}$ is locally free. Changing the (strictly adapted) orbifold structure yields compatible sheaves in the sense of [31, Def. 3.2], hence we will often denote this sheaf by $\Omega^{[1]}_{(X,\Delta)}$.*

The same construction can be carried out for any integer $p \ge 0$, yielding orbi-sheaves $\Omega^{[p]}_{(X,\Delta)}$. For $p = 0$, we obtain the structure sheaf $\mathscr{O}_{(X,\Delta)}$, *which is nothing but $\mathscr{O}_{X_\alpha}$ in each chart f_α.*

Lemma 20. *Let (X,Δ) be a projective klt pair with standard coefficients, and let X° be endowed with a smooth orbi-étale orbi-structure $\mathscr{C}$ as in Lemma 14. Let H be an ample line bundle on X and pick a complete intersection surface*

$$S = D_1 \cap \cdots \cap D_{n-2}$$

of $n-2$ general hypersurfaces $D_i \in |mH|$ for $m \gg 1$. Then $S \subset X^\circ$ and the restriction of $\mathscr{C}$ to $(S, \Delta|_S)$ induces a smooth orbi-étale orbi-structure on $(S,\Delta|_S)$. In particular, $(S,\Delta|_S)$ has quotient singularities.

Proof. We have $S \subset X^\circ$ for dimensional and genericity reasons. Next, if we express the structure $\mathscr{C}$ as $\mathscr{C} = \{(X_\alpha, f_\alpha, U_\alpha)\}$, set $S_\alpha := S \cap U_\alpha$, $T_\alpha := f_\alpha^{-1}(S_\alpha)$, $g_\alpha := f_\alpha|_{T_\alpha}$, and define $\mathscr{C}|_S := \{(T_\alpha, g_\alpha, S_\alpha)\}$. We claim that T_α is smooth, which would prove the lemma. Indeed, since f_α is quasi-finite (as the composition of an étale map with a finite map), one can find an open immersion $X_\alpha \hookrightarrow \overline{X_\alpha}$ and a finite extension $\overline{f_\alpha}\colon \overline{X_\alpha} \to X$ of f_α as follows:

$$\begin{tikzcd} T_\alpha \arrow[r, hook] \arrow[d, "g_\alpha"] & X_\alpha \arrow[r, hook] \arrow[d, "f_\alpha"] & \overline{X_\alpha} \arrow[d, "\overline{f_\alpha}"] \\ S_\alpha \arrow[r, hook] & U_\alpha \arrow[r, hook] & X \end{tikzcd}$$

Since $\overline{f_\alpha}^*|mH|$ is basepoint-free, Bertini's theorem guarantees that if $\overline{T_\alpha}$ is a general intersection of $(n-2)$ hypersurfaces in $\overline{f_\alpha}^*|mH|$, then $\overline{T_\alpha} \cap \overline{X_\alpha}^{\mathrm{reg}}$ is smooth. Since $X_\alpha \subset \overline{X_\alpha}^{\mathrm{reg}}$, this shows that T_α is smooth, hence the lemma. □

2.3. *The orbifold fundamental group*

Let (X,Δ) be a klt pair with standard coefficients as before, and set $X^* := X_{\mathrm{reg}} \setminus \operatorname{supp}\Delta$.

Definition 21 (Fundamental group). *The* (orbifold) fundamental group *of* (X,Δ) *is defined as*

$$\pi_1^{\mathrm{orb}}(X,\Delta) := \pi_1(X^*)/\langle\langle \gamma_i^{m_i},\, i \in I\rangle\rangle.$$

Here, for each $i \in I$*, the element* γ_i *is a "loop around* Δ_i*", i.e. a loop in the normal circle bundle of* $(\Delta_i)_{\mathrm{reg}} \cap X_{\mathrm{reg}} \subset X_{\mathrm{reg}}$*, and* $\langle\langle \cdots \rangle\rangle$ *denotes the normal subgroup generated by a given subset.*

Note that if $D = \emptyset$, then $\pi_1^{\mathrm{orb}}(X,\emptyset) = \pi_1(X_{\mathrm{reg}})$ is in general different from $\pi_1(X)$.

Definition 22 (Covers branched at Δ, cf. [14, Def. 1.3]). *A cover of* X branched at most at Δ *is a holomorphic map* $\pi\colon Y \to X$*, where:*

1. Y *is a normal connected complex space (not necessarily quasi-projective),*
2. π *has discrete fibres and* $\pi^{-1}(X^*) \to X^*$ *is étale,*
3. *at each irreducible component* $\widetilde{\Delta}_{j,k} \subset \pi^{-1}(\Delta_j)$*, the ramification index* $r_{j,k}$ *of* π *divides* m_j*,*
4. *every* $x \in X$ *has a connected neighborhood* $V \subset X$ *such that every connected component* U *of* $\pi^{-1}(V)$ *meets the fibre* $\pi^{-1}(x)$ *in only one point, and* $\pi|_U\colon U \to V$ *is finite.*

We say that π *is* branched exactly at Δ *if in* (3)*, we have* $r_{j,k} = m_j$ *for all* j,k*.*

Note that if Y is quasi-projective and π is Galois, then saying that π is branched exactly at Δ is the same as saying that π is orbi-étale.

Theorem 23 (Covers and the fundamental group). *There exists a natural one-to-one correspondence between subgroups* $G \subset \pi_1^{\mathrm{orb}}(X,\Delta)$ *and covers* $\pi\colon Y \to X$ *branched at most at* Δ*. Furthermore:*

1. G *is of finite index if and only if* π *is finite.*
2. G *is a normal subgroup if and only if* π *is Galois.*
3. *Let* $Y_{1,2} \to X$ *be two covers branched at most at* Δ*, with corresponding subgroups* $G_{1,2} \subset \pi_1^{\mathrm{orb}}(X,\Delta)$*. Then there is a factorization*

$$\begin{tikzcd} & Y_2 \arrow[d] \\ Y_1 \arrow[ur, dashed, "\exists"] \arrow[r] & X \end{tikzcd}$$

if and only if $G_1 \subset G_2$*.*

Proof. The proof is the same as in the snc case, cf. [14, Thm. 1.1], with one important difference: in order to extend (possibly non-finite) étale covers of X^* to branched covers of X, we would like to apply [19, Thm. 3.4]. In order to do this, we must invoke the finiteness of local orbifold fundamental groups of klt pairs, as proved in [11, Thm. 1]. (Note that [11] works in the algebraic category, but in view of [22, Thm. 1.7] and [16, Rem. 6.10] his result extends to complex spaces as well.) □

Definition 24 (Universal cover). *The (*orbifold*)* universal cover *of* (X,Δ) *is the cover* $\pi\colon \widetilde{X}_\Delta \to X$ *corresponding to the trivial subgroup* $\{1\} \subset \pi_1^{\mathrm{orb}}(X,\Delta)$ *under the correspondence from Theorem 23.*

Let $\widetilde{\Delta}$ be the divisor on $\widetilde{X}_\Delta$ which is supported on $\pi^{-1}(\operatorname{supp}\Delta)$ and satisfies

$$K_{\widetilde{X}_\Delta} + \widetilde{\Delta} = \pi^*(K_X + \Delta).$$

It is easy to see that the pair $(\widetilde{X}_\Delta, \widetilde{\Delta})$ is again klt with standard coefficients. Also, $\widetilde{\Delta} = 0$ if and only if π is branched exactly at Δ.

Definition 25 (Developable pairs). *We say that* (X,Δ) *is* developable *if in the above notation,* $\widetilde{X}_\Delta$ *is smooth and* $\widetilde{\Delta} = 0$*.*

Intuitively, being developable means that the universal cover is a manifold.

Example 26. Consider the klt pair (X,Δ), where $X = \mathbb{P}^1$ and

$$\Delta = \left(1-\frac{1}{n}\right)\cdot[0] + \left(1-\frac{1}{m}\right)\cdot[\infty]$$

with $n, m \geq 2$. Set $d = \gcd(n,m)$. Then $\pi_1^{\mathrm{orb}}(X,\Delta) = \mathbb{Z}/d\mathbb{Z}$, and the universal cover $\pi\colon \widetilde{X}_\Delta = \mathbb{P}^1 \to \mathbb{P}^1$ is given by $[z_0 : z_1] \mapsto [z_0^d : z_1^d]$. We have

$$\widetilde{\Delta} = \left(1-\frac{1}{n/d}\right)\cdot[0] + \left(1-\frac{1}{m/d}\right)\cdot[\infty].$$

In particular, (X,Δ) is developable if and only if $n = m$.

Corollary 27 (Galois closure). *Let $Y \to X$ be a finite cover branched at most at Δ. Then there is a finite cover $Y' \to Y$ such that the composition $Y' \to X$ is finite, Galois, and branched at most at Δ. If additionally $Y \to X$ is branched exactly at Δ, then the same is true of $Y' \to X$, and $Y' \to Y$ is quasi-étale.*

We call $Y' \to X$ the *Galois closure* of $Y \to X$.

Proof. Using the correspondence from Theorem 23, the statement boils down to the following: for a group G and a subgroup $H \subset G$ of finite index, there is a normal subgroup $N \trianglelefteq G$ of finite index such that $N \subset H$. But this is easy (and well-known): simply set

$$N := \bigcap_{g\in G/H} gHg^{-1}.$$

The last statement is easily seen to be true by comparing the ramification indices of $Y \to X$ and $Y' \to X$ over the components Δ_i. $\square$

3. Orbifold Chern classes of klt pairs

In this section, we recall the definition of the first and second orbifold Chern classes for klt pairs, in the spirit of [24]. We then explain how to compute them concretely in two cases: in the projective setting by a cutting-down argument (Section 3.3), and when we have an "orbi-resolution" at our disposal (Section 3.4).

3.1. *The general Kähler case*

Let us begin by recalling how to define Chern numbers associated with the first and second Chern classes. This is nothing but a slight generalization of [24, Def. 5.2] that takes into account the presence of a boundary. The construction relies on the Chern–Weil formalism in the orbifold setting. We will not recall the basic definitions and properties for the differential geometry of orbifolds (e.g. Hermitian metrics on orbifold bundles, orbifold Chern classes, orbifold de Rham cohomology, and so on). A good reference is [8, §2].

Let (X,Δ) as in Setup 5 and let $X^\circ \subset X$ be the largest open subset of X such that (X,Δ) admits a smooth orbi-étale orbi-structure $\mathscr{C}^\circ$, and set $Z := X \setminus X^\circ$. As proved in Lemma 14, $\dim Z \leq n-3$. Next, let $\alpha \in \mathrm{H}^{2n-4}(X,\mathbb{R})$ where that cohomology space is understood as the cohomology of the locally constant sheaf $\underline{\mathbb{R}}_X$. For dimensional reasons, we have an isomorphism $\mathrm{H}_{\mathrm{c}}^{2n-4}(X^\circ,\mathbb{R}) \xrightarrow{\sim} \mathrm{H}^{2n-4}(X,\mathbb{R})$. Next, the de Rham complex of orbifold differential forms on X° yields a de Rham–Weil isomorphism $\mathrm{H}^\bullet_{\mathrm{dR,c}}(X^\circ,\mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{c}}(X^\circ,\mathbb{R})$, so that in the end we get a natural isomorphism

$$\psi : \mathrm{H}^{2n-4}_{\mathrm{dR,c}}\big(X^\circ,\mathbb{R}\big) \xrightarrow{\ \sim\ } \mathrm{H}^{2n-4}(X,\mathbb{R}). \tag{6}$$

Now, let $E \to X^\circ$ be an orbifold bundle for the pair (X°,Δ°). We can equip it with an orbifold Hermitian metric h and form the Chern classes $\mathrm{c}_i^{\mathrm{orb}}(E,h)$ which are orbifold differential forms of bidegree (i,i). We can use the isomorphism (6) to define real numbers when $i = 2$. If $\alpha \in \mathrm{H}^{2n-4}(X,\mathbb{R})$, the class $\psi^{-1}(\alpha)$ can be represented by a compactly supported orbifold $(2n-4)$-form Ω on X°, so that $\mathrm{c}_2^{\mathrm{orb}}(E,h) \wedge \Omega$ is a compactly supported orbifold (n,n)-form on X°.

Definition 28. *The orbifold second Chern class $\widetilde{\mathrm{c}}_2(E)$ is the unique element in the dual space $\mathrm{H}^{2n-4}(X,\mathbb{R})^\vee$ which under $\psi^\vee$ corresponds to the Poincaré dual of the class $\mathrm{c}_2^{\mathrm{orb}}(E) \in \mathrm{H}^4_{\mathrm{dR}}(X^\circ,\mathbb{R})$, where the latter is taken with respect to (but independent of) the orbi-structure $\mathscr{C}^\circ$. The quantity*

$$\widetilde{\mathrm{c}}_2(E) \cdot \alpha := \int_{X^\circ} \mathrm{c}_2^{\mathrm{orb}}(E,h) \wedge \Omega$$

is thus a well defined real number for any class $\alpha \in \mathrm{H}^{2n-4}(X,\mathbb{R})$.

Let us apply the above construction to $\Omega^1_{(X^\circ,\Delta^\circ)}$ the orbifold bundle of differential forms. For the first Chern class, one can avoid the use of orbistructures and define it directly as a cohomology class as follows.

Definition 29. *For a klt pair (X,Δ), we set*

$$\widetilde{\mathrm{c}}_1(X,\Delta) := \frac{1}{m}\mathrm{c}_1\big(\big(\omega_X{}^{[m]} \otimes \mathscr{O}_X(m\Delta)\big)^{\vee\vee}\big) \in \mathrm{H}^2(X,\mathbb{R})$$

where $m \geq 1$ is an integer such that the reflexive rank 1 sheaf $\big(\omega_X{}^{[m]} \otimes \mathscr{O}_X(m\Delta)\big)^{\vee\vee}$ is a line bundle.

Now let us consider the case of the second Chern class.

Definition 30. *The orbifold second Chern class $\widetilde{\mathrm{c}}_2(X,\Delta) \in \mathrm{H}^{2n-4}(X,\mathbb{R})^\vee$ of the pair (X,Δ) is the second Chern class of the orbi-bundle $\Omega^1_{(X^\circ,\Delta^\circ)}$ on X° defined in Definition 19.*

Remark 31. As already observed in [24, p. 893], the object constructed in Definition 30 is naturally a homology class:

$$\widetilde{\mathrm{c}}_2(X,\Delta) \in \mathrm{H}_{2n-4}(X,\mathbb{R}).$$

3.2. *The projective case — Mumford's construction*

Let (X,Δ) be a projective dlt pair with standard coefficients such that each component Δ_i of Δ is $\mathbb{Q}$-Cartier. In [31, §3.1, p. 1458], the orbifold Chern classes $\widetilde{\mathrm{c}}_2(X,\Delta)$ and $\widetilde{\mathrm{c}}_1^2(X,\Delta)$ were defined as multilinear forms on $\mathrm{N}^1(X)_{\mathbb{Q}}$. Here we would like to observe that this procedure can also be carried out without the assumption that the Δ_i be $\mathbb{Q}$-Cartier. Our argument follows the proof of [29, Thm. 3.13] closely. We will restrict attention to the case of klt pairs, as we are only concerned with those in this paper.

So let (X,Δ) be an n-dimensional projective klt pair with standard coefficients. Applying Lemma 14, we obtain an open subset $X^\circ \subset X$ whose complement has codimension ≥ 3 and such that $(X^\circ,\Delta|_{X^\circ})$ admits a smooth orbi-étale orbi-structure $\mathscr{C}$. Consider the "big global cover" $\gamma\colon \widehat{X^\circ} \to X^\circ$ associated to $\mathscr{C}$, cf. [41, §§2–3], which up to shrinking X° may be assumed to be Cohen–Macaulay. The locally free orbi-sheaf $\Omega^{[1]}_{\mathscr{C}}$ from Definition 19 induces a genuine locally free sheaf $\mathscr{F}$ on $\widehat{X^\circ}$. The Chern classes of $\mathscr{F}$ induce classes $\mathrm{c}_i\big(\Omega^{[1]}_{\mathscr{C}}\big) \in \mathrm{A}_{n-i}(X^\circ)$. Since $\mathrm{A}_*(X^\circ)$ is equipped with a ring structure, we also have $\mathrm{c}_1^2\big(\Omega^{[1]}_{\mathscr{C}}\big) \in \mathrm{A}_{n-2}(X^\circ)$. For dimensional reasons, $\mathrm{A}_{n-i}(X) \xrightarrow{\sim} \mathrm{A}_{n-i}(X^\circ)$ is an isomorphism for $i \leq 2$. We obtain classes $\mathrm{c}_2\big(\Omega^{[1]}_{\mathscr{C}}\big)$ and $\mathrm{c}_1^2\big(\Omega^{[1]}_{\mathscr{C}}\big) \in \mathrm{A}_{n-2}(X)$, which are independent of the choice of $\mathscr{C}$ by [31, Prop. 3.5]. The orbifold Chern classes $\widetilde{\mathrm{c}}_2(X,\Delta)$ and $\widetilde{\mathrm{c}}_1^2(X,\Delta)$ are then given by cap product with Chern classes of line bundles on X:

$$\widetilde{\mathrm{c}}_2(X,\Delta) \cdot \mathscr{L}_1 \cdots \mathscr{L}_{n-2} := \deg\big(\mathrm{c}_2\big(\Omega^{[1]}_{\mathscr{C}}\big) \cap \mathrm{c}_1(\mathscr{L}_1) \cap \cdots \cap \mathrm{c}_1(\mathscr{L}_{n-2})\big),$$
$$\widetilde{\mathrm{c}}_1^2(X,\Delta) \cdot \mathscr{L}_1 \cdots \mathscr{L}_{n-2} := \deg\big(\mathrm{c}_1^2\big(\Omega^{[1]}_{\mathscr{C}}\big) \cap \mathrm{c}_1(\mathscr{L}_1) \cap \cdots \cap \mathrm{c}_1(\mathscr{L}_{n-2})\big),$$

and these maps factors via $\mathrm{N}^1(X)_{\mathbb{Q}}$.

3.3. *The projective case — cutting down*

If (X,Δ) is a projective klt pair with standard coefficients, then Lemma 14 allows one to generalize Mumford's construction of $\mathbb{Q}$-Chern classes [41] to this setting as explained above. The fact that the Chern–Weil construction from Definition 30 and Mumford's definition of $\mathbb{Q}$-Chern classes are equivalent is given in [24, Claim 6.5] in the case where $\Delta = 0$. It extends readily to the more general setting of klt pairs with standard coefficients.

Since ψ is an abstract isomorphism, it is in practice difficult to actually compute these numbers. There is, however, an important situation where things get much more explicit and that is when $\alpha = \mathrm{c}_1(L)^{n-2}$ where L is an ample line bundle on X (we could also have $(n-2)$ different ample line bundles, but let us stick to the former case for simplicity). By homogeneity of the intersection product, we can assume that L is very ample and induces an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $L \cong i^*\mathscr{O}_{\mathbb{P}^N}(1)$. We pick $(n-2)$ hyperplanes $H_1,\dots,H_{n-2}$ in general position. In particular, one has that $\sum H_i$ has simple normal crossings and $S := H_1 \cap \cdots \cap H_{n-2} \cap X \subset X^\circ$.

Lemma 32. *With the notation as above, the Chern number from Definition 28 can be computed with the following formula:*

$$\widetilde{\mathrm{c}}_2(E)\cdot \mathrm{c}_1(L)^{n-2} = \int_S \mathrm{c}_2^{\mathrm{orb}}(E,h)\big|_S. \tag{7}$$

Proof. To begin with, let us choose sections $s_i \in \mathrm{H}^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)\big)$ such that $H_i = \{s_i = 0\}$, and we equip $\mathscr{O}_{\mathbb{P}^N}(1)$ with the Fubini–Study metric. Next, we choose cut-off functions $\chi_i : \mathbb{P}^N \to [0,1]$ such that

$$\chi_i = \begin{cases} 0 & \text{on} \quad \{|s_i| \le \delta\} \\ 1 & \text{on} \quad \{|s_i| \ge 2\delta\} \end{cases}$$

for some $\delta > 0$ small enough so that

$$\bigcap_{i=1}^{n-2} \{|s_i| \le 2\delta\} \cap X \subset X^\circ.$$

For any $\varepsilon \in (0,1]$, one defines $\varphi_{i,\varepsilon} := \chi_i \log|s_i|^2 + (1-\chi_i)\log(|s_i|^2+\varepsilon^2)$ and set $\omega_{i,\varepsilon} := \omega_{\mathrm{FS}} + \mathrm{dd}^c\varphi_{i,\varepsilon}$. Clearly, $\omega_{i,\varepsilon}$ is supported on $\{|s_i| \le 2\delta\}$ and $\omega_{i,\varepsilon} \to [H_i]$ as $\varepsilon \to 0$, both weakly as currents on $\mathbb{P}^N$ and locally smoothly away from H_i. We set $\Omega_\varepsilon := \bigwedge_{i=1}^{n-2}\omega_{i,\varepsilon}$, which is supported on $\bigcap_{i=1}^{n-2}\{|s_i| \le 2\delta\}$.

The immersion $i : X^\circ \hookrightarrow \mathbb{P}^N$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{2n-4}_{\mathrm{dR}}\big(\mathbb{P}^N,\mathbb{R}\big) & \xrightarrow{\sim} & \mathrm{H}^{2n-4}\big(\mathbb{P}^N,\mathbb{R}\big) \\ \big\downarrow{\scriptstyle i^*} & & \big\downarrow{\scriptstyle i^*} \\ \mathrm{H}^{2n-4}_{\mathrm{dR}}(X^\circ,\mathbb{R}) & \xrightarrow{\sim} & \mathrm{H}^{2n-4}(X^\circ,\mathbb{R}). \end{array}$$

and by our choices the image $i_*[\Omega_\varepsilon]$ lands in the image of the natural map

$$\mathrm{H}^{2n-4}_{\mathrm{dR,c}}\big(X^\circ,\mathbb{R}\big) \to \mathrm{H}^{2n-4}_{\mathrm{dR}}\big(X^\circ,\mathbb{R}\big)$$

and satisfies $\psi(i_*[\Omega_\varepsilon]) = \mathrm{c}_1(\mathscr{O}_{\mathbb{P}^N}(1))^{n-2}|_X = \mathrm{c}_1(L)^{n-2}$. Therefore, we have for any $\varepsilon > 0$ the identity

$$\widetilde{\mathrm{c}}_2(E)\cdot \mathrm{c}_1(L)^{n-2} = \int_{X^\circ} \mathrm{c}_2^{\mathrm{orb}}(E,h)\wedge\Omega_\varepsilon. \tag{8}$$

Now, since $\sum H_i$ has simple normal crossings, an easy local computation shows that Ω_ε converges to the current of integration along the submanifold $W := \bigcap_{i=1}^{n-2} H_i$, both weakly on $\mathbb{P}^N$ and locally smoothly away from W. Since the support of $\Omega_\varepsilon|_X$ is contained in a fixed compact subset of X°, ones sees that $\Omega_\varepsilon|_{X^\circ}$ converges weakly to $[S] = [W \cap X^\circ]$ in the sense of currents on the orbifold X°. Letting ε tend to 0 in (8), we finally get the formula (7). □

3.4. *Orbi-resolutions and Chern numbers*

When X is smooth in codimension two, one can compute Chern numbers on a resolution of singularities, cf. e.g. [15]. In the presence of singularities in codimension two, it is explained in loc. cit. that a resolution does not compute Chern numbers anymore in general. The substitute of a resolution in that setting is an *orbi-resolution* as defined below.

Definition 33 (Orbi-resolutions). *Let (X,Δ) be a pair, where X is a normal complex space, Δ has standard coefficients and let $X^\circ \subset X$ be the orbifold locus of (X,Δ). An orbi-resolution of (X,Δ) is a surjective, proper bimeromorphic map $\pi\colon \widehat{X} \to X$ from a normal complex space $\widehat{X}$ such that:*

(1) $\big(\widehat{X}, \widehat{\Delta} := \pi_*^{-1}(\Delta)\big)$ *has only quotient singularities, and*
(2) π *is isomorphic over X°.*

The existence of orbi-resolutions can be established[1] for quasi-projective varieties (with $\Delta = 0$), using deep results about stacks as Chenyang Xu has showed in [37, §3]. However, the construction proposed there is highly non-canonical (or non-functorial) and this makes it difficult to generalize it to the complex analytic setting, even assuming algebraic singularities.

One important application of the existence of orbi-resolutions is highlighted by the following lemma, which shows that we can use such partial resolutions to compute the orbifold second Chern class of (X,Δ) against a class in $\mathrm{H}^{2n-4}(X,\mathbb{R})$.

Lemma 34. *Let (X,Δ) be a pair as in Setup 5. Assume that (X,Δ) admits an orbi-resolution $\pi\colon (\widehat{X},\widehat{\Delta}) \to (X,\Delta)$ as in Definition 33. Given any $a \in \mathrm{H}^{2n-4}(X,\mathbb{R})$, one has the formula*

$$\widetilde{\mathrm{c}}_2(X,\Delta)\cdot a = \mathrm{c}_2^{\mathrm{orb}}\big(\widehat{X},\widehat{\Delta}\big)\cdot \psi(\pi^* a),$$

where on the right-hand side, $\mathrm{c}_2^{\mathrm{orb}}\big(\widehat{X},\widehat{\Delta}\big) \in \mathrm{H}^4_{\mathrm{dR}}\big(\widehat{X},\mathbb{R}\big)$ is the usual orbifold second Chern class of $(\widehat{X},\widehat{\Delta})$ and $\psi\colon \mathrm{H}^\bullet\big(\widehat{X},\mathbb{R}\big) \to \mathrm{H}^\bullet_{\mathrm{dR}}\big(\widehat{X},\mathbb{R}\big)$ is the orbifold de Rham–Weil isomorphism.

Proof. With the notation from Definition 33, let us denote $\widehat{X}\setminus E := \pi^{-1}(X^\circ)$ and $j\colon \widehat{X}\setminus E \to \widehat{X}$ the natural inclusion; for simplicity we set $k := 2n-4$ and skip the reference to $\mathbb{R}$ in the cohomology spaces below. Finally, we set $\pi_0 := \pi|_{\widehat{X}\setminus E}\colon \widehat{X}\setminus E \to X^\circ$.

We then have the following diagram

$$\begin{tikzcd}
 & & \mathrm{H}^k_{\mathrm{dR}}\big(\widehat{X}\big) \\
\mathrm{H}^k_{\mathrm{dR,c}}\big(\widehat{X}\setminus E\big) \arrow[urr, "j_*^{\mathrm{dR}}"] \arrow[r, "\phi"] & \mathrm{H}^k_{\mathrm{c}}\big(\widehat{X}\setminus E\big) \arrow[r, "j_*"] & \mathrm{H}^k\big(\widehat{X}\big) \arrow[u, "\psi"] \\
\mathrm{H}^k_{\mathrm{dR,c}}(X^\circ) \arrow[u, "(\pi_0^{\mathrm{dR}})^*"] \arrow[r, "\phi"] & \mathrm{H}^k_{\mathrm{c}}(X^\circ) \arrow[u, "\pi_0^*"] \arrow[r, "i_*"] & \mathrm{H}^k(X) \arrow[u, "\pi^*"]
\end{tikzcd}$$

where all arrows except for j_*, j_*^{dR} and π^* are isomorphisms. Now, one can pick an orbifold Hermitian metric $\widehat{h}$ on $T_{\widehat{X},\widehat{\Delta}}$ and descend it to an orbifold Hermitian metric h on T_{X° since π

[1] The proof of [37, Thm. 3] applies verbatim when $\Delta \neq 0$, but we will only use the existence of orbi-resolutions when $\Delta = 0$.

is an isomorphism $\widehat{X} \setminus E \to X^\circ$. Then, if as before α is an orbifold representative of $\phi^{-1}(i_*^{-1}(a))$ with compact support in X°, we have

$$\begin{aligned}\widetilde{\mathrm{c}}_2(X,\Delta)\cdot a &= \int_{X^\circ} \mathrm{c}_2^{\mathrm{orb}}\big(X^\circ, h\big)\wedge\alpha \\ &= \int_{\widehat{X}\setminus E} \mathrm{c}_2^{\mathrm{orb}}\big(\widehat{X},\widehat{h}\big)\wedge\pi^*\alpha \\ &= \mathrm{c}_2^{\mathrm{orb}}\big(\widehat{X},\widehat{\Delta}\big)\cdot[\pi^*\alpha]_{\mathrm{dR}} \\ &= \mathrm{c}_2^{\mathrm{orb}}\big(\widehat{X},\widehat{\Delta}\big)\cdot\psi(\pi^* a)\end{aligned}$$

since we have $\psi(\pi^* a) = (j_*)^{\mathrm{dR}}([\pi^*\alpha]_{\mathrm{dR}})$ from the commutativity of the diagram above. □

We conclude this paragraph with a remark on the non-orbifold locus. For the sake of clarity (and also since we will use only this case), we stick to the case $\Delta = 0$.

If X is a normal complex space that admits an orbi-resolution $\pi\colon \widehat{X} \to X$ in the sense of Definition 33, it is immediate that its non-orbifold locus $X \setminus X^{\mathrm{orb}}$ coincides with $\pi(E)$, where $E \subset \widehat{X}$ is the exceptional locus of π. In particular, the non-orbifold locus is an analytic subset of X. This latter statement is very natural and should be true regardless of the existence of orbi-resolutions. Unfortunately, we are neither able to prove it in the general analytic setting nor able to locate a suitable reference. We can, however, prove it under the additional assumption that the singularities of X are algebraic. This is sufficient for the application in Section 7.

Lemma 35 (Analyticity of the non-orbifold locus). *Let X be a normal complex space having only algebraic singularities (in the sense of [16, Def. 2.4]). Then its non-orbifold locus $Z := X \setminus X^{\mathrm{orb}}$ is a closed analytic subset.*

In particular, this applies if X is a compact klt Kähler space with $\mathrm{c}_1(X) = 0$.

Proof. When X is algebraic, this is a straightforward consequence of [3, Cor. 2.6]. If $U \subset X$ is a euclidean open subset of X being isomorphic through a map $\varphi\colon U \xrightarrow{\sim} V$ to an open subset $V \subset Y$ of an algebraic variety, then we have $\varphi(Z \cap U) = V \setminus V^{\mathrm{orb}}$, and this is an analytic subset of V by the algebraic case. The subset $Z \cap U$ is then given by the vanishing of a family of holomorphic functions, i.e. it is analytic in U.

The last statement is a consequence of [5, Thm. B]: X can be realized as a member of a locally trivial family which also has projective fibers. The family being locally trivial (over a smooth connected base), all the fibers are locally isomorphic and such an X then has locally algebraic singularities (cf. [16, Ex. 2.5]). □

4. Uniformization of canonical models

In this section, we prove Theorem A. Let us first introduce notation. We set $A := K_X + \Delta$ and pick a complete intersection surface $S = D_1 \cap \cdots \cap D_{n-2}$ of $n-2$ general hypersurfaces $D_i \in |mA|$, where m is sufficiently large and divisible. The proof is divided into four steps.

Step 1: The orbi Higgs-sheaf $(\mathscr{E}_X, \vartheta_X)$

Using the notation introduced in the proof of Lemma 14, we can find a (a priori non-smooth) orbi-étale structure $\mathscr{C} = \{U_\alpha, g_\alpha, U'_\alpha\}$ with respect to (X,Δ) on the whole X. Then, one can define the reflexive orbi-Higgs sheaf $(\mathscr{E}_X, \vartheta_X)$ with respect to $\mathscr{C}$ as follows:

$$\vartheta_X\colon \mathscr{E}_X := \Omega^{[1]}_{(X,\Delta)} \oplus \mathscr{O}_{(X,\Delta)} \longrightarrow \mathscr{E}_X \otimes \Omega^{[1]}_{(X,\Delta)}, \tag{9}$$

where on each chart U'_α, we define $\vartheta_{U'_\alpha}(a, f) := (0, a)$ where (a, f) is a section of $\mathscr{E}_{U'_\alpha} := \Omega^{[1]}_{U'_\alpha} \oplus \mathscr{O}_{U'_\alpha}$. Cf. also Definition 19 and [31, §5.1, Step 2].

In order to compute Chern numbers involving $\mathscr{E}_X$, one needs to introduce a global cover $f\colon Y \to X$ and an actual reflexive sheaf $\mathscr{E}_Y$ on Y as we now explain. Thanks to Proposition 13, there exists a finite morphism $f\colon Y \to X$ that is strictly adapted for (X,Δ) and whose extra ramification in codimension one (i.e. away from $\operatorname{supp}(\Delta)$) is supported over a general element H of a very ample linear system on X. Let N be the ramification order along H; we have

$$K_Y = f^*\left(K_X + \Delta + \left(1 - \frac{1}{N}\right)H\right). \tag{10}$$

We set $D := \Delta + \left(1-\frac{1}{N}\right)H$ and define $(X,D)_{\mathrm{orb}}$ to be the largest open subset of X where the pair (X,D) admits a smooth orbi-étale orbi-structure $\mathscr{C}^\circ$; we know that $\operatorname{codim}_X(X \setminus (X,D)_{\mathrm{orb}}) \geq 3$ by Lemma 14. One can be a bit more precise about the shape of $\mathscr{C}^\circ$, which will be useful later. Recall from the proof of Lemma 14 that if we set $K := I \times J$ and $\alpha := (\beta,\gamma) \in K$, then we have a diagram

$$\begin{tikzcd} X'_\alpha \arrow[r, "f_\alpha"] \arrow[rr, bend left, "h_\alpha"] & U'_\alpha \arrow[r, "g_\alpha"] \arrow[d, hook] & U_\alpha \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "\mathrm{id}"] \\ & U'_\beta \arrow[r, "g_\beta"] & U_\beta \arrow[r, hook] & X \end{tikzcd}$$

where X'_α is smooth and f_α is quasi-étale. Note that one can "restrict" $\mathscr{E}_X$ to the orbifold locus $\bigcup_\alpha U_\alpha \subset X$ of (X,Δ) to get a *locally free* orbi-Higgs sheaf with respect to the smooth orbi-étale structure $\{U_\alpha, h_\alpha, X'_\alpha\}_{\alpha\in K}$ for the pair (X,Δ) in codimension two, given by $\mathscr{E}_{X'_\alpha} := f_\alpha^{[*]}(\mathscr{E}_{U'_\beta}|_{U'_\alpha}) \simeq \Omega^1_{X'_\alpha} \oplus \mathscr{O}_{X'_\alpha}$. In particular, one can define the Chern number $\widetilde{c}_2(\mathscr{E}_X)\cdot A^{n-2}$ as explained in Section 3.1.

By choosing H general, one can arrange that $h_\alpha^* H$ is smooth for all indices $\alpha \in K$ thanks to Bertini's theorem, so that a further Kawamata cover $\kappa_\alpha\colon X_\alpha \to X'_\alpha$ orbi-étale with respect to $(X'_\alpha, h_\alpha^*(1-\frac{1}{N})H)$ yields the expected smooth orbi-étale orbi-structure $\mathscr{C}^\circ := \{U_\alpha, p_\alpha, X_\alpha\}_{\alpha\in K}$ for the pair (X,D) in codimension two where $p_\alpha = h_\alpha \circ \kappa_\alpha$. We end up with the following factorization:

$$\begin{tikzcd} X_\alpha \arrow[rr, "p_\alpha"] \arrow[dr, "\kappa_\alpha"'] & & U_\alpha \arrow[r, "\text{étale}"] & X \\ & X'_\alpha \arrow[ur, "h_\alpha"'] & & \end{tikzcd}$$

Next, set

$$Y^\circ := f^{-1}\big((X,D)_{\mathrm{orb}}\big) \cap (Y,\emptyset)_{\mathrm{orb}} \subset Y.$$

Since f is finite, and by Lemma 14 applied to $(Y,\emptyset)$, we have $\operatorname{codim}_Y(Y \setminus Y^\circ) \geq 3$. The map f restricts to $f^\circ\colon Y^\circ \to X^\circ := (X,D)_{\mathrm{orb}}$.

Finally, we set $T := f^{-1}(S)$. Since the linear system $|mA|$ (resp. $f^*|mA|$) is basepoint-free and S is general, we have $S \subset X^\circ$ (resp. $T \subset Y^\circ$). Also, recall from Lemma 20 that $(S, D|_S)$ has quotient singularities. The following diagram summarizes the situation:

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "f|_T"'] & Y^\circ \arrow[r, hook] \arrow[d, "f^\circ"] & Y \arrow[d, "f"] \\ S \arrow[r, hook] & X^\circ \arrow[r, hook] & X \end{tikzcd}$$

Moreover, the ramification formula $K_T = f^*(K_S + D|_S)$ shows that T is klt as well, i.e. it is a surface with quotient singularities.

Step 2: Computing Chern numbers for $\mathscr{E}_X$.

Set $\Delta^\circ := \Delta|_{X^\circ}$ and $D^\circ := D|_{X^\circ}$. Consider the locally free orbi-sheaf for the pair (X°, D°) with respect to the orbi-structure $\mathscr{C}^\circ$ constructed in Step 1 above, defined by

$$\mathscr{E}_{X_\alpha} = \Omega^{[1]}_{(X^\circ,\Delta^\circ,p_\alpha)} \oplus \mathscr{O}_{X_\alpha}. \tag{11}$$

Since $\big(X_\alpha, p_\alpha^{-1}(H)\big)$ is log smooth, the subsheaf $\Omega^{[1]}_{(X^\circ,\Delta^\circ,p_\alpha)} \subset \Omega^1_{X_\alpha}$ has a very explicit expression in terms of local coordinates. More precisely, if $(z_1,\dots,z_n)$ is a local chart such that $p_\alpha^{-1}(H) = \{z_1 = 0\}$ on that chart, then the bundle at play is the subbundle of $\Omega^1_{X_\alpha}$ generated by $z_1^{N-1}\mathrm{d}z_1, \mathrm{d}z_2, \dots, \mathrm{d}z_n$. In particular, it agrees with $\Omega^1_{X_\alpha}$ outside of $p_\alpha^{-1}(H)$.

Now set $\mathscr{E}_Y := \Omega^{[1]}_{(X,\Delta,f)} \oplus \mathscr{O}_Y \subset \Omega^{[1]}_Y \oplus \mathscr{O}_Y$, which we should think of as the reflexive pull back of $\mathscr{E}_X$ by f. We equip this sheaf with the usual Higgs field ϑ_Y, and denote by $\mathscr{E}_{Y^\circ}$ its restriction to Y°. Note that by (2), $\mathscr{E}_Y = \Omega^{[1]}_Y \oplus \mathscr{O}_Y$ holds on $Y \setminus f^{-1}(H)$. Let $\big\{(V_\beta, q_\beta, Y_\beta)\big\}_{\beta\in K}$ be a smooth orbi-étale (i.e. quasi-étale, in this case) orbi-structure for $(Y^\circ, \emptyset)$, which exists by (3) and Lemma 14 again, at least after shrinking Y°. Set $\mathscr{E}_{Y_\beta} := q_\beta^{[*]}\mathscr{E}_Y$ and consider the diagram

$$\begin{tikzcd}
W_{\alpha\beta} \arrow[r, "r_{\alpha\beta}"] \arrow[dd, "g_{\alpha\beta}"'] & Y_\beta \arrow[d, "q_\beta"] \\
 & Y^\circ \arrow[d, "f"] \\
X_\alpha \arrow[r, "p_\alpha"] & X^\circ
\end{tikzcd} \tag{12}$$

where $W_{\alpha\beta}$ is the normalization of $X_\alpha \times_{X^\circ} Y_\beta$. Since p_α is orbi-étale with respect to D°, the map $r_{\alpha\beta}$ is étale over $X^\circ_{\mathrm{reg}} \setminus \operatorname{supp}(D^\circ)$. Moreover, since q_β is quasi-étale, it follows that $f\circ q_\beta$ and p_α ramify to the same order along each component of D. In other words, the smooth orbi-étale orbi-structures $\mathscr{C}^\circ$ and $\big\{\big(f(V_\beta), f\circ q_\beta, Y_\beta\big)\big\}$ are compatible. In particular, $g_{\alpha\beta}$ and $r_{\alpha\beta}$ are étale so that $W_{\alpha\beta}$ is smooth, and we have additionally $g^*_{\alpha\beta}\mathscr{E}_{X_\alpha} \cong r^*_{\alpha\beta}\mathscr{E}_{Y_\beta}$ by (3). Since $\mathscr{E}_{X_\alpha}$ is locally free, so is $\mathscr{E}_{Y_\beta}$, so that the reflexive sheaf $\mathscr{E}_{Y^\circ}$ is a genuine orbifold bundle on the orbifold Y°.

Let ω be an orbifold Kähler metric adapted to (X°, Δ°), as given by Lemma 11. It is defined on an arbitrarily large relatively compact open subset of X°. In particular, it is defined in a neighborhood of S and this will be enough for our purposes. Set $S^* := S_{\mathrm{reg}} \setminus \operatorname{supp} D$. By definition, one has

$$\widetilde{\mathrm{c}}_2\Big(\Omega^{[1]}_{(X,\Delta)}\big|_S\Big) = \int_{S_{\mathrm{reg}}\setminus\operatorname{supp}(\Delta)} \mathrm{c}_2(\Omega^1_{X_{\mathrm{reg}}}, \omega) = \int_{S^*} \mathrm{c}_2(\Omega^1_{X_{\mathrm{reg}}}, \omega)$$

and the last two integrals on the right are well-defined since ω pulls back to a smooth Kähler metric across points in $S_{\mathrm{sing}} \cup \operatorname{supp}(\Delta)$ via the finite maps h_α. The smooth form $p_\alpha^*\omega = f_\alpha^* h_\alpha^*\omega$ is semipositive, degenerate along $p_\alpha^{-1}(H)$. More precisely, if $p_\alpha^{-1}(H) \cap U = \{z_1 = 0\}$ for some coordinate chart $U \subset X_\alpha$, then

$$\begin{aligned}
p_\alpha^*\omega|_U = a_{1\bar{1}}|z_1|^{2(N-1)} i\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \sum_{k=2}^n a_{1\bar{k}} z_1^{N-1} \mathrm{d}z_1 \wedge i\mathrm{d}\bar{z}_k \\
+ \sum_{k=2}^n a_{k\bar{1}} \bar{z}_1^{N-1} \mathrm{d}z_k \wedge i\mathrm{d}\bar{z}_1 + \sum_{j,k=2}^n a_{j\bar{k}} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k
\end{aligned}$$

where $(a_{j\bar{k}})$ is smooth and definite positive. In particular, $p_\alpha^*\omega$ defines a smooth Hermitian metric on $\Omega^{[1]}_{(X^\circ,\Delta^\circ,p_\alpha)}$. Said otherwise, $g^*_{\alpha\beta}p^*_\alpha\omega$ induces a smooth Hermitian metric on $g^*_{\alpha\beta}\Omega^{[1]}_{(X^\circ,\Delta^\circ,p_\alpha)} \cong r^*_{\alpha\beta}\Omega^{[1]}_{(X^\circ,\Delta^\circ,f\circ q_\beta)}$. Hence, $q_\beta^* f^*\omega$ is a smooth Hermitian metric on the vector bundle $\Omega^{[1]}_{(X^\circ,\Delta^\circ,f\circ q_\beta)} = q_\beta^{[*]}\Omega^{[1]}_{(X^\circ,\Delta^\circ,f)}$, so that $f^*\omega$ induces an orbifold metric on the orbi-bundle $\Omega^{[1]}_{(X^\circ,\Delta^\circ,f)}$. By the definition of the Chern classes of orbifold vector bundles, we have

$$\begin{aligned}\widetilde{\mathrm{c}}_2\Big(\Omega^{[1]}_{(X^\circ,\Delta^\circ,f)}\big|_T\Big) &= \int_{f^{-1}(S^*)} \mathrm{c}_2(\Omega^1_{Y_{\mathrm{reg}}}, f^*\omega)\\ &= \deg(f|_T)\cdot\int_{S^*}\mathrm{c}_2(\Omega^1_{X_{\mathrm{reg}}},\omega)\\ &= \deg(f)\cdot\widetilde{\mathrm{c}}_2\Big(\Omega^{[1]}_{(X,\Delta)}\big|_S\Big)\end{aligned}$$

where the last identity follows from $\deg(f|_T) = \deg(f)$ since S is general. All in all, we find by Lemma 32

$$\widetilde{\mathrm{c}}_2(\mathscr{E}_Y)\cdot(f^*A)^{n-2} = \deg(f)\,\widetilde{\mathrm{c}}_2(\mathscr{E}_X)\cdot A^{n-2}. \tag{13}$$

The same arguments show the similar identity

$$\widetilde{\mathrm{c}}_1^2(\mathscr{E}_Y)\cdot(f^*A)^{n-2} = \deg(f)\,\widetilde{\mathrm{c}}_1^2(\mathscr{E}_X)\cdot A^{n-2}. \tag{14}$$

Step 3: (X,Δ) *has quotient singularities*

Consider on X the orbi-Higgs sheaf $(\mathscr{F}_X,\Theta_X) := \mathrm{End}(\mathscr{E}_X,\vartheta_X)$. It satisfies:

$$\widetilde{\mathrm{c}}_1^2(\mathscr{F}_X)\cdot A^{n-2} = \widetilde{\mathrm{c}}_2(\mathscr{F}_X)\cdot A^{n-2} = 0,$$

as follows from the assumption on the Chern classes of (X,Δ), i.e. the assumption that equality holds in (2). Combined with (13)–(14), the latter identity implies that the (genuine) Higgs sheaf $(\mathscr{F}_Y,\Theta_Y) := \mathrm{End}(\mathscr{E}_Y,\vartheta_Y)$ on Y satisfies

$$\widetilde{\mathrm{c}}_1^2(\mathscr{F}_Y)\cdot(f^*A)^{n-2} = \widetilde{\mathrm{c}}_2(\mathscr{F}_Y)\cdot(f^*A)^{n-2} = 0.$$

Moreover, by [31, §4.4, proof of Thm. C], the sheaf $\Omega^{[1]}_{(X,\Delta,f)}$ is (f^*A)-semistable. Recall that $\mathrm{c}_1(\Omega^{[1]}_{(X,\Delta,f)}) = f^*A$ by [17, (3.11.5)]. It follows that $(\mathscr{E}_Y,\vartheta_Y)$ is (f^*A)-Higgs-stable, cf. the calculations in [29, proof of Cor. 7.2]. This in turn implies that the endomorphism sheaf $(\mathscr{F}_Y,\Theta_Y)$ is (f^*A)-Higgs-polystable. Indeed, the last assertion can be deduced from the usual smooth case by restricting to a general complete intersection curve and using the Mehta–Ramanathan theorem for Higgs sheaves [29, Thm. 5.22]. Cf. also [30, Lem. 4.7].

By the Simpson correspondence for klt spaces [30, Thm. 5.1], the Higgs sheaf $(\mathscr{F}_Y,\Theta_Y)\big|_{Y_{\mathrm{reg}}}$ is locally free and is induced by a tame, purely imaginary harmonic bundle. By [30, Prop. 3.17], the reflexive pull-back $g^{[*]}\mathscr{F}_Y$ of $\mathscr{F}_Y$ to a maximally quasi-étale cover $g\colon Z\to Y$ (whose existence is guaranteed by [27, Thm. 1.5]) is locally free.

Now, set $W := X\setminus H\subset X$ and $h := f\circ g\colon Z\to X$. On $h^{-1}(W)$, we have that

$$g^{[*]}\mathscr{E}_Y \cong g^{[*]}\big(\Omega^{[1]}_Y\oplus\mathscr{O}_Y\big)\cong\Omega^{[1]}_Z\oplus\mathscr{O}_Z.$$

It follows that $g^{[*]}\mathscr{F}_Y\cong\mathrm{End}\big(\Omega^{[1]}_Z\oplus\mathscr{O}_Z\big)$, which contains the tangent sheaf $\mathscr{T}_Z$ as a direct summand (again, only on $h^{-1}(W)$). Since direct summands of locally free sheaves are locally free by Nakayama's lemma, the resolution of the Lipman–Zariski Conjecture for klt spaces [20, 25, 26] implies that $h^{-1}(W)$ is smooth.

By construction, the map $h^{-1}(W)\to W$ is branched exactly at $\Delta|_W$. By Corollary 27, its Galois closure $\widetilde{W}\to W$ also has this property, and $\widetilde{W}$ is smooth, being a quasi-étale (hence étale) cover of the smooth space $h^{-1}(W)$. This shows that $(W,\Delta|_W)$ has quotient singularities. So far, we have only imposed that H is general in its (basepoint-free) linear system. We can therefore repeat the argument by choosing general elements $H_1,\dots,H_{n+1}\in|H|$ and conclude that (X,Δ) has quotient singularities. This means that (X,Δ) is a "complex orbifold" in the sense of [10, p. 109].

Step 4: (X,Δ) *is a ball quotient*

Since (X,Δ) is a complex orbifold with $K_X+\Delta$ ample, there is an orbifold Kähler–Einstein metric ω such that $\operatorname{Ric}\omega=-\omega$, cf. [10, Thm. 5.2.2]. Set $X^*:=X_{\mathrm{reg}}\setminus\operatorname{supp}(\Delta)$, so that ω is a genuine Kähler metric on X^*. One can compute the orbifold Chern classes using ω, and, in particular, one has from the usual Chern form computations

$$\begin{aligned}0&=\big(2(n+1)\widetilde{\mathrm{c}}_2(X,\Delta)-n\,\widetilde{\mathrm{c}}_1^2(X,\Delta)\big)\cdot[K_X+\Delta]^{n-2}\\&=\int_{X^*}\big(2(n+1)\mathrm{c}_2(X,\omega)-n\mathrm{c}_1^2(X,\omega)\big)\wedge\omega^{n-2}\\&=C_n\int_{X^*}|\Theta^\circ(T_X,\omega)|^2_\omega\,\omega^n,\end{aligned}$$

where $C_n>0$ is a dimensional constant, while

$$\Theta^\circ(T_X,\omega):=\Theta(T_X,\omega)-\frac{1}{n}\mathrm{tr}_{\mathrm{End}}(\Theta(T_X,\omega))\cdot\mathrm{id}_{T_X}$$

is the trace-free Chern curvature tensor of (T_X,ω).

As a result, ω has constant negative bisectional curvature. This implies that ω has negative Riemannian sectional curvature on X^* by e.g. [23, §2.4.2]. (Note that one could also have said that (X^*,ω) is locally isometric to the complex hyperbolic space $(\mathbb{B}^n,\omega_{\mathrm{hyp}})$ by [9, Thm. 6] and conclude by the usual curvature properties of the complex hyperbolic metric.)

Let $\pi\colon\widetilde{X}_\Delta\to X$ be the orbifold universal cover of (X,Δ), cf. Definition 24. By the previous paragraph, (X,Δ,ω) is an orbifold of nonpositive Riemannian sectional curvature. It then follows from [12, Cor. 2.16 on p. 603] that (X,Δ) is developable. Now, $(\widetilde{X}_\Delta,\pi^*\omega)$ is a simply connected Kähler manifold with constant negative bisectional curvature, so it is holomorphically isometric to $(\mathbb{B}^n,\omega_{\mathrm{hyp}})$ by [34, Thm. 7.9]. In particular, $\widetilde{X}_\Delta\cong\mathbb{B}^n$, proving Theorem A. □

5. Characterization of ball quotients

In this section, we prove Corollary 3. We prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (1) separately.

(1)⇒(2). This is Theorem A.

(2)⇒(3). Let $\pi\colon\mathbb{B}^n\to X$ be the orbifold universal cover of (X,Δ). (In particular, (X,Δ) is developable.) By (2), the map π is Galois, with Galois group $\Gamma\cong\pi_1^{\mathrm{orb}}(X,\Delta)$. Note that $\Gamma\subset\operatorname{Aut}(\mathbb{B}^n)=\mathrm{PU}(1,n)$ is a finitely generated linear group. Furthermore, the stabilizers of the action $\Gamma\circlearrowright\mathbb{B}^n$ are finite by (4). By Selberg's lemma [2], there is a finite index normal subgroup $\Gamma'\subset\Gamma$ which is torsion-free. This implies that Γ' acts freely on $\mathbb{B}^n$. We obtain the following factorization of π:

$$\mathbb{B}^n\longrightarrow\mathbb{B}^n/\Gamma'\xrightarrow{\ f\ }\mathbb{B}^n/\Gamma=X,$$

where f is the quotient by the action of the finite group $G:=\Gamma/\Gamma'$ on the projective manifold $Y:=\mathbb{B}^n/\Gamma'$. Since the first map is étale, it exhibits $\mathbb{B}^n$ as the universal cover of Y. Combining this with the fact that π is branched exactly at Δ, we infer that f is orbi-étale.

(3)⇒(1). Recall that K_Y is ample and that Y satisfies equality in the Miyaoka–Yau inequality, cf. e.g. [35, (8.8.3)]. As $f\colon Y\to X$ is orbi-étale, it follows that also $K_X+\Delta$ is ample and equality likewise holds in the Miyaoka–Yau inequality for (X,Δ). □

6. Uniformization of minimal models

This section has two (related) purposes: first, to remove the assumption about the irreducible components of Δ being $\mathbb{Q}$-Cartier from Theorem 2. And second, to prove Corollary 4.

6.1. *Orbifold Miyaoka–Yau inequality*

In Theorem 2, or more generally in [31, Thm. B], the assumption that the Δ_i be $\mathbb{Q}$-Cartier can be dropped without replacement. We give two proofs of this result, the first one relying on [7] and the second one on Proposition 13.

Theorem 36 (Miyaoka–Yau inequality). *Let (X,Δ) be an n-dimensional projective klt pair with standard coefficients, and assume that $K_X+\Delta$ is big and nef. Then the following inequality holds:*

$$\big(2(n+1)\,\widetilde{c}_2(X,\Delta)-n\,\widetilde{c}_1^2(X,\Delta)\big)\cdot[K_X+\Delta]^{n-2}\geq 0. \tag{15}$$

First proof. Consider a $\mathbb{Q}$-factorialization $f\colon X'\to X$, cf. [7, Cor. 1.4.3] applied with $\mathfrak{E}=\emptyset$. Set $\Delta':=f_*^{-1}\Delta$. The map f is small, meaning that $\mathrm{Exc}(f)\subset X'$ has codimension at least two. Therefore (X',Δ') reproduces all the assumptions made on (X,Δ), and in addition X' is $\mathbb{Q}$-factorial. In particular, $K_{X'}+\Delta'=f^*(K_X+\Delta)$ is big and nef. Furthermore, $f(\mathrm{Exc}(f))\subset X$ has codimension ≥ 3, therefore $f_*\big(\widetilde{c}_2(X',\Delta')\big)=\widetilde{c}_2(X,\Delta)$ as homology classes, and likewise for $\widetilde{c}_1^2(X',\Delta')$ (cf. Remark 31). By the projection formula, we obtain

$$\big(2(n+1)\,\widetilde{c}_2(X,\Delta)-n\,\widetilde{c}_1^2(X,\Delta)\big)\cdot[K_X+\Delta]^{n-2}=\big(2(n+1)\,\widetilde{c}_2(X',\Delta')-n\,\widetilde{c}_1^2(X',\Delta')\big)\cdot[K_{X'}+\Delta']^{n-2}.$$

The right-hand side is non-negative by [31, Thm. B]. □

Second proof. Observe that in [31], the assumption that the Δ_i be $\mathbb{Q}$-Cartier is only used in order to construct a strictly adapted morphism whose extra ramification is supported on a general very ample divisor (cf. Ex. 2.11 of that paper). However, using Proposition 13 we can construct such a cover even without that assumption. After that, the proof of [31, Thm. B] applies verbatim. □

6.2. *Uniformization of minimal models*

In order to prove Corollary 4, we use the strategy explained in [30, Step 1, p. 1086]. This means we first have to prove the following lemma.

Lemma 37. *In the setting of Corollary 4, the canonical model $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ also satisfies equality in* (2).

Assuming Lemma 37 for the moment, we then apply Theorem A on $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ to conclude. This finishes the proof of Corollary 4.

Remark 38. If we had proved Theorem A only in the setting of [31] (that is, assuming that the Δ_i are $\mathbb{Q}$-Cartier), then the above argument would break down. This is because the irreducible components of Δ_{can} may not be $\mathbb{Q}$-Cartier (even if the same is true of Δ).

Proof of Lemma 37. As in the statement of Corollary 4, let $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ denote the canonical model of the pair (X,Δ) and $\pi\colon(X,\Delta)\to(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ the canonical morphism ($K_X+\Delta$ being big and nef, some multiple is basepoint-free and so π is a morphism). By construction, $K_{X_{\mathrm{can}}}+\Delta_{\mathrm{can}}$ is ample and π is crepant:

$$K_X+\Delta=\pi^*\big(K_{X_{\mathrm{can}}}+\Delta_{\mathrm{can}}\big). \tag{16}$$

The pair $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ still has klt singularities. From Theorem 2, we know that the inequality (2) holds for $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ and we are led to checking that:

$$\begin{aligned}&\big(2(n+1)\,\widetilde{c}_2(X,\Delta)-n\,\widetilde{c}_1^2(X,\Delta)\big)\cdot[K_X+\Delta]^{n-2}\\&\qquad\geq\big(2(n+1)\,\widetilde{c}_2(X_{\mathrm{can}},\Delta_{\mathrm{can}})-n\,\widetilde{c}_1^2(X_{\mathrm{can}},\Delta_{\mathrm{can}})\big)\cdot[K_{X_{\mathrm{can}}}+\Delta_{\mathrm{can}}]^{n-2}.\end{aligned} \tag{17}$$

In view of (16), this amounts to showing

$$\widetilde{c}_2(X,\Delta)\cdot[K_X+\Delta]^{n-2}\geq\widetilde{c}_2(X_{\mathrm{can}},\Delta_{\mathrm{can}})\cdot[K_{X_{\mathrm{can}}}+\Delta_{\mathrm{can}}]^{n-2}. \tag{18}$$

At this point, let us consider a general surface $\Sigma \subset X_{\mathrm{can}}$ cut out by the linear system $|m(K_{X_{\mathrm{can}}} + \Delta_{\mathrm{can}})|$ (for $m \gg 1$ sufficiently divisible) and let us look at its preimage $S := \pi^{-1}(\Sigma) \subset X$ in X. The pairs[2] (S, Δ) and $(\Sigma, \Delta_{\mathrm{can}})$ are orbifold surfaces and contained in the orbifold loci of (X, Δ) and $(X_{\mathrm{can}}, \Delta_{\mathrm{can}})$ respectively. Obviously, $(\Sigma, \Delta_{\mathrm{can}})$ is nothing but $(S, \Delta)_{\mathrm{can}}$ and we can apply [40, Thm. 4.2]. This yields

$$4\,\widetilde{c}_2(\Sigma, \Delta_{\mathrm{can}}) - \widetilde{c}_1^2(\Sigma, \Delta_{\mathrm{can}}) \le 4\,\widetilde{c}_2(S, \Delta) - \widetilde{c}_1^2(S, \Delta)\,.$$

The morphism $\pi|_S \colon (S, \Delta) \to (\Sigma, \Delta_{\mathrm{can}})$ being crepant, the above inequality reads as

$$\widetilde{c}_2(\Sigma, \Delta_{\mathrm{can}}) \le \widetilde{c}_2(S, \Delta)\,. \tag{19}$$

With the notation introduced, the inequality (18) boils down to the following:

$$\widetilde{c}_2\big(\mathscr{T}_{(X,\Delta)}\big|_S\big) \ge \widetilde{c}_2\big(\mathscr{T}_{(X_{\mathrm{can}},\Delta_{\mathrm{can}})}\big|_\Sigma\big)\,.$$

This last inequality can be checked as in [30, pp. 1086–1087] by considering the (orbifold) normal sequences

$$0 \longrightarrow \mathscr{T}_{(S,\Delta)} \longrightarrow \mathscr{T}_{(X,\Delta)}\big|_S \longrightarrow \mathscr{N}_{(S,\Delta)|(X,\Delta)} \longrightarrow 0, \tag{20}$$

$$0 \longrightarrow \mathscr{T}_{(\Sigma,\Delta_{\mathrm{can}})} \longrightarrow \mathscr{T}_{(X_{\mathrm{can}},\Delta_{\mathrm{can}})}\big|_\Sigma \longrightarrow \mathscr{N}_{(\Sigma,\Delta_{\mathrm{can}})|(X_{\mathrm{can}},\Delta_{\mathrm{can}})} \longrightarrow 0. \tag{21}$$

It is worth noting that both sequences (20) and (21) are exact sequences of orbifold vector bundles, since the surface S (resp. Σ) is contained in the orbifold locus of (X, Δ) (resp. $(X_{\mathrm{can}}, \Delta_{\mathrm{can}})$) and the terms in the middle are thus genuine orbifold bundles. Now it is enough to remark that the normal bundles $\mathscr{N}_{(S,\Delta)|(X,\Delta)}$ and $\mathscr{N}_{(\Sigma,\Delta_{\mathrm{can}})|(X_{\mathrm{can}},\Delta_{\mathrm{can}})}$ satisfy

$$\mathscr{N}_{(S,\Delta)|(X,\Delta)} \cong \pi^*\big(\mathscr{N}_{(\Sigma,\Delta_{\mathrm{can}})|(X_{\mathrm{can}},\Delta_{\mathrm{can}})}\big)\,. \tag{22}$$

Together with (16) and (19), this finally proves that the inequality (18) holds true. This concludes the proof of Lemma 37. □

Remark. In general, the canonical morphism $\pi|_S \colon (S, \Delta) \to (\Sigma, \Delta_{\mathrm{can}})$ is *not* an orbifold morphism, but the normal bundles are actual locally free sheaves defined on S (resp. on Σ) and not only on the orbifold (S, Δ) (resp. $(\Sigma, \Delta_{\mathrm{can}})$). The Chern classes of $\mathscr{N}_{(\Sigma,\Delta_{\mathrm{can}})|(X_{\mathrm{can}},\Delta_{\mathrm{can}})}$ thus come from Σ and can be pulled back to S in the usual way.

7. Characterization of torus quotients

In this final section, we first establish the positivity of the orbifold second Chern class for Calabi–Yau and for irreducible holomorphic symplectic varieties. Using the Decomposition Theorem [5], we can then easily deduce Theorem 6 and Theorem B. Finally, we prove Corollary 7.

7.1. *Positivity of the second Chern class — the projective case*

If X is projective, then we know that it has an orbi-resolution in the sense of Definition 33, and we can use this to understand the orbifold second Chern class of X.

Proposition 39. *Let X be a projective irreducible Calabi–Yau (resp. irreducible holomorphic symplectic) variety of dimension n with klt singularities and let $\beta \in \mathrm{H}^2(X, \mathbb{R})$ be a Kähler class. Then we have*

$$\widetilde{c}_2(X) \cdot \beta^{n-2} > 0.$$

[2] To avoid cumbersome notation, the restriction of the divisors Δ and Δ_{can} to S and Σ is not written out.

Proof. Let $\pi\colon \widehat{X} \to X$ be an orbi-resolution, whose existence is garanteed by [37] since X is projective. Let $\widehat{\beta}$ be a Kähler class on $\widehat{X}$ and let $\omega \in \beta$ (resp. $\widehat{\omega} \in \widehat{\beta}$) be a Kähler form. Recall that it follows easily from the Bochner principle [16, Thm. A] that T_X is stable with respect to β. This implies that $T_{\widehat{X}}$ is stable with respect to $\pi^*\beta$, hence $T_{\widehat{X}}$ is stable with respect to $\pi^*\beta + \varepsilon\widehat{\beta}$ for $\varepsilon > 0$ small enough, cf e.g. [15, Prop. 3.4]. In particular, as explained in [21, Thm. 4.2], there exists an orbifold Hermite–Einstein metrics h_ε on $T_{\widehat{X}}$ with respect to $\omega_\varepsilon := \pi^*\omega + \varepsilon\widehat{\omega}$. From Lemma 34, we have

$$\widetilde{c}_2(X)\cdot\beta^{n-2} = \lim_{\varepsilon\to 0}\int_{\widehat{X}} c_2^{\mathrm{orb}}(T_{\widehat{X}}, h_\varepsilon)\wedge\omega_\varepsilon^{n-2}.$$

The exact same arguments as in [15, Prop. 3.11] using orbifold forms instead of usual forms shows that the latter quantity is non-negative, and if it is zero, then we have $\widetilde{c}_2(X)\cdot\gamma^{n-2} = 0$ for *any* Kähler class γ on X. We claim that this cannot happen. Indeed, since X is projective, this applies to classes of the form $c_1(H)$ for an ample divisor H on X. Then [38] would imply that X is the quotient of an Abelian variety, clearly a contradiction. □

7.2. *Positivity of the second Chern class — the IHS case*

We will derive the general Kähler case from the projective one using a deformation argument, as in [15, Prop. 4.4].

Proposition 40. *Let X be an irreducible holomorphic symplectic variety of dimension n with klt singularities and let $\beta \in H^2(X,\mathbb{R})$ be a Kähler class. Then we have*

$$\widetilde{c}_2(X)\cdot\beta^{n-2} > 0.$$

Proof. We will first prove that there exists a constant $C_X \in \mathbb{R}$ such that

$$\widetilde{c}_2(X)\cdot a = C_X\, q_X(a)^{\frac{n}{2}-1} \tag{23}$$

for any $a \in H^2(X,\mathbb{R})$, where $q_X\colon H^2(X,\mathbb{R}) \to \mathbb{C}$ is the Beauville–Bogomolov–Fujiki quadratic form. Moreover, we will see that C_X is constant when X moves in a locally trivial family.

The result follows from standard arguments (see e.g. [15, Prop. 4.4] and references therein) once one has proved that the formation of $\widetilde{c}_2(X)\cdot a$ is invariant under parallel transport along a locally trivial deformation, which we now prove.

Let $\pi\colon \mathfrak{X} \to \mathbb{D}$ be a proper surjective map which is a locally trivial deformation of $X = \pi^{-1}(0)$. We denote by $\mathfrak{X}^{\mathrm{orb}}$ (resp. X_t^{orb}) the orbifold locus of $\mathfrak{X}$ (resp. X_t), which is a Zariski open subset of $\mathfrak{X}$ (resp. X_t) according to Lemma 35. Next, we set $Z := \mathfrak{X}\setminus\mathfrak{X}^{\mathrm{orb}}$ and $Z_t = Z\cap X_t$. The family being locally trivial, we infer that $\mathfrak{X}^{\mathrm{orb}}\cap X_t = X_t^{\mathrm{orb}}$ and thus that $Z_t = X_t\setminus X_t^{\mathrm{orb}}$.

Claim 41. *Up to shrinking $\mathbb{D}$, there exists a $\mathscr{C}^\infty$ diffeomorphism $F\colon \mathfrak{X} \to X_0\times\mathbb{D}$ commuting with the projection to $\mathbb{D}$ such that*

(i) *F preserves the orbifold locus, i.e. $F(X_t^{\mathrm{orb}}) = X_0^{\mathrm{orb}}\times\{t\}$.*
(ii) *$F|_{X_t^{\mathrm{orb}}}\colon X_t^{\mathrm{orb}} \to X_0^{\mathrm{orb}}$ is smooth in the orbifold sense.*

In this singular context, we mean that F is the restriction of a smooth map under local embeddings in $\mathbb{C}^N$ which induces an homeomorphism between $\mathfrak{X}$ and $X_0\times\mathbb{D}$.

Proof of Claim 41. Let us start with the existence of the diffeomorphism F. To do so, one can find a proper $\mathscr{C}^\infty$ embedding $\iota\colon \mathfrak{X} \hookrightarrow \mathbb{C}^N$ thanks to [1]. Next, extend π smoothly to a smooth map f with support in a neighborhood of $\iota(X)$. Since $\pi\colon \mathfrak{X}\to\mathbb{D}$ is locally trivial, one can stratify $\mathfrak{X}$ such that the restriction of π to each stratum is proper and smooth (in the analytic sense, i.e. it is a submersion). The existence of F then follows from Thom's first isotopy lemma, cf [39, Prop. 11.1].

In order to prove the two items in the claim, let us briefly recall the construction of F in loc. cit. while emphasizing on the important points for our purposes. Start with local holomorphic trivializations $g_\alpha : U_\alpha \to (U_\alpha \cap X_0) \times \mathbb{D}$ for a covering of analytic open sets $(U_\alpha)_{\alpha\in A}$ of $\mathfrak{X}$, and let $Z = \bigsqcup Z^{(k)}$ be the standard stratification of the analytic set $Z \subset \mathfrak{X}$. The maps g_α induces a local biholomorphism between $Z^{(k)}$ and $Z_0^{(k)} \times \mathbb{D}$ for all k; in particular the holomorphic vector fields $v_\alpha := g_\alpha^* \frac{\partial}{\partial t}$ satisfy

$$v_\alpha\big|_{Z^{(k)}} \in \mathrm{H}^0\Big(Z^{(k)}, \mathscr{T}_{Z^{(k)}}\Big)$$

Next, let (χ_α) be a partition of unity subordinate to the open cover $(U_\alpha)_{\alpha\in A}$. The $\mathscr{C}^\infty$ vector field $v := \sum \chi_\alpha v_\alpha$ still satisfies

$$v|_{Z^{(k)}} \in \mathscr{C}^\infty(Z^{(k)}, T_{Z^{(k)}}).$$

As showed in [39], its flow (F_t) is well-defined over $\pi^{-1}(\mathbb{D}_{1/2})$ for $|t| < 1/2$, and it preserves $Z^{(k)}$ for all k, hence it preserves Z as well. Equivalently, the flow of v preserves $\mathfrak{X}^{\mathrm{orb}}$, which proves (i).

Moreover, $v|_{\mathfrak{X}^{\mathrm{orb}}}$ is smooth in the orbifold sense (i.e. when pulled back to the local smooth covers), a property which need not be true for arbitrary vector fields. This is straightforward since the v_α satisfy this property (they lift to holomorphic vector fields on the quasi-étale local covers), and multiplying by smooth functions is harmless. In order to prove (ii), let $x_0 \in X_0^{\mathrm{orb}}$ be an arbitrary point and let $U \subset \mathfrak{X}^{\mathrm{orb}}$ be a small connected open neighborhood of x_0 admitting a smooth quasi-étale cover $p : \widehat{U} \to U$. We can find $U' \Subset U$ such that for $|t| \le s$ (with $s > 0$ small enough) the flow F_t is defined on U and satisfies $F_t(U') \subset U$. Remember that $\widehat{v} := p^* v|_{U_{\mathrm{reg}}}$ extends to a smooth vector field on $\widehat{U}$ which we still denote by $\widehat{v}$, and whose flow we denote by $\widehat{F}_t$. Since p is étale over U_{reg}, uniqueness of flow ensures that we have a commutative diagram

$$\begin{array}{ccc} p^{-1}(U') & \xrightarrow{\widehat{F}_t} & p^{-1}(F_t(U')) \\ \big\downarrow p & & \big\downarrow p \\ U' & \xrightarrow{F_t} & F_t(U'). \end{array}$$

Indeed, since p is a local diffeomorphism over U_{reg}, we get

$$F_t \circ p = p \circ \widehat{F}_t \text{ on } p^{-1}(U_{\mathrm{reg}}),$$

hence everywhere by continuity of the above maps. In summary, $F_t : U' \to F_t(U')$ is an homeomorphism which therefore lifts to the diffeomorphism $\widehat{F}_t$ between the manifolds $p^{-1}(U')$ and its image $p^{-1}(F_t(U'))$. That is, F_t induces an orbifold diffeomorphism between U' and $F_t(U')$. Item (ii) is now proved. □

Let us now consider the orbifold diffeomorphisms $F_t^{\mathrm{orb}} : X_t^{\mathrm{orb}} \to X_0^{\mathrm{orb}}$, and let h_0 be an orbifold Hermitian metric on $T_{X_0^{\mathrm{orb}}}$. Finally, let α_0 be a closed orbifold form with compact support on X_0^{orb} representing a class $a_0 \in \mathrm{H}^{2n-4}(X_0, \mathbb{R})$. We have

$$\begin{aligned} \widetilde{\mathrm{c}}_2(X_0) \cdot a_0 &= \int_{X_0^{\mathrm{orb}}} \mathrm{c}_2^{\mathrm{orb}}(X_0^{\mathrm{orb}}, h_0) \wedge \alpha_0 \\ &= \int_{X_t^{\mathrm{orb}}} (F_t^{\mathrm{orb}})^* \Big(\mathrm{c}_2^{\mathrm{orb}}(X_0^{\mathrm{orb}}, h_0) \wedge \alpha_0\Big) \\ &= \int_{X_t^{\mathrm{orb}}} \mathrm{c}_2^{\mathrm{orb}}(X_t^{\mathrm{orb}}, (F_t^{\mathrm{orb}})^* h_t) \wedge (F_t^{\mathrm{orb}})^* \alpha_0 \\ &= \widetilde{\mathrm{c}}_2(X_t) \cdot F_t^* a_0 \end{aligned}$$

where the last line comes from the fact that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{2n-4}_{\mathrm{dR,c}}\big(X_t^{\mathrm{orb}}, \mathbb{C}\big) & \xrightarrow{\sim} & \mathrm{H}^{2n-4}(X_t, \mathbb{C}) \\ {\scriptstyle (F_t^{\mathrm{orb}})^*}\big\uparrow & & {\scriptstyle F_t^*}\big\uparrow \\ \mathrm{H}^{2n-4}_{\mathrm{dR,c}}\big(X_0^{\mathrm{orb}}, \mathbb{C}\big) & \xrightarrow{\sim} & \mathrm{H}^{2n-4}(X_0, \mathbb{C}) \end{array}$$

so that (23) is proved.

Finally, we must show that $C_X > 0$. Since C_X is invariant under locally trivial deformation, one can use [6, Cor. 1.3] and [5, Cor. 3.10] to deform X locally trivially to a projective IHS variety Y. Proposition 39 shows that $C_Y > 0$, which concludes the proof of the proposition. □

7.3. *Simultaneous proof of Theorem 6 and Theorem B*

Here we closely follow the arguments from [15, proof of Thm. 5.2].

Let (X,Δ) be as in Setup 5 and such that $\widetilde{c}_1(X,\Delta) = 0$. We denote by $X^\circ := (X,\Delta)_{\mathrm{orb}}$ the open locus where the pair has quotient singularities, and set $\Delta^\circ := \Delta|_{X^\circ}$. It has been proved in [13, Cor. 1.18] that abundance holds for such a pair and in particular $K_X + \Delta$ is torsion. We can then apply Proposition 12 and infer the existence of an orbi-étale map $f\colon Y \to X$ such that

$$\mathscr{O}_Y \cong K_Y \cong f^*(K_X + \Delta).$$

Arguing as in the proof of formula (13), one has:

Lemma 42. *We have the identity*

$$\widetilde{c}_2(Y)\cdot f^*(\alpha)^{n-2} = \deg(f)\,\widetilde{c}_2(X,\Delta)\cdot\alpha^{n-2}. \tag{24}$$

Proof. Let a be an orbifold differential form of degree $2n-4$ with compact support in X° representing α^{n-2} and let h be an orbifold Hermitian metric on $\Omega^1_{(X^\circ,\Delta^\circ)}$. Consider the space $Y^\circ = f^{-1}(X^\circ)$; by taking a fiber product with local smooth charts of X°, it follows easily from purity of branch locus that Y° admits a smooth orbistructure and that f^*h induces an smooth Hermitian metric on Ω_{Y°. In particular, we have

$$\begin{aligned}\widetilde{c}_2(Y)\cdot f^*(\alpha)^{n-2} &= \int_{Y^\circ} c_2(\Omega_{Y^\circ}, f^*h)\wedge f^*a\\ &= \int_{Y^\circ\setminus f^{-1}(\operatorname{supp}\Delta)} c_2(\Omega_{Y^\circ}, f^*h)\wedge f^*a\\ &= \deg(f)\int_{X^\circ\setminus\operatorname{supp}\Delta} c_2(\Omega_{(X^\circ,\Delta^\circ)}, h)\wedge a\\ &= \deg(f)\int_{X^\circ} c_2(\Omega_{(X^\circ,\Delta^\circ)}, h)\wedge a\\ &= \deg(f)\,\widetilde{c}_2(X,\Delta)\cdot\alpha^{n-2},\end{aligned}$$

which proves the lemma. □

Both members of the equation (24) being simultaneously non-negative or zero (and $f^*(\alpha)$ still being a Kähler class on Y), we shall replace X with Y and assume from now on that there is no orbifold structure in codimension one, i.e. that $\Delta = 0$.

By [5, Thm. A], there exists a finite, Galois quasi-étale cover $f\colon X' \to X$ such that $X' \cong T\times\prod_{i\in I} Y_i \times \prod_{j\in J} Z_j$ where T is a torus, Y_i are CY varieties and Z_j are IHS varieties. By [24, Prop. 5.6], we have

$$\widetilde{c}_2(X')\cdot f^*\beta^{n-2} = \deg(f)\,\widetilde{c}_2(X)\cdot\beta^{n-2},$$

while $f^*\beta$ is still a Kähler class by [24, Prop. 3.5]. All in all, there is no loss in generality assuming that $X = X'$ is split, which we do from now on.

Since $H^1(Y_i,\mathbb{R}) = H^1(Z_j,\mathbb{R}) = 0$, the Künneth decomposition on the space $H^2(X,\mathbb{R})$ enables us to write

$$\beta = p_T^*\beta_T + \sum_{i\in I} p_{Y_i}^*\beta_{Y_i} + \sum_{j\in J} p_{Z_j}^*\beta_{Z_j}$$

where β_T, β_{Y_i} and β_{Z_j} are Kähler classes on T, Y_i and Z_j respectively. In particular, we get

$$\widetilde{c}_2(X)\cdot\beta^{n-2}=\sum_{i\in I}\lambda_i\,\widetilde{c}_2(Y_i)\cdot\beta_{Y_i}^{\dim(Y_i)-2}+\sum_{j\in J}\mu_j\,\widetilde{c}_2\big(Z_j\big)\cdot\beta_{Z_j}^{\dim(Z_j)-2},$$

where $\lambda_i,\mu_j>0$ are positive combinatorial coefficients. Proposition 39 and Proposition 40 imply that the above quantity is non-negative, and strictly positive unless $I=J=\emptyset$; i.e. unless $X=T$ is a torus. Theorem 6 and Theorem B are now proved. □

7.4. *Proof of Corollary 7*

To finish, we prove Corollary 7 by proving both implications separately, similar to Corollary 3.

(1)⇒(2). This is what we have just proved in the above lines.

(2)⇒(1). If $f\colon T\to X$ is a Galois orbi-étale map (for the pair (X,Δ)) from a complex torus, the section $(\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n)^{\otimes m}$ is G-invariant, where $G:=\mathrm{Gal}(f)$ and $m:=|G|$. This proves that $m(K_X+\Delta)\sim 0$ and thus that $c_1(K_X+\Delta)=0$. Let ω_T be any Kähler metric on T and let us consider

$$\omega_f:=\sum_{g\in G}g^*\omega_T.$$

It descends to an orbifold Kähler metric ω_X on (X,Δ) and, the map f being orbi-étale, we have:

$$\widetilde{c}_2(X,\Delta)\cdot[\omega_X]^{n-2}=\frac{1}{\deg(f)}\,\widetilde{c}_2(T)\cdot[\omega_f]^{n-2}=0.$$

Since $[\omega_X]$ is a Kähler class, this ends the proof. □

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