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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 799-806

Online since: 17 September 2024

https://doi.org/10.5802/crmath.600

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Comptes Rendus. Mathématique
2024, Vol. 362, p. 799-806
https://doi.org/10.5802/crmath.600

Research article / *Article de recherche*
Numerical analysis / *Analyse numérique*

Some properties of a modified Hilbert transform

Quelques propriétés d'une transformée de Hilbert modifiée

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Abstract. Recently, Steinbach et al. introduced a novel operator $\mathscr{H}_T : L^2(0,T) \to L^2(0,T)$, known as the modified Hilbert transform. This operator has shown its significance in space-time formulations related to the heat and wave equations. In this paper, we establish a direct connection between the modified Hilbert transform $\mathscr{H}_T$ and the canonical Hilbert transform $\mathscr{H}$. Specifically, we prove the relationship $\mathscr{H}_T\varphi = -\mathscr{H}\widetilde{\varphi}$, where $\varphi \in L^2(0,T)$ and $\widetilde{\varphi}$ is a suitable extension of φ over the entire $\mathbb{R}$. By leveraging this crucial result, we derive some properties of $\mathscr{H}_T$, including a new inversion formula, that emerge as immediate consequences of well-established findings on $\mathscr{H}$.

Résumé. Récemment, Steinbach et al. ont introduit un nouvel opérateur $\mathscr{H}_T : L^2(0,T) \to L^2(0,T)$, connu sous le nom de transformée de Hilbert modifiée. Cet opérateur a montré son importance dans les formulations spatio-temporelles liées aux équations de la chaleur et des ondes. Dans cet article, nous établissons un lien direct entre la transformée de Hilbert modifiée $\mathscr{H}_T$ et la transformée de Hilbert canonique $\mathscr{H}$. Plus précisément, nous prouvons la relation $\mathscr{H}_T\varphi = -\mathscr{H}\widetilde{\varphi}$, où $\varphi \in L^2(0,T)$ et $\widetilde{\varphi}$ est une extension appropriée de φ sur l'ensemble de $\mathbb{R}$. En tirant parti de ce résultat crucial, nous déduisons certaines propriétés de $\mathscr{H}_T$, y compris une nouvelle formule d'inversion, qui émergent comme des conséquences immédiates de résultats bien établis sur $\mathscr{H}$.

2020 Mathematics Subject Classification. 44A15.

Funding. This work is partially supported by the INdAM-GNCS project "Metodi numerici per lo studio di strutture geometriche parametriche complesse" (CUP E53C22001930001) and by the MIUR project "Dipartimenti di Eccellenza 2018-2022" (CUP E11G18000350001). Moreover, this research was funded in part by the Austrian Science Fund (FWF) project 10.55776/F65.

Manuscript received 25 September 2023, accepted 15 December 2023.

1. Introduction and main result

In [10], a modified Hilbert transform $\mathscr{H}_T$, associated with the bounded interval $(0,T)$, has been defined. Given the Fourier series of $\varphi \in L^2(0,T)$

$$\varphi(t) = \sum_{k=0}^{\infty} \varphi_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) \quad \text{with} \quad \varphi_k = \frac{2}{T}\int_0^T \varphi(s)\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s}{T}\right)\mathrm{d}s,$$

the modified Hilbert transform is defined as

$$\mathscr{H}_T\varphi(t) = \sum_{k=0}^{\infty} \varphi_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \quad t\in(0,T). \tag{1}$$

This operator has been employed in the context of space-time discretizations of PDEs using both finite element [2, 7, 10] and boundary element [9] methods. It is particularly well-suited because, by defining the Sobolev space

$$H^{1/2}_{0,}(0,T) = \left\{\varphi\in H^{1/2}(0,T) : \varphi(0)=0\right\},$$

it can be shown (see [10]) that $-\partial_t\mathscr{H}_T : H^{1/2}_{0,}(0,T)\to[H^{1/2}_{0,}(0,T)]'$ induces an equivalent norm in $H^{1/2}_{0,}(0,T)$. Furthermore, it holds $\langle\varphi,\mathscr{H}_T\varphi\rangle_{L^2(0,T)}\geq 0$ for all $\varphi\in L^2(0,T)$.

An alternative integral representation of $\mathscr{H}_T$ suited for numerical schemes has been presented in [11, Lemma 2.1].

Lemma 1. *For $\varphi\in L^2(0,T)$, the operator $\mathscr{H}_T$ allows the integral representation*

$$\mathscr{H}_T\varphi(t) = \frac{1}{2T}\mathscr{P}\varphi\int_0^T \varphi(s)\left[\csc\left(\frac{\pi(s+t)}{2T}\right)+\csc\left(\frac{\pi(s-t)}{2T}\right)\right]\mathrm{d}s, \quad t\in(0,T) \tag{2}$$

as Cauchy principal value integral. Moreover, if $\varphi\in H^1(0,T)$ it holds

$$\mathscr{H}_T\varphi(t) = -\frac{2}{\pi}\varphi(0)\log\left(\tan\left(\frac{\pi t}{4T}\right)\right) - \frac{1}{\pi}\int_0^T \partial_t\varphi(s)\log\left(\tan\left(\frac{\pi(s+t)}{4T}\right)\tan\left(\frac{\pi|s-t|}{4T}\right)\right)\mathrm{d}s, \tag{3}$$

for $t\in(0,T)$ as a weakly singular integral.

The Hilbert transform $\mathscr{H}$ of a function φ is defined as Cauchy principal value integral

$$\mathscr{H}\varphi(t) = \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}}\frac{\varphi(s)}{t-s}\,ds,$$

whenever it exists (see [1, Chapter 9], [5] and references therein).

The following relationship between the Hilbert transform $\mathscr{H}$ and its modified version $\mathscr{H}_T$ has been established in [8, Theorem 4.3].

Theorem 2. *For $\varphi\in L^2(0,T)$, it holds*

$$\mathscr{H}_T\varphi = -\mathscr{H}\bar{\varphi} + B\varphi \qquad \text{in } L^2(0,T),$$

where $B : L^2(0,T)\to L^2(0,T)$ is a compact operator, and $\bar{\varphi}$ is the reflection (see Figure 1)

$$\bar{\varphi}(s) = \begin{cases} -\varphi(s+2T) & s\in(-2T,-T),\\ -\varphi(-s) & s\in(-T,0),\\ \varphi(s) & s\in(0,T),\\ \varphi(2T-s) & s\in(T,2T),\\ 0 & \text{else.}\end{cases} \tag{4}$$

In this paper, we prove that $\mathscr{H}_T$ is, in fact, precisely the Hilbert transform $\mathscr{H}$ applied to a specific odd periodic extension with alternating signs.

Theorem 3. *For $\varphi\in L^2(0,T)$, it holds*

$$\mathscr{H}_T\varphi = -\mathscr{H}\widetilde{\varphi} \qquad \text{in } L^2(0,T),$$

where $\widetilde{\varphi}$ is the periodic reflection (see Figure 2)

$$\widetilde{\varphi}(s) = \begin{cases} -\varphi(s+2T-4kT) & s\in((4k-2)T,(4k-1)T),\ k\in\mathbb{Z},\\ -\varphi(4kT-s) & s\in((4k-1)T,4kT),\ k\in\mathbb{Z},\\ \varphi(s-4kT) & s\in(4kT,(4k+1)T),\ k\in\mathbb{Z},\\ \varphi(4kT+2T-s) & s\in((4k+1)T,(4k+2)T),\ k\in\mathbb{Z}.\end{cases} \tag{5}$$

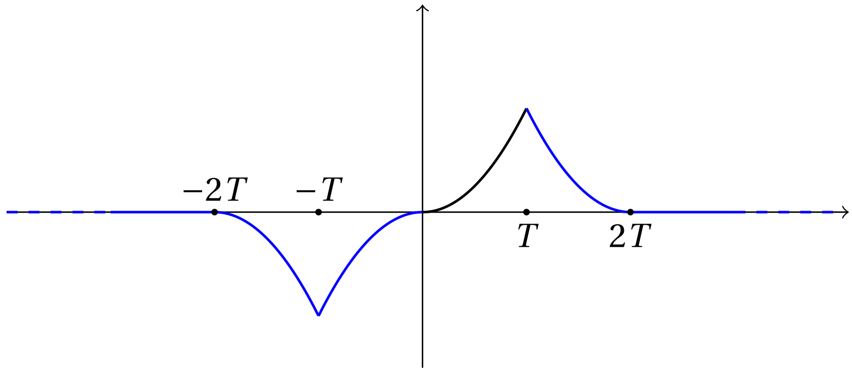

Figure 1. For $\varphi(x) = x^2$, $\varphi : [0, T] \to \mathbb{R}$, it is plotted $\bar{\varphi} : \mathbb{R} \to \mathbb{R}$ defined in (4).

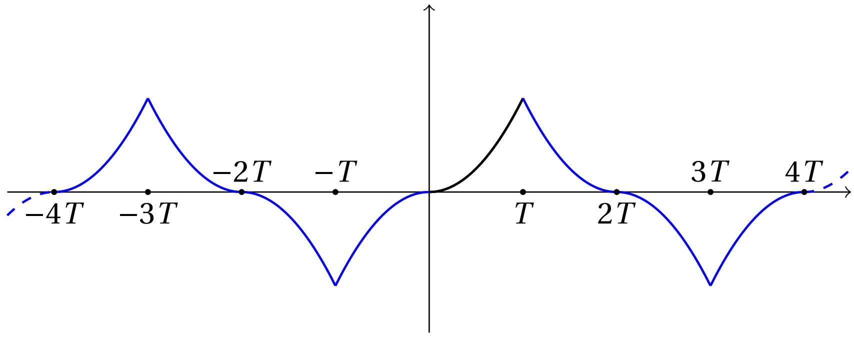

Figure 2. For $\varphi(x) = x^2$, $\varphi : [0, T] \to \mathbb{R}$, it is plotted $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ defined in (5).

Example 4. From (3), we obtain that for $\varphi(s) = 1$ with $s \in (0, T)$, we have $\mathscr{H}_T\varphi(t) = -\frac{2}{\pi}\log|\tan(\frac{\pi t}{4T})|$. It can be readily seen that, for $s \in \mathbb{R}$, $\widetilde{\varphi}(s) = \operatorname{sgn}(\sin(as))$ with $a = \frac{\pi}{2T}$, and for this function, it is well-known that $\mathscr{H}\widetilde{\varphi}(t) = \frac{2}{\pi}\log|\tan(\frac{at}{2})|$ (see [5, Equation 6.9]).

Example 5. A function that satisfies a periodicity of the form (5) is $\varphi(s) = \sin(\pi s)$ with $T = \frac{1}{2}$. In this case, on $\mathbb{R}$, we have the trivial extension $\widetilde{\varphi}(s) = \sin(\pi s)$. It is well-known that $\mathscr{H}\widetilde{\varphi}(t) = -\cos(\pi t)$ (see [5, Equation 1.8]). From (2), we can calculate

$$\begin{aligned}\mathscr{H}_T\varphi(t) &= \text{p.v.}\int_0^{\frac{1}{2}} \sin(\pi s)\big[\csc(\pi(s+t)) + \csc(\pi(s-t))\big]\mathrm{d}s \\ &= \text{p.v.}\int_t^{t+\frac{1}{2}} \frac{\sin(\pi(s-t))}{\sin(\pi s)}\mathrm{d}s + \text{p.v.}\int_{-t}^{\frac{1}{2}-t} \frac{\sin(\pi(s-t))}{\sin(\pi s)}\mathrm{d}s = \cos(\pi t).\end{aligned}$$

2. Proofs of the main result

In this section, we present three distinct proofs of Theorem 3. The first proof relies on the integral representation 2, while the second proof utilizes an alternative definition of the Hilbert transform $\mathscr{H}$ specifically designed for periodic functions. Finally, the third proof employs Fourier series.

2.1. *Proof based on the integral representation*

The first proof is established by leveraging the Laurent series expansion of the cosecant function, as presented in [4, Formula (9.3.30)]

$$\csc(z) = \frac{1}{z} + 2z\sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - k^2\pi^2}. \tag{6}$$

This series converges absolutely and uniformly for $|z| < \pi$, in fact for each $N \in \mathbb{N}$ it holds

$$\left|\csc(z) - \frac{1}{z}\right| \le \frac{2|z|}{\pi^2} \sum_{k=1}^{N} \frac{1}{\left|\left(\frac{z}{k\pi}\right)^2 - 1\right|} \frac{1}{k^2} \le \frac{|z|}{3}.$$

Starting from the definitions of Hilbert transform $\mathscr{H}$ and $\widetilde{\varphi}$ in (5), we can write

$$\begin{aligned}
\mathscr{H}\widetilde{\varphi}(t) &= \frac{1}{\pi}\left[\ldots + \text{p.v.}\int_{-3T}^{-T} \frac{\widetilde{\varphi}(s)}{t-s}\mathrm{d}s + \text{p.v.}\int_{-T}^{T} \frac{\widetilde{\varphi}(s)}{t-s}\mathrm{d}s + \text{p.v.}\int_{T}^{3T} \frac{\widetilde{\varphi}(s)}{t-s}\mathrm{d}s + \ldots\right] \\
&= \frac{1}{\pi}\left[\ldots - \text{p.v.}\int_{-T}^{T} \frac{\widetilde{\varphi}(s)}{t-s-2T}\mathrm{d}s + \text{p.v.}\int_{-T}^{T} \frac{\widetilde{\varphi}(s)}{t-s}\mathrm{d}s - \text{p.v.}\int_{-T}^{T} \frac{\widetilde{\varphi}(s)}{t-s+2T}\mathrm{d}s + \ldots\right]
\end{aligned}$$

where we have used the alternating signs periodicity of $\widetilde{\varphi}$. Therefore, we can compactly write

$$\mathscr{H}\widetilde{\varphi}(t) = \frac{1}{\pi}\sum_{k=-\infty}^{+\infty} \text{p.v.}\int_{-T}^{T} \widetilde{\varphi}(s)\frac{(-1)^k}{t-s+2kT}\mathrm{d}s.$$

By interchanging the integral and the summation (which is possible due to the uniform convergence of the series), and utilizing (6), we obtain

$$\begin{aligned}
\mathscr{H}\widetilde{\varphi}(t) &= \frac{1}{\pi}\text{p.v.}\int_{-T}^{T} \widetilde{\varphi}(s)\left[\sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{t-s+2kT}\right]\mathrm{d}s \\
&= \frac{1}{\pi}\text{p.v.}\int_{-T}^{T} \widetilde{\varphi}(s)\left[\frac{1}{t-s} + 2(t-s)\sum_{k=1}^{+\infty} \frac{(-1)^k}{(t-s)^2 - 4k^2T^2}\right]\mathrm{d}s \\
&= \frac{1}{2T}\text{p.v.}\int_{-T}^{T} \widetilde{\varphi}(s)\csc\left(\frac{\pi(t-s)}{2T}\right)\mathrm{d}s.
\end{aligned}$$

From the latter, we obtain the modified Hilbert transform $\mathscr{H}_T$ in integral form (2) by recalling the definition (5), and writing

$$\begin{aligned}
\mathscr{H}\widetilde{\varphi}(t) &= -\frac{1}{2T}\text{p.v.}\int_{-T}^{0} \varphi(-s)\csc\left(\frac{\pi(t-s)}{2T}\right)\mathrm{d}s + \frac{1}{2T}\text{p.v.}\int_{0}^{T} \varphi(s)\csc\left(\frac{\pi(t-s)}{2T}\right)\mathrm{d}s \\
&= -\frac{1}{2T}\text{p.v.}\int_{0}^{T} \varphi(s)\csc\left(\frac{\pi(t+s)}{2T}\right)\mathrm{d}s - \frac{1}{2T}\text{p.v.}\int_{0}^{T} \varphi(s)\csc\left(\frac{\pi(t-s)}{2T}\right)\mathrm{d}s \\
&= -\mathscr{H}_T\varphi(t).
\end{aligned}$$

2.2. *Proof based on the Hilbert transform for periodic functions*

Given $\varphi \in L^2(0,T)$, we observe that $\widetilde{\varphi}$ as in (5) is actually periodic with a period of $4T$, and it is known that the Hilbert transform $\mathscr{H}$ for a periodic function can be calculated using (see [6]):

$$\mathscr{H}\widetilde{\varphi}(t) = \frac{1}{4T}\text{p.v.}\int_{-2T}^{2T} \widetilde{\varphi}(s)\cot\left(\frac{\pi(t-s)}{4T}\right)\mathrm{d}s. \tag{7}$$

Let us expand the four contributions of the integral above to recover $\mathscr{H}_T\varphi$

$$\begin{aligned}\mathscr{H}\widetilde{\varphi}(t) = &-\frac{1}{4T}\Bigg[\text{p.v.}\int_{-2T}^{-T}\varphi(s+2T)\cot\left(\frac{\pi(t-s)}{4T}\right)\mathrm{d}s - \text{p.v.}\int_{-T}^{0}\varphi(-s)\cot\left(\frac{\pi(t-s)}{4T}\right)\mathrm{d}s \\ &+ \text{p.v.}\int_0^T\varphi(s)\cot\left(\frac{\pi(t-s)}{4T}\right)\mathrm{d}s + \text{p.v.}\int_T^{2T}\varphi(2T-s)\cot\left(\frac{\pi(t-s)}{4T}\right)\mathrm{d}s\Bigg] \\ = &-\frac{1}{4T}\,\text{p.v.}\int_0^T\varphi(s)\cot\left(\frac{\pi(t-s+2T)}{4T}\right)\mathrm{d}s - \frac{1}{4T}\,\text{p.v.}\int_0^T\varphi(s)\cot\left(\frac{\pi(t+s)}{4T}\right)\mathrm{d}s \\ &+ \frac{1}{4T}\,\text{p.v.}\int_0^T\varphi(s)\cot\left(\frac{\pi(t-s)}{4T}\right)\mathrm{d}s + \frac{1}{4T}\,\text{p.v.}\int_0^T\varphi(s)\cot\left(\frac{\pi(t+s-2T)}{4T}\right)\mathrm{d}s.\end{aligned}$$

Finally, using the trigonometric formula

$$2\csc(2x) = -\cot\left(x\pm\frac{\pi}{2}\right) + \cot(x), \quad |x| < \frac{\pi}{2}, \tag{8}$$

we again conclude that $\mathscr{H}_T\varphi = -\mathscr{H}\widetilde{\varphi}$ in $L^2(0,T)$.

2.3. *Proof based on Fourier series*

For f with period $2T$ in $L^2(-T,T)$, the circular Hilbert transform is defined as (see [5, Section 6.4])

$$\mathscr{H}f(t) = \mathrm{i}\sum_{k=1}^{\infty}\left[f_{-k}e^{-\mathrm{i}k\pi\frac{t}{T}} - f_k e^{\mathrm{i}k\pi\frac{t}{T}}\right] \quad \text{with} \quad f_k = \frac{1}{2T}\int_{-T}^{T}f(s)e^{-\mathrm{i}k\pi\frac{s}{T}}\mathrm{d}s.$$

Let us verify that this definition is consistent with the original definition of the modified Hilbert transform $\mathscr{H}_T$ by Steinbach and Zank (1). That is, we will demonstrate that for a given function $\varphi\in L^2(0,T)$, we have $\mathscr{H}\widetilde{\varphi} = -\mathscr{H}_T\varphi$ in $L^2(0,T)$, with $\widetilde{\varphi}$ as in (5). Recalling that $\widetilde{\varphi}$ is a periodic function with period $4T$, let us begin by writing

$$\mathscr{H}\widetilde{\varphi}(t) = \mathrm{i}\sum_{k=1}^{\infty}\left[\widetilde{\varphi}_{-k}e^{-\mathrm{i}k\pi\frac{t}{2T}} - \widetilde{\varphi}_k e^{\mathrm{i}k\pi\frac{t}{2T}}\right],$$

and readily one can verify

$$\begin{aligned}\widetilde{\varphi}_k = \frac{1}{4T}\int_{-2T}^{2T}\widetilde{\varphi}(s)e^{-\mathrm{i}k\pi\frac{s}{2T}}\mathrm{d}s &= \frac{1}{4T}\int_0^{2T}\widetilde{\varphi}(s)\left[e^{-\mathrm{i}k\pi\frac{s}{2T}} - e^{\mathrm{i}k\pi\frac{s}{2T}}\right]\mathrm{d}s \\ &= -\frac{\mathrm{i}}{2T}\int_0^{2T}\widetilde{\varphi}(s)\sin\left(k\pi\frac{s}{2T}\right)\mathrm{d}s = -\widetilde{\varphi}_{-k}.\end{aligned}$$

Hence, we continue

$$\begin{aligned}\mathscr{H}\widetilde{\varphi}(t) &= -\sum_{k=1}^{\infty}\left[e^{-\mathrm{i}k\pi\frac{t}{2T}} + e^{\mathrm{i}k\pi\frac{t}{2T}}\right]\frac{1}{2T}\int_0^{2T}\widetilde{\varphi}(s)\sin\left(k\pi\frac{s}{2T}\right)\mathrm{d}s \\ &= -\sum_{k=1}^{\infty}2\cos\left(k\pi\frac{t}{2T}\right)\frac{1}{2T}\int_0^{2T}\widetilde{\varphi}(s)\sin\left(k\pi\frac{s}{2T}\right)\mathrm{d}s.\end{aligned}$$

Moreover, we can now write

$$\begin{aligned}\frac{1}{2T}\int_0^{2T}\widetilde{\varphi}(s)\sin\left(k\pi\frac{s}{2T}\right)\mathrm{d}s &= \frac{1}{2T}\int_0^{T}\varphi(s)\sin\left(k\pi\frac{s}{2T}\right)\mathrm{d}s + \frac{1}{2T}\int_T^{2T}\varphi(2T-s)\sin\left(k\pi\frac{s}{2T}\right)\mathrm{d}s \\ &= \frac{1}{2T}\int_0^T\varphi(s)\left[\sin\left(k\pi\frac{s}{2T}\right) - \sin(k\pi)\sin\left(k\pi\frac{s}{2T}\right)\right]\mathrm{d}s \\ &= \begin{cases} 0 & k \text{ even}, \\ \dfrac{1}{T}\displaystyle\int_0^T\varphi(s)\sin\left(k\pi\frac{s}{2T}\right)\mathrm{d}s & k \text{ odd}.\end{cases}\end{aligned}$$

Finally, we conclude

$$\mathscr{H}\widetilde{\varphi}(t) = -\sum_{k=0}^{\infty}\cos\left((2k+1)\pi\frac{t}{2T}\right)\frac{2}{T}\int_0^T \widetilde{\varphi}(s)\sin\left((2k+1)\pi\frac{s}{2T}\right)\mathrm{d}s = -\mathscr{H}_T\varphi(t).$$

3. Consequences of the main result

In this section we show simple consequences of Theorem 3.

3.1. *Inversion formula*

For $f \in L^2(-T,T)$ and periodic with period $2T$, the inversion formula holds (see [5, Formula (6.35)])

$$\mathscr{H}^2 f(t) = -f(t) + \frac{1}{2T}\int_{-T}^{T} f(s)\mathrm{d}s, \qquad \text{in } L^2(-T,T). \tag{9}$$

For $\varphi \in L^2(0,T)$, we can calculate

$$\mathscr{H}\left(\mathscr{H}_T\varphi\right)(t) = -\mathscr{H}^2\widetilde{\varphi}(t) = \widetilde{\varphi}(t) - \frac{1}{4T}\int_{-2T}^{2T}\widetilde{\varphi}(s)\mathrm{d}s = \varphi(t), \qquad \text{in } L^2(0,T),$$

since $\widetilde{\varphi}$ is an odd function.

3.2. *Alternative formula*

For an odd function $f \in L^2(-T,T)$ with period $2T$, it can be shown that

$$\mathscr{H}f(t) = \frac{1}{T}\,\text{p.v.}\int_0^T f(s)\frac{\sin\left(\pi\frac{s}{T}\right)}{\cos\left(\pi\frac{s}{T}\right)-\cos\left(\pi\frac{t}{T}\right)}\mathrm{d}s, \qquad \text{in } L^2(-T,T).$$

Therefore, we obtain the alternative formula in $L^2(0,T)$

$$\begin{aligned}\mathscr{H}_T\varphi(t) = -\mathscr{H}\widetilde{\varphi}(t) &= -\frac{1}{2T}\,\text{p.v.}\int_0^{2T}\widetilde{\varphi}(s)\frac{\sin\left(\pi\frac{s}{2T}\right)}{\cos\left(\pi\frac{s}{2T}\right)-\cos\left(\pi\frac{t}{2T}\right)}\mathrm{d}s\\ &= -\frac{\cos\left(\pi\frac{t}{2T}\right)}{T}\,\text{p.v.}\int_0^T\varphi(s)\frac{\sin\left(\pi\frac{s}{2T}\right)}{\cos^2\left(\pi\frac{s}{2T}\right)-\cos^2\left(\pi\frac{t}{2T}\right)}\mathrm{d}s.\end{aligned}$$

This formula can also be deduced from equation (2) using trigonometric identities.

3.3. *Integral representation*

Let suppose that $\varphi \in H^1(0,T)$, then the derivative of $\widetilde{\varphi}$ is

$$\partial_t\widetilde{\varphi}(s)_{|(-2T,2T)} = \begin{cases} -\partial_t\varphi(s+2T) & s\in(-2T,-T),\\ \partial_t\varphi(-s) & s\in(-T,0),\\ \partial_t\varphi(s) & s\in(0,T),\\ -\partial_t\varphi(2T-s) & s\in(T,2T).\end{cases} \tag{10}$$

We note that if $\varphi \in H^1(0,T)$, the only possible discontinuity of $\widetilde{\varphi}_{|(-2T,2T)}$ is in the point $s=0$.

Starting from (7), integrating by parts in the intervals of continuity $(-2T,0)$ and $(0,2T)$ of $\widetilde{\varphi}$, we obtain

$$\begin{aligned}\mathscr{H}_T\varphi(t) = -\mathscr{H}\widetilde{\varphi}(t) &= -\frac{1}{\pi}\int_{-2T}^{2T}\partial_t\widetilde{\varphi}(s)\log\left(\sin\left(\frac{\pi|t-s|}{4T}\right)\right)\mathrm{d}s \\ &\quad+\frac{1}{\pi}\left[\left(\widetilde{\varphi}(0^-)-\widetilde{\varphi}(0^+)\right)\log\left(\sin\left(\frac{\pi t}{4T}\right)\right)+\left(\widetilde{\varphi}(2T^-)-\widetilde{\varphi}(-2T^+)\right)\log\left(\cos\left(\frac{\pi t}{4T}\right)\right)\right] \\ &= -\frac{1}{\pi}\int_{-2T}^{2T}\partial_t\widetilde{\varphi}(s)\log\left(\sin\left(\frac{\pi|t-s|}{4T}\right)\right)\mathrm{d}s-\frac{2}{\pi}\varphi(0)\log\left(\tan\left(\frac{\pi t}{4T}\right)\right).\end{aligned}$$

This formula, in the case $\varphi(0)=0$ is the original form in which Hilbert wrote the Hilbert transform in [3]. Let split the integral in the four usual intervals, and use (10),

$$\begin{aligned}\mathscr{H}_T\varphi(t) &= \frac{1}{\pi}\int_0^T\partial_t\varphi(s)\log\left(\cos\left(\frac{\pi(t+s)}{4T}\right)\right)\mathrm{d}s-\frac{1}{\pi}\int_0^T\partial_t\varphi(s)\log\left(\sin\left(\frac{\pi(t+s)}{4T}\right)\right)\mathrm{d}s \\ &\quad-\frac{1}{\pi}\int_0^T\partial_t\varphi(s)\log\left(\sin\left(\frac{\pi|t-s|}{4T}\right)\right)\mathrm{d}s+\frac{1}{\pi}\int_0^T\partial_t\varphi(s)\log\left(\cos\left(\frac{\pi(t-s)}{4T}\right)\right)\mathrm{d}s \\ &\quad-\frac{2}{\pi}\varphi(0)\log\left(\tan\left(\frac{\pi t}{4T}\right)\right) \\ &= -\frac{1}{\pi}\int_0^T\partial_t\varphi(s)\log\left(\tan\left(\frac{\pi(t+s)}{4T}\right)\tan\left(\frac{\pi|t-s|}{4T}\right)\right)\mathrm{d}s-\frac{2}{\pi}\varphi(0)\log\left(\tan\left(\frac{\pi t}{4T}\right)\right).\end{aligned}$$

We have obtained exactly the result of Lemma 3 in an alternative way.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organisation that could benefit from this article, and have declared no affiliation other than their research organisations.

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