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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 841-849

Online since: 7 October 2024

https://doi.org/10.5802/crmath.602

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Comptes Rendus. Mathématique
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https://doi.org/10.5802/crmath.602

Research article / *Article de recherche*
Number theory / *Théorie des nombres*

Tori and surfaces violating a local-to-global principle for rationality

Tores et surfaces violant le principe local-global pour la rationalité

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Abstract. We show that even within a class of varieties where the Brauer–Manin obstruction is the only obstruction to the local-to-global principle for the existence of rational points (Hasse principle), this obstruction, even in a stronger, base change invariant form, may be insufficient for explaining counter-examples to the local-to-global principle for rationality. We exhibit examples of toric varieties and rational surfaces over an arbitrary global field k each of those, in the absence of the Brauer obstruction to rationality, is rational over all completions of k but is not k-rational.

Résumé. Nous démontrons que même dans une classe des variétés où l'obstruction de Brauer–Manin est la seule obstruction à l'existence de points rationnels (le principe de Hasse) cette obstruction, même sous sa forme la plus forte invariante par rapport au changement de base, peut être insuffisant pour expliquer des contre-exemples au principe local-global pour la rationalité. Nous présentons des exemples de variétés toriques et de surfaces rationnelles sur un corps global arbitraire k dont chacune est rationnelle partout localement mais n'est pas k-rationnelle, en absence d'obstruction de Brauer à la rationalité.

Keywords. Algebraic torus, toric variety, rational surface, conic bundle, rationality, Brauer group.

Mots-clés. Tore algébrique, variété torique, surface rationnelle, fibré en coniques, rationalité, groupe de Brauer.

2020 Mathematics Subject Classification. 14E08, 14G12, 14G25, 14J20, 14J26, 14M20, 11G35, 20G30.

Funding. This research was supported by the ISF grant 1994/20 and accomplished during the author's visit to the IHES (Bures-sur-Yvette). Support of these institutions is gratefully acknowledged.

Manuscript received 9 May 2023, revised 16 December 2023, accepted 22 December 2023.

1. Introduction

This short note is inspired by a recent result by Sarah Frei and Lena Ji [12] where the authors have constructed a smooth, projective, $\mathbb{Q}$-unirational threefold X, a smooth intersection of two quadrics in $\mathbb{P}^5$, such that $X_v := X \times_{\mathbb{Q}} \mathbb{Q}_v$ is $\mathbb{Q}_v$-rational for all places v of $\mathbb{Q}$ but X is not $\mathbb{Q}$-rational (the first version of [12] was conditional on the Birch and Swinnerton-Dyer conjecture for the Jacobian of a certain genus 2 curve, in the second version the proof is unconditional). Additional properties of X are the absence of the Brauer obstruction to rationality (i.e. one has

$\mathrm{Br}(X \times_{\mathbb{Q}} K) = \mathrm{Br}(K)$ for all field extensions $K/\mathbb{Q}$), and the existence of an integral model $\mathscr{X}$ of X whose special fibres $\mathscr{X}_p$ at all odd primes p are $\mathbb{F}_p$-rational.

Colliot-Thélène asked the author whether one can find an example with similar properties among algebraic tori. The first result of the present note consists in an (unconditional) affirmative answer to this question.

Theorem 1. *For any global field k there exists a smooth, projective toric k-variety X for which the Brauer obstruction to rationality is absent, $X_v := X \times_k k_v$ is k_v-rational for all places v of k but X is not k-rational.*

Remark 2. We do not impose any conditions on the reductions because of the presence of disconnected fibres of integral models of tori at the ramified places. The unirationality condition is satisfied automatically since any k-torus is k-unirational [3, 8.13(2)]. One can choose X in Theorem 1 to be of dimension 3.

The second result states that one can do better and reduce the dimension of counter-examples to two.

Theorem 3. *For any global field k of characteristic $\neq 2$ there exists a smooth, projective, k-unirational, geometrically rational k-surface X for which the Brauer obstruction to rationality is absent, $X_v := X \times_k k_v$ is k_v-rational for all places v of k but X is not k-rational.*

It turns out that the celebrated example of a cubic k-surface which is stably k-rational but not k-rational [2] also works in our context.

Remark 4. A simple-minded meaning of Theorems 1 and 3 can be formulated as follows: even within a class of varieties where the Brauer–Manin obstruction is the only obstruction to the local-to-global principle for the existence of rational points (Hasse principle), this obstruction, even in a stronger, base change invariant form, may be insufficient for explaining counter-examples to the local-to-global principle for rationality.

In more technical terms, for X as in each of the above theorems we have

$$\mathrm{Br}(X)/\mathrm{Br}(k) = H^1(k, N) = 0$$

where $N = \mathrm{Pic}(X \times_k \bar{k})$ is the geometric Picard group viewed as a Galois module (for the first equality in the above formula see, e.g., [6, Proposition 5.4.2]). Moreover, this vanishing property holds after any extension K/k of the ground field. However, in the set-up of Theorem 1, there are X for which there exists a subtler obstruction: N is not a stably permutation module thus preventing X from being k-rational or even stably k-rational.

As to examples in Theorem 3, N is a stably permutation module but there is another obstruction to the k-rationality which is of geometric nature. Namely, X is birationally k-equivalent to a conic bundle over the projective line with sufficiently many degenerate fibres, and one can use deep results of Iskovskikh [15] which rely on the classical method of linear systems with base points going back to Segre (see also Manin's book [21]).

To prove the rationality over all completions, both for tori and surfaces we use a theorem of Shafarevich from the inverse Galois theory.

Explicit examples implying the statement of Theorem 1 are contained in Section 3. Section 4 is devoted to the proof of Theorem 3. Section 2 contains some necessary preliminaries.

2. Preliminaries

The monographs [6, 25] and the survey [22] can serve as general references for algebraic tori, Brauer group, and rational surfaces, respectively. Below we collect some basic facts on tori indispensable for our considerations. We also recall a not very well known fact related to Shafarevich's theorem on the realizability of solvable groups as Galois groups of extensions of global fields.

We first fix some standing notation and recall some basic definitions. Unless stated otherwise, throughout below k is an arbitrary field, $\bar{k}$ is a fixed separable closure of k, $\Gamma = \mathrm{Gal}(\bar{k}/k)$ is the absolute Galois group of k. For a k-variety X, we shorten $X \times_k \bar{k}$ to $\overline{X}$.

Remark 5. To justify the above notation, note that the main two objects of our attention in this note, algebraic tori and geometrically rational surfaces, split over a separable closure of the field of definition. For tori this is a well-known fact for which several different proofs have been produced by Ono, Borel, Springer, Tate, Tits; see [26] for a nice overview. For rational surfaces this was proved by Coombes [7].

The (cohomological) Brauer group of a k-variety X is denoted $\mathrm{Br}(X)$. For a smooth non-projective variety V we also consider the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(k(V)/k)$, which is isomorphic to $\mathrm{Br}(X)$ where X is a smooth projective variety containing V as an open subset; see [6] for details.

Remark 6. For the classes of varieties considered in this paper, we have $\mathrm{Br}_{\mathrm{nr}}(k(V)/k) \cong H^1(k, \mathrm{Pic}(\overline{X}))$, where $\mathrm{Pic}(\overline{X})$ is the Picard group of $\overline{X}$ viewed as a Galois module, see, e.g. [6, Propositions 5.4.2 and 6.2.7].

Definition 7. *We say that a k-variety V is*

(i) k-rational *if V is birationally k-equivalent to $\mathbb{A}^n$;*
(ii) stably k-rational *if $V \times \mathbb{A}^m$ is k-rational for some $m \geq 0$;*
(iii) Br-trivial *if $\mathrm{Br}_{nr}(K(V)/K)$ is isomorphic to $\mathrm{Br}(K)$ for all field extensions K/k.*

We have irreversible implications (i) ⇒ (ii) ⇒ (iii). In case (iii), we sometimes say that the Brauer obstruction is absent.

2.1. *Algebraic tori*

Given a k-torus T, the free abelian group $M = \widehat{T} = \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$ viewed as a Γ-module is called the character module of T. The category of k-tori is dual to the category of finitely generated **Z**-free Γ-modules (for the sake of brevity, we just say "modules" throughout below).

Every k-torus splits over a finite Galois field extension L of k, i.e. $T \times_k L$ is isomorphic to the split L-torus $\mathbb{G}_{\mathrm{m},L}^d$, $d = \dim T$. The smallest such L is called the splitting field of T, we denote $\Pi = \mathrm{Gal}(L/k)$ and call it the splitting group of T. Accordingly, we replace Γ-modules with Π-modules.

Definition 8. *We say that a Π-module N is*

(i) permutation *if it has a $\mathbb{Z}$-base permuted by Π;*
(ii) stably permutation *if $N \oplus S_1 \cong S_2$ for some permutation modules S_1, S_2;*
(iii) invertible *if N is a direct summand of a permutation module;*
(iv) H^1-trivial *(aka coflasque) if $H^1(\Pi', N) = 0$ for all subgroups $\Pi' \leq \Pi$;*
(v) flasque *if the dual module $N^\circ := \mathrm{Hom}(N, \mathbb{Z})$ is H^1-trivial;*
(vi) H-trivial *if it is both flasque and coflasque.*

Accordingly, we sometimes say that a torus T is flasque (coflasque) if so is its character module $\widehat{T}$.

We have irreversible implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ∩ (v) = (vi) ⇒ (iv) (or (v)).

Any module M can be embedded into a short exact sequence

$$0 \longrightarrow M \longrightarrow S \longrightarrow F \longrightarrow 0, \tag{1}$$

where S is permutation and F is flasque; such a sequence is called a flasque resolution of M.

Note that if $M = \widehat{T}$ is the character module of a torus T, we have $\mathrm{Br}_{\mathrm{nr}}(k(T)/k) \cong H^1(\Pi, F)$. Indeed, any torus T can be embedded into a smooth projective model X as an open subset (this is true even in positive characteristic, see [4]), the module $F := \mathrm{Pic}(\overline{X})$ is flasque [25, p. 4.6], [5, Proposition 6], and we can argue as in Remark 6. More generally, the properties of F encode the rationality properties of T as follows. Let us add to the list given in Definition 7 one more property: we say that T is

(ii′) *retract rational* if $T \times T'$ is k-rational for some torus T'.

We then have the implications (i) ⇒ (ii) ⇒ (ii′) ⇒ (iii) for the properties listed in Definition 7, all irreversible except possibly for the leftmost one, whose reversibility is a notoriously difficult long-standing problem. The rightmost implication is a consequence of the following relation between the properties in Definitions 7 and 8: T is stably rational (resp. retract rational, resp. Br-trivial) if and only if the module F in a flasque resolution of $M = \widehat{T}$ is stably permutation (resp. invertible, resp. H-trivial).

We shall use the notation $R_{K/k}$ for Weil's restriction of scalars from K to k, and in particular, the kernel of the norm map $R_{K/k}\mathbb{G}_{\mathrm{m},K} \to \mathbb{G}_{\mathrm{m},k}$ will be called a norm torus and denoted $R^1_{K/k}\mathbb{G}_{\mathrm{m}}$.

2.2. *Shafarevich's theorem*

We shall systematically use the following fact contained in the proof of the celebrated Shafarevich's theorem on the realizability of all solvable groups as Galois groups; see [23, Section 9.6] and particularly [24] for details.

Theorem 9 (Shafarevich). *Let k be a global field, and let G be a finite solvable group. Then there exists a Galois field extension K/k with group G such that all decomposition groups G_v are cyclic.*

3. Toric examples

First note that if one drops the requirement of the absence of the Brauer obstruction, the task becomes very easy. Say, the norm torus $R^1_{L/k}\mathbb{G}_{\mathrm{m}}$ corresponding to a biquadratic extension $L = k(\sqrt{a}, \sqrt{b})$ all of whose decomposition groups are cyclic (which is easily achieved with the help of the quadratic reciprocity law) fulfils the job: T is not k-rational but all the $T_v := T \times_k k_v$ are k_v-rational. A well-known particular example is $L = \mathbf{Q}(\sqrt{13}, \sqrt{17})$.

In this section we exhibit three different examples of tori with the trivial Brauer obstruction which violate the local-global principle for rationality. In each example, we first provide a finite solvable group Π and a Π-module M with the needed properties and then choose a Galois extension L/k of a global field k with group Π which satisfies the conditions of Shafarevich's theorem.

Example 10. Let $\Pi = \mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, the nontrivial semidirect product of the cyclic group of order 5 by the cyclic group of order 4. This group is oftentimes denoted by F_{20} and called the Frobenius group of degree 5. It is isomorphic to the group of affine transformations of $\mathbb{F}_5$ and is a part of the family of Frobenius groups.

Choose an element $\sigma \in \Pi$ of order 4 and define $M = \mathbb{Z}[\Pi/\sigma]/\mathbb{Z}$. If L/k is a Galois extension with group Π, then M is the character module of the norm torus $T = R^1_{K/k}\mathbb{G}_{\mathrm{m}}$ corresponding to the extension K/k where $K = L^{\sigma}$.

Let $0 \to M \to S \to F \to 0$ be a flasque resolution of M. It is known (see [5, R4, (d2)], [9, 10]) that the module F is invertible but not stably permutation, so that the torus T is retract k-rational but not stably k-rational. The absence of the Brauer obstruction follows from the retract rationality.

Let now L/k be an F_{20}-extension of global fields all of whose decomposition groups Π_v are cyclic (such an extension exists by Shafarevich's theorem). Then every torus $T_v = T \times_k k_v$ splits over a cyclic extension of degree 2^s5^t, and all such tori are rational [1].

Remark 11. Starting from a global field k, it is not an obvious task to exhibit an explicit F_{20}-extension L/k all of whose decomposition groups are cyclic. If $\operatorname{char}(k) = 0$, one can try to use the generic F_{20}-polynomial constructed by Lecacheux [20].

Although Example 10 is sufficient for our goals, we present another two, each having certain advantages.

Example 12. The torus in Example 10 is not so far from being rational. One can construct an example with the same properties where T is not retract rational, at the expense of small dimension and explicit construction.

The construction is based on the work of Endo and Miyata [11] who classified the finite groups Π such that there exists an H-trivial Π-module which is not invertible. It turned out that this class consists of all groups except those whose p-Sylow subgroups are cyclic for all odd p and the 2-Sylow subgroups are cyclic or dihedral. Thus we can consider the simplest example $\Pi = (\mathbb{Z}/2\mathbb{Z})^3$. By [11, Theorem 2.1], there exists an H-trivial Π-module N which is not invertible. Then any k-torus T such that the module F in a flasque resolution (1) of $M = \widehat{T}$ is isomorphic to N is Br-trivial but not retract rational.

To construct such a T, one can embed F into a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}[\Pi]^r \longrightarrow F \longrightarrow 0$$

with a $\mathbb{Z}$-free module M and consider T with character module $\widehat{T} = M$.

Choosing a Galois extension L/k of global fields with group Π which satisfies Shafarevich's theorem, we conclude that all tori T_v are rational. Indeed, any such torus is split by a quadratic extension of k and is therefore a direct product of tori of dimensions 1 and 2, hence k-rational [25, Section 4.9].

Remark 13. For $k = \mathbb{Q}$, one can easily construct a required extension $L/\mathbb{Q}$ using quadratic reciprocity, say, one can take $L = \mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{89})$.

Remark 14. The construction of the module N presented above is somewhat implicit. To do this in a more or less explicit way, one can use the construction in [11, Section 2].

Namely, let $J = \mathbb{Z}[\Pi]/\mathbb{Z}$, then one can write a flasque resolution of J in the form

$$0 \longrightarrow J \longrightarrow \mathbb{Z}[\Pi]^7 \longrightarrow N_0 \longrightarrow 0, \tag{2}$$

see [11, p. 233]. Let now

$$0 \longrightarrow N_0 \longrightarrow S_0 \longrightarrow N_1 \longrightarrow 0 \tag{3}$$

be a flasque resolution of N_0. By [11, Lemma 2.4], the module N_1 is not invertible.

At this point, it is convenient to use the following general statement, which might be interesting in its own right. It is implicitly contained in a different form in [11], as a parenthetical note in the last paragraph of Section 2 (without proof).

Lemma 15. *Every flasque k-torus is stably k-equivalent to some* Br*-trivial k-torus.*

Proof. Let T be a flasque torus with character module $\widehat{T} = M$. By [11, Lemma 1.1(2)], one can embed M into an exact sequence

$$0 \longrightarrow M \longrightarrow M' \longrightarrow S \longrightarrow 0 \tag{4}$$

with M' coflasque and S permutation. Since M was supposed to be flasque, M' is also flasque, hence H-trivial.

It remains to notice that since S is a permutation module, sequence (4) gives rise to the stable equivalence of tori T and T' corresponding to M and M', respectively. As M' is H-trivial, T' is Br-trivial. □

We can now finish the construction. Let us apply Lemma 15 to the flasque module N_0 appearing in (2) and (3). We obtain an H-trivial module N such that the k-tori T_0 and T corresponding to the modules N_0 and N, respectively, are stably k-equivalent. Hence one can choose a flasque resolution of N of the form

$$0 \longrightarrow N \longrightarrow S_1 \longrightarrow N_1 \longrightarrow 0$$

with N_1 the same as in (3). As the module N_1 is not invertible, so is N. Thus N is an H-trivial, non-invertible Π-module.

Note that in this example $\dim(T)$ is huge (even at the starting point we have the module N_0 of rank 49), in sharp contrast with the 4-dimensional torus in Example 10.

In the next example we exhibit a 3-dimensional torus with the needed properties. One cannot do better from the point of view of dimension because all tori of dimension 1 or 2 are rational by a theorem of Voskresenskiĭ [25, Section 4.9]. However, the construction is a bit more complicated compared to Example 10.

Example 16. The construction is based on the author's paper [16]. As in Example 12, let $\Pi = (\mathbb{Z}/2\mathbb{Z})^3 = \langle \alpha, \beta, \gamma \rangle$. Choose a subgroup of Π of order 4, say, $\Pi_0 = \langle \alpha, \beta \rangle$. Let $I := \ker[\mathbb{Z}[\Pi_0] \to \mathbb{Z}]$ denote the augmentation ideal of $\mathbb{Z}[\Pi_0]$.

Given a field k and a Galois extension L/k with group Π, let $L_0 = k(\sqrt{a}, \sqrt{b}) = L^{\gamma}$, $L_1 = k(\sqrt{c}) = L^{\Pi_0}$. The extensions L_0 and L_1 are linearly disjoint and $L = L_0 L_1 = k(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is their compositum.

Further, denote by T_0 the k-torus with character module I split by L_0. Let $N\colon L_1 \to k$ denote the norm map. We denote by the same letter the norm map

$$N\colon R_{L_1/k}(T_0 \times_k L_1) \longrightarrow T_0 \tag{5}$$

and define $T := \ker N$ (which is isomorhic to the quotient $R_{L_1/k}(T_0 \times_k L_1)/T_0$).

It is convenient to describe this construction translating it into the language of finite groups of integral matrices. Indeed, given any n-dimensional k-torus T_0 with minimal splitting field L_0, where $\mathrm{Gal}(L_0/k) = \Pi_0$, one can attach to the Π_0-module $\widehat{T}$ the finite subgroup W_0 in $\mathrm{GL}(n, \mathbb{Z})$, and to isomorphic tori there correspond conjugate subgroups. If now L_1 is a quadratic extension of k linearly disjoint from L_0, as in our example, then denoting $L = L_0 L_1$ and defining T as the kernel of the map (5), denote by Π the Galois group of L/k and by $W \subset \mathrm{GL}(n, \mathbb{Z})$ the isomorphic image of Π. Then we have $W = W_0 \times \langle \pm I_n \rangle$, where I_n stands for the identity $n \times n$-matrix, see [19, Lemme 4.6].

Applying this observation to our set-up, we conclude that the subgroup W is isomorphic to W_1 on the list of [16, Theorem 1]. In Section 4 of [16] (see also [13, Section 7]) it is shown that T is not retract rational. It is also noted at the end of Section 3 of [16] (see [13] for computer verification of this fact) that T is Br-trivial.

If now we choose an extension L/k of global fields so that all decomposition groups are cyclic, we conclude that all k_v-tori T_v are rational, as in the previous example.

To finish the proof of Theorem 1, it remains to construct a smooth projective k-variety X containing the torus T from any of Examples 10, 12, or 16 as a dense open subset. Recall that such an X exists for any torus defined over any field, see [4]. Being birationally equivalent to T, the k-variety X keeps all rationality properties of T. By the definition of the unramified Brauer group, we have $\mathrm{Br}(X \times_k K) = \mathrm{Br}_{\mathrm{nr}}(K(T)/K) = \mathrm{Br}(K)$ for all extensions K/k. Theorem 1 is proven. □

Remark 17. The examples constructed above give more than stated in Theorem 1, namely they provide a variety X which is not only k-irrational but is not even stably k-rational.

4. Rational surfaces

Before proving Theorem 3, recall that the story started with the affine $\mathbb{Q}$-variety $V \subset \mathbb{A}^3$ given by

$$y^2 - 221z^2 = (x^2 - 13)(x^2 - 17). \tag{6}$$

This example first appeared in Tsfasman's PhD thesis in 1982. It was explored there and in the subsequent papers [8, 18] (the latter was mentioned in the introduction to [12]). Among many interesting properties, a smooth projectivization X of V satisfies the property of being $\mathbb{Q}_p$-rational for all p, $\mathbb{R}$-rational but not $\mathbb{Q}$-rational. However, $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z}$, so this counter-example is explained by the Brauer obstruction.

The surface X arising from equation (6) belongs to a family of Châtelet surfaces. It has a structure of conic bundle over the projective line (by projecting onto the x-coordinate) with 4 degenerate fibres (pairs of transversally intersecting lines) corresponding to the zeros of the polynomial on the right-hand side.

Proof of Theorem 3. The example we are going to use is somewhat similar. We start with the affine cubic surface $V \subset \mathbb{A}^3$ given over an arbitrary global field k with $\mathrm{char}(k) \neq 2$ by

$$y^2 - az^2 = f(x), \tag{7}$$

where a is not a square in k and $f \in k[x]$ is a separable irreducible polynomial of degree 3 with discriminant a. This surface was explored in [2] where it was proved that it is stably k-rational but not k-rational.

The smooth projectivization X of the surface V given by (7) is also a Châtelet surface and also has 4 degenerate fibres (at the zeros of the right-hand side and at infinity). Let L denote the splitting field of f. Then $\mathrm{Gal}(L/k)$, the Galois group of f, is isomorphic to the symmetric group S_3, it acts on the collection $D = \{\ell_1, \bar{\ell}_1, \dots, \ell_4, \bar{\ell}_4\}$ of eight components of the degenerate fibres as $G = \langle (123)c_2c_3, (12)c_3c_4 \rangle$, where $(i\,j)$ swaps the i^{th} and j^{th} degenerate fibres, and c_i swaps the components of the i^{th} fibre, see [17, Theorem 4.19]. The Π-module $\mathrm{Pic}(\overline{X})$ is stably permutation [2, Theorem 2] (this is only one ingredient in the proof that these particular surfaces X are stably k-rational). Therefore, we have $H^1(\Pi', \mathrm{Pic}(\overline{X})) = 0$ for all $\Pi' \subseteq \Pi$, so that $\mathrm{Br}(X \times_k K) = \mathrm{Br}(K)$ for all extensions K/k.

Suppose that all decomposition groups G_v of L/k are cyclic (as S_3 is a solvable group, such an extension L/k exists for any global field k). Then G_v acting on D is conjugate either to a cyclic group $\langle (123)c_2c_3 \rangle$ of order 3, or to a cyclic group $\langle (12)c_3c_4 \rangle$ of order 2. In both cases, the resulting conic bundle k_v-surface X_v is not relatively minimal: in the first case one can blow down the 4^{th} degenerate fibre, and in the second case the first two ones. It remains to apply Iskovskikh's results on the structure and birational properties of conic bundle surfaces, Namely, we conclude that

- each surface X_v is birationally k_v-equivalent to a conic bundle with 2 or 3 degenerate fibres and is hence k_v-rational [14, Theorem 4.1];
- the surface X is a relatively minimal conic bundle with 4 degenerate fibres and hence is not k-rational [15, Theorem 2].

Theorem 3 is proven. □

Remark 18. The surface X from Theorem 3 is birationally k-equivalent to a Del Pezzo surface S of degree 4 with $\mathrm{Pic}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$, see [17, proof of Theorem 4.19]. In the course of this proof it is also shown that X is essentially the only example of a Br-trivial k-irrational surface within this class (it turned out that within this class the module $\mathrm{Pic}(\bar{S})$ is stably permutation if and only if it is H-trivial). However, in Theorem 5.20 of *loc. cit.* it is proven that within the class of Del Pezzo surfaces S of degree 4 with $\mathrm{Pic}(S) \cong \mathbb{Z}$ there are three more examples of Br-trivial k-irrational surfaces (once again, $\mathrm{Pic}(\bar{S})$ is stably permutation if and only if it is H-trivial). All three surfaces are k_v-rational if all decomposition groups of k are cyclic, and Shafarevich's theorem is also applicable in each of the three cases. It is not known whether these surfaces are stably k-rational.

Acknowledgement

I thank Jean-Louis Colliot-Thélène for bringing [12] to my attention, asking the question mentioned in the introduction, and fruitful discussions. I also thank the anonymous referee for useful remarks.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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