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Translational and great Darboux cyclides

Cyclides de Darboux translationnelles et Grandes cyclides de Darboux

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Abstract. A surface that is the pointwise sum of circles in Euclidean space is either coplanar or contains no more than 2 circles through a general point. A surface that is the pointwise product of circles in the unit-quaternions contains either 2, 3, 4, or 5 circles through a general point. A surface in a unit-sphere of any dimension that contains 2 great circles through a general point contains either 4, 5, 6, or infinitely many circles through a general point. These are some corollaries from our classification of translational and great Darboux cyclides. We use the combinatorics associated to the set of low degree curves on such surfaces modulo numerical equivalence.

Résumé. Une surface qui est la somme ponctuelle de cercles dans l'espace euclidien est soit coplanaire, soit ne contient pas plus de 2 cercles passant par un point général. Une surface qui est le produit ponctuel de cercles dans les quaternions unitaires contient soit 2, 3, 4, ou 5 cercles passant par un point général. Une surface dans une sphère unitaire de n'importe quelle dimension qui contient 2 grands cercles passant par un point général contient soit 4, 5, 6, ou une infinité de cercles passant par un point général. Ce sont quelques corollaires de notre classification des cyclides de translation et des cyclides de Darboux. Nous utilisons la combinatoire associée à l'ensemble des courbes de faible degré sur de telles surfaces modulo l'équivalence numérique.

Keywords. real surfaces, pencils of circles, singular locus, Darboux cyclides, Clifford torus, Möbius geometry, elliptic geometry, hyperbolic geometry, Euclidean geometry, Euclidean translations, Clifford translations, unit quaternions, weak del Pezzo surfaces, divisor classes, Néron–Severi lattice.

Mots-clés. surfaces réelles, faisceaux de cercles, lieu singulier, cyclides de Darboux, tore de Clifford, géométrie de Möbius, géométrie elliptique, géométrie hyperbolique, géométrie euclidienne, translations euclidiennes, translations de Clifford, quaternions unitaires, surfaces de del Pezzo faibles, classes de diviseurs, réseau de Néron–Severi.

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1. Introduction

In this article, we characterize surfaces in $\mathbb{R}^3$ that contain at least two circles through each point. Such surfaces are algebraic by [23, Theorem 2] and thus with *surface* we shall mean a real irreducible algebraic surface (see Section 2).

Surfaces that are a union of circles in two different ways have applications in architecture [20], kinematics [13, 17] and geometric modeling in general [1, 11, 19]. In particular, the "Darboux cyclides" have a long history [5, 12], and its various properties are still a topic of recent research [2, 8, 18, 21, 28, 31]. In order to clarify our main result and its relation to [27], we recall some definitions for the non-expert.

An *inversion* with respect to a sphere $O \subset \mathbb{R}^3$ with center c and radius r is the map $f\colon \mathbb{R}^3 \setminus \{c\} \to \mathbb{R}^3 \setminus \{c\}$ such that $\|x-c\| \cdot \|f(x)-c\| = r^2$ and the vectors $x-c$ and $f(x)-c$ are codirected for all $x \in \mathbb{R}^3 \setminus \{c\}$. Such a map exchanges the interior and exterior of O and takes generalized circles to generalized circles, where a *generalized circle* is either a circle or a line. We call two surfaces in $\mathbb{R}^3$ *Möbius equivalent* if one surface is mapped to the other by a composition of inversions.

Let $\mu\colon S^3 \dashrightarrow \mathbb{R}^3$ with $\mu(y) := (y_1, y_2, y_3)/(1-y_4)$ denote the *stereographic projection* from the point $(0,0,0,1)$ on the *3-dimensional unit-sphere* $S^3 \subset \mathbb{R}^4$. The *Möbius degree* of a surface $Z \subset \mathbb{R}^3$ is defined as $\deg \mu^{-1}(Z)$. A surface $Z \subset \mathbb{R}^3$ is called *λ-circled* if the Zariski closure of $\mu^{-1}(Z)$ contains at least $\lambda \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ circles through a general point. If $\lambda \in \mathbb{Z}_{\geq 0}$, then we assume that Z is not $(\lambda+1)$-circled. If $\lambda \geq 2$, then we call Z *celestial*. The Möbius degree and λ are both Möbius invariants.

We may identify the unit-sphere $S^3 \subset \mathbb{R}^4$ with the unit quaternions and we denote the Hamiltonian product by $\star$. We consider the following constructions where A and B are curves in $\mathbb{R}^3$ or S^3:

$$A + B := \left\{ a + b \in \mathbb{R}^3 \,\middle|\, a \in A \text{ and } b \in B \right\},$$
$$A \star B := \left\{ a \star b \in S^3 \,\middle|\, a \in A \text{ and } b \in B \right\}.$$

Suppose that $Z \subset \mathbb{R}^3$ is a surface. We call Z *Bohemian* or *Cliffordian* if there exist generalized circles A and B such that Z is the Zariski closure of $A+B$ and $\mu(A \star B)$, respectively. A surface that is either Bohemian or Cliffordian is called *translational*. If A and B are great circles such that $A \star B \subset S^3$ is a surface, then $A \star B$ is called a *Clifford torus*.

A *Darboux cyclide* in $\mathbb{R}^3$ is a surface of Möbius degree four. A *Q cyclide* is a Darboux cyclide that is Möbius equivalent to a quadric Q. For example, a *CH1 cyclide* is Möbius equivalent to a Circular Hyperboloid of 1 sheet (see Figure 1), where we used the following abbreviations:

E = elliptic/ellipsoid	P = parabolic/paraboloid	O = cone
C = circular	H = hyperbolic/hyperboloid	Y = cylinder

A CO cyclide and CY cyclide is also known as a *spindle cyclide* and *horn cyclide*, respectively. A *ring cyclide*, *Perseus cyclide* or *Blum cyclide* is a Darboux cyclide without real singularities that is 4-circled, 5-circled and 6-circled, respectively (see Figure 1). See Table 5 for a complete list of names for celestial Darboux cyclides.

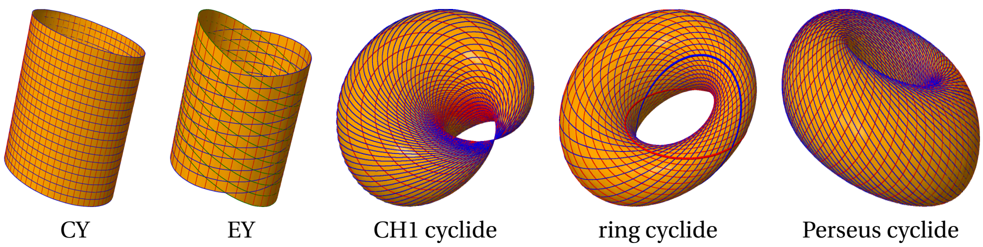


Figure 1. Examples of Darboux cyclides.

It follows from [27, Main Theorem 1.1] that a celestial surface in $\mathbb{R}^3$ is either a Darboux cyclide or Möbius equivalent to a Bohemian or Cliffordian surface. The following question arises: *what* *are the Bohemian and Cliffordian Darboux cyclides?* We shall provide necessary conditions using the combinatorics of divisor classes of curves on such surfaces. We also classify "great" celestial Darboux cyclides; we call a surface $Z \subset \mathbb{R}^3$ *great* if its inverse stereographic projection $\mu^{-1}(Z)$ is covered by great circular arcs.

We will use Theorems A and B and therefore build on [15].

Theorem 1. *Suppose that $Z \subset \mathbb{R}^3$ is a λ-circled surface of Möbius degree d such that $\lambda \geq 2$ and $(d,\lambda) \neq (8,2)$.*

(a) *The surface Z is Bohemian if and only if Z is either a plane, CY or EY.*
(b) *If Z is Cliffordian, then Z is either a Perseus cyclide, ring cyclide or CH1 cyclide. Conversely, if Z is a ring cyclide, then Z is Möbius equivalent to a Cliffordian surface.*
(c) *The surface Z is Möbius equivalent to a great celestial surface if and only if Z is either a plane, sphere, Blum cyclide, Perseus cyclide, ring cyclide, EO cyclide or CO cyclide.*

Remark 2. Theorem 1 (b) should be considered the main result of this article, since Theorem 1 (a) and (c) can be proven using alternative and well-known methods from [3, 21, 28] (see Remarks 33 and 36 for details). However, we propose that our proof methods for (a) and (c) provide additional geometric insight, some of which is made precise by Corollaries 3 to 6 below (see also Remarks 44, 50 and 51).

We summarized Theorem 1 in Table 1.

Table 1. Overview of λ-circled surfaces in $\mathbb{R}^3$ of Möbius degree d that are either Bohemian, Cliffordian, or great and celestial.

name	d	λ	possible types
plane/sphere	2	∞	Bohemian, great
Blum cyclide	4	6	great
Perseus cyclide	4	5	Cliffordian, great
ring cyclide	4	4	Cliffordian, great
CH1 cyclide	4	3	Cliffordian
EY	4	3	Bohemian
CY	4	2	Bohemian
EO cyclide	4	3	great
CO cyclide	4	2	great
	8	2	Bohemian, Cliffordian, great

See Examples 32, 47 and 48 for an example for each row and each possible type. In particular, we consider for $0 \leq i, j \leq 8$ the surface $Z_{ij} \subset \mathbb{R}^3$, which is defined as the Zariski closure of a stereographic projection of the surface

$$\left\{ C_i(\alpha) \star C_j(\beta) \,\middle|\, 0 \leq \alpha, \beta < 2\pi \right\} \subset S^3,$$

where the circle parametrizations $C_i(t)$ are defined in Table 2. We show in Example 32 that Z_{01}, Z_{23} and Z_{45} are a ring cyclide, Perseus cyclide and CH1 cyclide, respectively. The Cliffordian surfaces Z_{06} and Z_{78} are of degree 8 and illustrated in Figure 2.

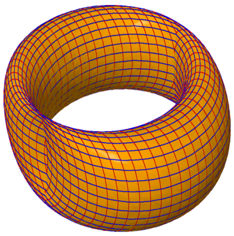
From Theorem 1 and its proof we recover the following four corollaries.

Corollary 3. *A Darboux cyclide is not Möbius equivalent to both a Bohemian surface and a Cliffordian surface.*

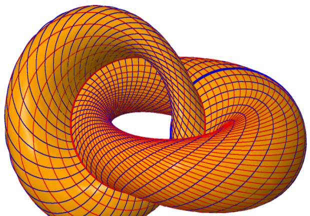
Corollary 4. *If $A, B \subset \mathbb{R}^3$ are circles such that $A + B$ is a non-planar λ-circled surface of Möbius degree d, then $(\lambda, d) = (2, 8)$.*

Table 2. Parametrizations of circles in S^3 with $0 \le t < 2\pi$. Only $C_0(t)$ and $C_1(t)$ define great circles.

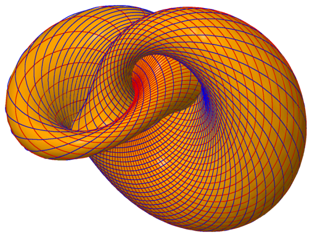
$$C_0(t) := \Big(\cos(t), \sin(t), 0, 0\Big),$$
$$C_1(t) := \tfrac{1}{5}\Big(5\cos(t), 4\sin(t), 3\sin(t), 0\Big),$$
$$C_2(t) := \tfrac{1}{3}\Big(\cos(t), \sin(t), -2, 2\Big),$$
$$C_3(t) := \tfrac{1}{3}\Big(2\cos(t), 2\sin(t), 2, 1\Big),$$
$$C_4(t) := \tfrac{1}{3-2\cos(t)}\Big(-2+2\cos(t), 2\sin(t), 0, 1-2\cos(t)\Big),$$
$$C_5(t) := \tfrac{1}{3+2\cos(t)}\Big(2+2\cos(t), 2\sin(t), 0, 1+2\cos(t)\Big),$$
$$C_6(t) := \tfrac{1}{17+12\cos(t)}\Big(12+8\cos(t), 8\sin(t), 0, 9+12\cos(t)\Big),$$
$$C_7(t) := \tfrac{1}{3-2\cos(t)}\Big(2-2\cos(t), -2\sin(t), 0, 1-2\cos(t)\Big),$$
$$C_8(t) := \tfrac{1}{3+2\cos(t)}\Big(2+2\cos(t), 2\sin(t), 0, 1+2\cos(t)\Big).$$



Bohemian



Cliffordian and great



Cliffordian

Figure 2. 2-circled surfaces of Möbius degree 8 (see Example 32).

Corollary 5. *If $Z \subset S^n$ with $n \ge 3$ is a surface that contains two great circles through a general point and is not contained in a hyperplane section, then $n = 3$ and its stereographic projection $\mu(Z)$ is either a Blum cyclide, Perseus cyclide, or ring cyclide.*

See Example 48 for a Blum cyclide, Perseus cyclide and ring cyclide that contain two great circles through each point. See Figures 1 and 9 for renderings of these cyclides.

Corollary 6. *If $Z \subset \mathbb{R}^3$ is a great ring cyclide, then $Z = \mu(A \star B)$ for some great circles $A, B \subset S^3$. Great Perseus cyclides are not Cliffordian.*

Notice that a ring cyclide is by Corollary 6 Möbius equivalent to the stereographic projection of a Clifford torus.

We conjecture the converse of Theorem 1 (b).

Conjecture 7. *If $Z \subset \mathbb{R}^3$ is either a Perseus cyclide or CH1 cyclide, then Z is Möbius equivalent to a Cliffordian surface.*

Overview

In Section 2, we propose a projective model for Möbius geometry which is a compactification of $\mathbb{R}^3$. In Section 3, we show that the intersection of Bohemian and Cliffordian surfaces with the boundary of this compactification consist of complex lines and/or base points of pencils of circles. In Section 4, we state a classification of possible incidences between complex lines and base points in Darboux cyclides. We use these results in Section 5, Section 6 and Section 7 and obtain a list of all possible candidates for Cliffordian, Bohemian and great Darboux cyclides, respectively. Moreover, we show that each candidate is realized by some example. In Section 8, we conclude the proof for Theorem 1.

2. A projective model for Möbius geometry

We define a *real variety* X to be a complex variety together with an antiholomorphic involution $\sigma\colon X \to X$ called the *real structure* and we denote its real points by

$$X_{\mathbb{R}} := \left\{ p \in X \,\middle|\, \sigma(p) = p \right\}.$$

Such varieties can always be defined by polynomials with real coefficients (see [26, Section I.1] and [24, Section 6.1]). In what follows, points, curves, surfaces and projective spaces $\mathbb{P}^n$ are real algebraic varieties and maps between such varieties are compatible with their real structures unless explicitly stated otherwise. In particular, curves and surfaces in this article are by default reduced and irreducible. We assume that the real structure $\sigma\colon \mathbb{P}^n \to \mathbb{P}^n$ sends x to $(\overline{x_0} : \cdots : \overline{x_n})$.

Let $f\colon X \dashrightarrow Y \subset \mathbb{P}^n$ be a rational map that is not defined at $U \subset X$. By abuse of notation we denote $f(X \setminus U) \subseteq Y$ by $f(X)$. We call f a *morphism* if it is everywhere defined and thus $U = \varnothing$.

We consider the following hyperquadric $\mathbb{S}^3 \subset \mathbb{P}^4$ and three different hyperplane sections $\mathbb{U}, \mathbb{E}, \mathbb{Y} \subset \mathbb{S}^3$:

- *Möbius quadric*: $\mathbb{S}^3 := \left\{ x \in \mathbb{P}^4 \,\middle|\, -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \right\}$,
- *Euclidean absolute*: $\mathbb{U} := \left\{ x \in \mathbb{S}^3 \,\middle|\, x_0 - x_4 = 0 \right\}$,
- *elliptic absolute*: $\mathbb{E} := \left\{ x \in \mathbb{S}^3 \,\middle|\, x_0 = 0 \right\}$, and
- *hyperbolic absolute*: $\mathbb{Y} := \left\{ x \in \mathbb{S}^3 \,\middle|\, x_4 = 0 \right\}$.

The following operators **P**, **C**, **R** and **S** are used to switch between different affine and projective models:

- If $Z \subset \mathbb{R}^n$, then $\mathbf{P}(Z) \subset \mathbb{P}^n$ denotes the Zariski closure of $\iota_n(Z)$, where the embedding $\iota_n\colon \mathbb{R}^n \hookrightarrow \mathbb{P}^n$ sends $(z_1, \ldots, z_n)$ to $(1 : z_1 : \cdots : z_n)$.
- If $Z \subset \mathbb{R}^n$, then $\mathbf{C}(Z) \subset \mathbb{C}^n$ denotes the Zariski closure of the embedding of Z into $\mathbb{C}^n$ via the standard embedding $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$.
- If $C \subset \mathbb{P}^n$, then $\mathbf{R}(C) \subset \mathbb{R}^n$ is defined as $\iota_n^{-1}(\{x \in C_{\mathbb{R}} \mid x_0 \neq 0\})$, where $\iota_n^{-1}\colon \mathbb{P}^n_{\mathbb{R}} \dashrightarrow \mathbb{R}^n$ sends $(x_0 : \cdots : x_n)$ to $(x_1, \ldots, x_n)/x_0$.
- If $Z \subset \mathbb{R}^3$, then $\mathbf{S}(Z) \subset \mathbb{S}^3$ is defined as $\mathbf{P}(\mu^{-1}(Z))$, where $\mu\colon S^3 \dashrightarrow \mathbb{R}^3$ is the stereographic projection at Section 1.

If $\mathbf{R}(\{a\}) = \{b\}$ for $a \in \mathbb{P}^n$, then we write $\mathbf{R}(a) = b$ instead.

Remark 8. We observe that, $\mathbf{P}(S^3) = \mathbb{S}^3$, $\mathbf{R}(\mathbb{S}^3) = S^3$, $\mathbf{S}(\mathbb{R}^3) = \mathbb{S}^3$, $\mathbf{R}(\mathbb{E}) = \mathbb{E}_{\mathbb{R}} = \varnothing$ and $\mathbf{R}(\mathbb{Y}) = S^2$. The quadrics $\mathbb{S}^3$, $\mathbb{E}$ and $\mathbb{Y}$ are smooth, and $\mathbb{U}$ is a quadratic cone with vertex in $\mathbb{U}_{\mathbb{R}} = \{(1:0:0:0:1)\}$.

We consider the following linear projections from $\mathbb{S}^3$ to $\mathbb{P}^3$:

- *stereographic projection* $\pi\colon \mathbb{S}^3 \dashrightarrow \mathbb{P}^3$, $\pi(x) := (x_0 - x_4 : x_1 : x_2 : x_3)$,
- *central projection* $\tau\colon \mathbb{S}^3 \to \mathbb{P}^3$, $\tau(x) := (x_1 : x_2 : x_3 : x_4)$, and
- *vertical projection* $\nu\colon \mathbb{S}^3 \to \mathbb{P}^3$, $\nu(x) := (x_0 : x_1 : x_2 : x_3)$.

Remark 9. The stereographic projection π corresponds via **P** with $\mu\colon S^3 \dashrightarrow \mathbb{R}^3$ as defined in Section 1. The projection center of π lies in $\mathbb{U}_{\mathbb{R}}$. Moreover, π defines a biregular isomorphism $\mathbb{S}^3 \setminus \mathbb{U} \cong \pi(\mathbb{S}^3 \setminus \mathbb{U})$ and $\pi(\mathbb{U}) = \left\{ x \in \mathbb{P}^3 \,\middle|\, x_0 = x_1^2 + x_2^2 + x_3^2 = 0 \right\}$ is an irreducible conic in $\mathbb{P}^3$ without real points. The central and vertical projections define 2:1 morphisms with ramification locus $\mathbb{E}$ and $\mathbb{Y}$, respectively. The branching loci $\tau(\mathbb{E})$ and $\nu(\mathbb{Y})$ are quadrics in $\mathbb{P}^3$. The central projection $\tau\colon \mathbb{S}^3 \to \mathbb{P}^3$ corresponds via $\mathbf{R}$ to a 2:1 linear map $S^3 \to \mathbb{R}^3$ whose fibers are antipodal points. For intuition, we remark that a linear projection $S^1 \to \mathbb{R}$ of the unit circle $S^1 \subset \mathbb{R}^2$ is via $\mathbf{P}$ a 1-dimensional analogue of π, τ and ν, if the center lies either on S^1, in the interior of S^1, or in the exterior of S^1, respectively.

The following two complex maps will be used for defining "translations" of $\mathbb{S}^3$:

- $\zeta_b\colon \mathbb{P}^4 \to \mathbb{P}^4$ with $b = (b_1, b_2, b_3) \in \mathbb{C}^3$ is the linear transformation corresponding to the following 5×5 matrix, where $\Delta := \frac{1}{2}(b_1^2 + b_2^2 + b_3^2)$:

$$\begin{pmatrix} 1+\Delta & b_1 & b_2 & b_3 & -\Delta \\ b_1 & 1 & 0 & 0 & -b_1 \\ b_2 & 0 & 1 & 0 & -b_2 \\ b_3 & 0 & 0 & 1 & -b_3 \\ \Delta & b_1 & b_2 & b_3 & 1-\Delta \end{pmatrix}.$$

- $\_\,\widehat{\star}\,\_\colon \mathbb{S}^3 \times \mathbb{S}^3 \dashrightarrow \mathbb{S}^3$ is the rational map defined by

$$(x,y) \mapsto (x_0y_0 : x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 : x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 :$$
$$x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2 : x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1).$$

We consider the following complex transformations of $\mathbb{S}^3$, where $\operatorname{Aut}_{\mathbb{C}}\mathbb{P}^4$ denotes the complex projective transformations of $\mathbb{P}^4$ and $H \in \{\mathbb{U}, \mathbb{E}, \mathbb{Y}\}$:

$$\begin{aligned} \operatorname{Aut}_{\mathbb{C}}\mathbb{S}^3 &:= \{\varphi \in \operatorname{Aut}_{\mathbb{C}}\mathbb{P}^4 \mid \varphi(\mathbb{S}^3) = \mathbb{S}^3\}, \\ \operatorname{Aut}_H\mathbb{S}^3 &:= \{\varphi \in \operatorname{Aut}_{\mathbb{C}}\mathbb{S}^3 \mid \varphi(H) = H\}, \\ \operatorname{UT}\mathbb{S}^3 &:= \{\zeta_b\colon \mathbb{P}^4 \to \mathbb{P}^4 \mid b \in \mathbb{C}^3\}, \\ \operatorname{LT}\mathbb{S}^3 &:= \{\varphi\colon \mathbb{S}^3 \dashrightarrow \mathbb{S}^3 \mid \varphi(x) = p\,\widehat{\star}\,x,\ p \in \mathbb{S}^3 \setminus \mathbb{E}\}, \text{ and} \\ \operatorname{RT}\mathbb{S}^3 &:= \{\varphi\colon \mathbb{S}^3 \dashrightarrow \mathbb{S}^3 \mid \varphi(x) = x\,\widehat{\star}\,p,\ p \in \mathbb{S}^3 \setminus \mathbb{E}\}. \end{aligned}$$

The *Möbius transformations* are defined as

$$\operatorname{Aut}\mathbb{S}^3 := \{\varphi \in \operatorname{Aut}_{\mathbb{C}}\mathbb{S}^3 \mid \varphi\circ\sigma = \sigma\circ\varphi\},$$

where $\sigma\colon \mathbb{P}^4 \to \mathbb{P}^4$ denotes the real structure. The *Euclidean transformations, elliptic transformations* and *hyperbolic transformations* of $\mathbb{S}^3$, are defined as

$$\operatorname{Aut}_{\mathbb{U}}\mathbb{S}^3 \cap \operatorname{Aut}\mathbb{S}^3, \quad \operatorname{Aut}_{\mathbb{E}}\mathbb{S}^3 \cap \operatorname{Aut}\mathbb{S}^3 \quad \text{and} \quad \operatorname{Aut}_{\mathbb{Y}}\mathbb{S}^3 \cap \operatorname{Aut}\mathbb{S}^3, \quad \text{respectively.}$$

The *Euclidean translations, left Clifford translations* and *right Clifford translations* are defined as

$$\operatorname{UT}\mathbb{S}^3 \cap \operatorname{Aut}\mathbb{S}^3, \quad \operatorname{LT}\mathbb{S}^3 \cap \operatorname{Aut}\mathbb{S}^3 \quad \text{and} \quad \operatorname{RT}\mathbb{S}^3 \cap \operatorname{Aut}\mathbb{S}^3, \quad \text{respectively.}$$

The *left generator* and *right generator* that pass through $p \in \mathbb{E}$ are defined as

$$\mathscr{L}_p := \{q\,\widehat{\star}\,p \mid q \in \mathbb{S}^3 \setminus \mathbb{E}\} \quad \text{and} \quad \mathscr{R}_p := \{p\,\widehat{\star}\,q \mid q \in \mathbb{S}^3 \setminus \mathbb{E}\}, \quad \text{respectively.}$$

We shall refer to the complex lines in $\mathbb{U}$ as *generators*.

The following proposition is classical and concerns translations in elliptic geometry (see [4, Section 7.9 and 7.93]). Our proof is based on [25, Proposition 1]. Recall from Section 1 that $\_\star\_\colon S^3 \times S^3 \to S^3$ denotes the Hamiltonian product for the unit quaternions.

Proposition 10.

(a) $\mathbf{R}(x\,\widehat{\star}\,y) = \mathbf{R}(x) \star \mathbf{R}(y)$ *for all* $x, y \in \mathbb{S}^3_{\mathbb{R}}$.
(b) $\operatorname{LT}\mathbb{S}^3, \operatorname{RT}\mathbb{S}^3 \subset \operatorname{Aut}_{\mathbb{E}}\mathbb{S}^3$.
(c) *For all* $p \in \mathbb{E}$, *the generators* $\mathscr{L}_p$ *and* $\mathscr{R}_p$ *are the two complex lines in* $\mathbb{E}$ *containing* p.
(d) *For all* $\varphi \in \operatorname{LT}\mathbb{S}^3$ *and* $p \in \mathbb{E}$, *we have* $\varphi(\mathscr{L}_p) = \mathscr{L}_p$.
For all $\varphi \in \operatorname{RT}\mathbb{S}^3$ *and* $p \in \mathbb{E}$, *we have* $\varphi(\mathscr{R}_p) = \mathscr{R}_p$.

Proof. We start by introducing some terminology, which is only needed in this proof. The algebra of *quaternions* consist of the vector space $\mathbb{H} := \langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle_{\mathbb{R}}$ together with the associative product $\_\star_{\mathbb{H}}\_ \colon \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ that is defined by

$$\mathbf{i} \star_{\mathbb{H}} \mathbf{i} = \mathbf{j} \star_{\mathbb{H}} \mathbf{j} = \mathbf{k} \star_{\mathbb{H}} \mathbf{k} = \mathbf{i} \star_{\mathbb{H}} \mathbf{j} \star_{\mathbb{H}} \mathbf{k} = -1$$

with $1 \in \mathbb{H}$ being the multiplicative unit. The algebra of *complex quaternions* is defined as $\mathbb{H}_{\mathbb{C}} := \langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle_{\mathbb{C}}$ and $\mathrm{i} \in \mathbb{C}$ denotes the imaginary unit. The product induced by $\star_{\mathbb{H}}$ is denoted by $\_\bullet\_ \colon \mathbb{H}_{\mathbb{C}} \times \mathbb{H}_{\mathbb{C}} \to \mathbb{H}_{\mathbb{C}}$.

The *conjugate* of a quaternion or complex quaternion $h = h_1 + h_2\,\mathbf{i} + h_3\,\mathbf{j} + h_4\,\mathbf{k}$ is defined as $h^* := h_1 - h_2\,\mathbf{i} - h_3\,\mathbf{j} - h_4\,\mathbf{k}$. A direct calculation shows that

$$h \bullet h^* = h_1^2 + h_2^2 + h_3^2 + h_4^2.$$

We observe that $S^3 = \{ h \in \mathbb{H} \mid h \star_{\mathbb{H}} h^* = 1 \} \subset \mathbb{R}^4 = \mathbb{H}$, and thus the Hamiltonian product $\_\star\_ \colon S^3 \times S^3 \to S^3$ is induced by $\star_{\mathbb{H}}$. Similarly,

$$\mathbf{C}(S^3) = \{ h \in \mathbb{H}_{\mathbb{C}} \mid h \bullet h^* = 1 \} \subset \mathbb{C}^4 = \mathbb{H}_{\mathbb{C}}.$$

A direct calculation shows that the product $\_\star_S\_ \colon \mathbf{C}(S^3) \times \mathbf{C}(S^3) \to \mathbf{C}(S^3)$ induced by $\bullet$ extends to the rational map $\_\widehat{\star}\_ \colon \mathbb{S}^3 \times \mathbb{S}^3 \dashrightarrow \mathbb{S}^3$ defined before. This implies that Assertion (a) holds.

If we identify $\mathbb{P}^3$ with the projectivized vector space $\mathbb{P}(\langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle_{\mathbb{C}})$, then $\bullet$ extends to the following rational map $\_\star_{\mathbb{P}}\_ \colon \mathbb{P}^3 \times \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, where $x = (x_1 : \cdots : x_4)$ and $y = (y_1 : \cdots : y_4)$:

$$\begin{aligned}(x, y) \mapsto (&x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4 : x_1 y_2 + x_2 y_1 + x_3 y_4 - x_4 y_3 : \\ &x_1 y_3 - x_2 y_4 + x_3 y_1 + x_4 y_2 : x_1 y_4 + x_2 y_3 - x_3 y_2 + x_4 y_1).\end{aligned}$$

We define $\iota \colon \mathbf{C}(S^3) \hookrightarrow \mathbb{S}^3$ and $\kappa \colon \mathbb{H}_{\mathbb{C}} \dashrightarrow \mathbb{P}^3$ as follows, where the complex quaternion $h = h_1 + h_2\,\mathbf{i} + h_3\,\mathbf{j} + h_4\,\mathbf{k} \in \mathbb{H}_{\mathbb{C}}$ is non-zero:

$$\iota(h) := (1 : h_1 : h_2 : h_3 : h_4) \qquad \text{and} \qquad \kappa(h) := (h_1 : h_2 : h_3 : h_4).$$

Notice that $(\tau \circ \iota)(h) = \kappa(h)$, where $\tau \colon \mathbb{S}^3 \to \mathbb{P}^3$ denotes the central projection.

Suppose that $q \in \mathbf{C}(S^3)$ and that either

- $\psi_1(x) := q \star_S x$, $\quad\psi_2(x) := \iota(q) \,\widehat{\star}\, x$, $\quad\psi_3(x) := \kappa(q) \star_{\mathbb{P}} x$, $\quad\psi_4(x) := q \bullet x$, or
- $\psi_1(x) := x \star_S q$, $\quad\psi_2(x) := x \,\widehat{\star}\, \iota(q)$, $\quad\psi_3(x) := x \star_{\mathbb{P}} \kappa(q)$, $\quad\psi_4(x) := x \bullet q$.

In both cases the diagram in Table 3 commutes as a direct consequence of the definitions.

Table 3. See the proof of Proposition 10.

$$\begin{tikzcd}
\mathbf{C}(S^3) \arrow[r, hook, "\iota"] \arrow[d, "\psi_1"'] & \mathbb{S}^3 \arrow[r, "\tau"] \arrow[d, "\psi_2"'] & \mathbb{P}^3 \arrow[d, "\psi_3"] & \mathbb{H}_{\mathbb{C}} \arrow[l, dashed, "\kappa"'] \arrow[d, "\psi_4"] \\
\mathbf{C}(S^3) \arrow[r, hook, "\iota"'] & \mathbb{S}^3 \arrow[r, "\tau"'] & \mathbb{P}^3 & \mathbb{H}_{\mathbb{C}} \arrow[l, dashed, "\kappa"]
\end{tikzcd}$$

We are now ready to prove the remaining Assertions (b), (c) and (d).

(b). Since $q \star_S q^* = q^* \star_S q = 1$, we find that $\psi_1^{-1}(x)$ is equal to either $q^* \star_S x$ or $x \star_S q^*$ for all $x \in \mathbf{C}(S^3)$. It follows that $\psi_2 \in \operatorname{Aut}_{\mathbb{C}} \mathbb{S}^3$. Moreover, $\iota(\mathbf{C}(S^3)) = \mathbb{S}^3 \setminus \mathbb{E}$, which implies that $\psi_2(\mathbb{E}) = \mathbb{E}$. Therefore, $\operatorname{LT} \mathbb{S}^3, \operatorname{RT} \mathbb{S}^3 \subset \operatorname{Aut}_{\mathbb{E}} \mathbb{S}^3$ and thus we concluded the proof of Assertion (b).

We set $E := \{ h \in \mathbb{H}_{\mathbb{C}} \mid h \bullet h^* = 0 \}$ and for all $\alpha \in E$ we define

$$L_\alpha := \{ h \in \mathbb{H}_{\mathbb{C}} \mid h \bullet \alpha = 0 \} \quad \text{and} \quad R_\alpha := \{ h \in \mathbb{H}_{\mathbb{C}} \mid \alpha \bullet h = 0 \}.$$

(c). Suppose that $\alpha \in E \setminus \{0\}$ and $\beta := \alpha^*$. The map $\mathbb{H}_\mathbb{C} \to \mathbb{H}_\mathbb{C}$ that sends h to $h \bullet \alpha$ is linear with respect to the underlying vector space $\langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle_\mathbb{C}$ and has kernel L_α. Let $V_\beta := \langle \beta,\ \mathbf{i} \bullet \beta,\ \mathbf{j} \bullet \beta,\ \mathbf{k} \bullet \beta \rangle_\mathbb{C}$. By assumption, $\alpha \bullet \alpha^* = \beta \bullet \beta^* = \beta \bullet \alpha = 0$ and thus $V_\beta \subseteq L_\alpha$. Now suppose by contradiction that $\dim V_\beta < 2$. In this case,

$$\beta = c_1 \bullet \mathbf{i} \bullet \beta = c_2 \bullet \mathbf{j} \bullet \beta = c_3 \bullet \mathbf{k} \bullet \beta$$

for some $c_1, c_2, c_3 \in \mathbb{C}$. Since

$$\mathbf{i} = \mathbf{j} \bullet \mathbf{k} = -\mathbf{k} \bullet \mathbf{j},\ \mathbf{j} = \mathbf{k} \bullet \mathbf{i} = -\mathbf{i} \bullet \mathbf{k},\ \mathbf{k} = \mathbf{i} \bullet \mathbf{j} = -\mathbf{j} \bullet \mathbf{i},\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

we find that there exists $\beta_0, \beta_1, \beta_2, \beta_3 \in \mathbb{C}$ such that

$$\begin{aligned}
\beta &= \beta_0 + \beta_1 \bullet \mathbf{i} + \beta_2 \bullet \mathbf{j} + \beta_3 \bullet \mathbf{k},\\
c_1 \bullet \mathbf{i} \bullet \beta &= -c_1 \bullet \beta_1 + c_1 \bullet \beta_0 \bullet \mathbf{i} - c_1 \bullet \beta_3 \bullet \mathbf{j} + c_1 \bullet \beta_2 \bullet \mathbf{k},\\
c_2 \bullet \mathbf{j} \bullet \beta &= -c_2 \bullet \beta_2 + c_2 \bullet \beta_3 \bullet \mathbf{i} + c_2 \bullet \beta_0 \bullet \mathbf{j} - c_2 \bullet \beta_1 \bullet \mathbf{k},\\
c_3 \bullet \mathbf{k} \bullet \beta &= -c_3 \bullet \beta_3 - c_3 \bullet \beta_2 \bullet \mathbf{i} + c_3 \bullet \beta_1 \bullet \mathbf{j} + c_3 \bullet \beta_0 \bullet \mathbf{k}.
\end{aligned}$$

By comparing the coefficients, we see that $\beta_0 = -c_i \bullet \beta_i$, $\beta_i = c_i \bullet \beta_0$ and thus $\beta_0 \neq 0$ and $c_i^2 = -1$ for all $1 \leq i \leq 3$. This implies that $\beta_1 = \beta_2 = \beta_3 = \pm\mathrm{i} \bullet \beta_0$ and thus

$$\beta \bullet \beta^* = \beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = -2 \bullet \beta_0^2 = 0.$$

We arrived at a contradiction as $\beta_0 \neq 0$. We established that

$$\dim L_\alpha \geq \dim V_\beta \geq 2.$$

Thus, $\kappa(L_\alpha)$ and $\kappa(R_\alpha)$ are the two complex lines in the projective quadric $\kappa(E)$ that contain the complex point $\kappa(\alpha^*) \in \kappa(L_\alpha) \cap \kappa(R_\alpha)$. Since $\tau(\mathbb{E}) = \kappa(E)$ as a direct consequence of the definitions, we may assume without loss of generality that $\kappa(\alpha^*) = \tau(p)$. It follows that $\tau(\mathscr{L}_p) = \kappa(L_\alpha)$ and $\tau(\mathscr{R}_p) = \kappa(R_\alpha)$. The central projection τ is a 2:1 morphism with ramification locus $\mathbb{E}$ and thus Assertion (c) is true.

(d). We may assume without loss of generality that $\psi_2 = \varphi$ and $\alpha \in E$ such that $\kappa(\alpha^*) = \tau(p)$. Thus, $\psi_2(x) = \iota(q) \,\widehat{\star}\, x$ for some $q \in \mathbf{C}(S^3)$. If $\beta \in L_\alpha$, then $h \bullet \beta \in L_\alpha$ for all $h \in \mathbb{H}_\mathbb{C}$. This implies that $\psi_4(L_\alpha) = L_\alpha$. As Table 3 commutes, we deduce that $\psi_2(\mathscr{L}_p) = \varphi(\mathscr{L}_p) = \mathscr{L}_p$ as was to be shown. The prove of the second statement is analogous and thus we concluded the proof. □

The following proposition shows that the Euclidean translations of $\mathbb{S}^3$ correspond to Euclidean translations of $\mathbb{R}^3$ and leave the generators of $\mathbb{U}$ invariant.

Proposition 11.

(a) $\mathbf{R}(\pi \circ \zeta_v(x)) = \mathbf{R}(\pi(x)) + v$ *for all* $x \in \mathbb{S}^3_\mathbb{R} \setminus \mathbb{U}$ *and* $v \in \mathbb{R}^3$.
(b) $\mathrm{UT}\,\mathbb{S}^3 \subset \mathrm{Aut}_\mathbb{U}\,\mathbb{S}^3$.
(c) *If* $L \subset \mathbb{U}$ *is a generator, then* $\varphi(L) = L$ *for all* $\varphi \in \mathrm{UT}\,\mathbb{S}^3$.

Proof. (a). It follows from a straightforward calculation (see [16, `cyclides`]) that

$$(\pi \circ \zeta_b)(x) = (x_{04} : x_1 + x_{04}\, b_1 : x_2 + x_{04}\, b_2 : x_3 + x_{04}\, b_3) \tag{1}$$

for all $b \in \mathbb{C}^3$, where $x_{04} := x_0 - x_4$. Thus, $\mathbf{R}(\pi \circ \zeta_b(x)) = \mathbf{R}(\pi(x)) + b$ for all $b \in \mathbb{R}^3$ and $x \in \mathbb{S}^3_\mathbb{R}$ such that $x_{04} \neq 0$.

(b). Suppose that $\varphi \in \mathrm{UT}\,\mathbb{S}^3$ so that $\varphi = \zeta_b$ for some $b \in \mathbb{C}^3$. Let M be the 5×5 matrix associated to φ and let J be the diagonal matrix with $(-1, 1, 1, 1, 1)$ on its diagonal. We verify that $M^\top \cdot J \cdot M = c \cdot J$ for some $c \in \mathbb{C}$ and thus $\varphi \in \mathrm{Aut}_\mathbb{C}\,\mathbb{S}^3$. Since $\varphi(\mathbb{U}) = \mathbb{U}$, it follows that $\varphi \in \mathrm{Aut}_\mathbb{U}\,\mathbb{S}^3$. See [16, `cyclides`] for an automatic verification.

(c). By assumption, $\varphi = \zeta_b$ for some $b \in \mathbb{C}^3$. Let $\iota(z) := (1 : z_1 : z_2 : z_3)$,

$$\psi_2(x) := (x_0 : x_1 + x_0\, b_1 : x_2 + x_0\, b_2 : x_3 + x_0\, b_3), \quad \psi_3(z) := (z_1 + b_1, z_2 + b_2, z_3 + b_3).$$

It follows from Equation (1) that the diagram in Table 4 commutes.

Table 4. See the proof of Proposition 11.

$$\begin{tikzcd} \mathbb{S}^3 \arrow[r, dashed, "\pi"] \arrow[d, "\varphi"'] & \mathbb{P}^3 \arrow[d, "\psi_2"'] & \mathbb{C}^3 \arrow[l, hook', "\iota"'] \arrow[d, "\psi_3"] \\ \mathbb{S}^3 \arrow[r, dashed, "\pi"'] & \mathbb{P}^3 & \mathbb{C}^3 \arrow[l, hook', "\iota"] \end{tikzcd}$$

Let $H \subset \mathbb{C}^3$ be a complex plane. Since $\psi_3(H)$ is parallel to H, we deduce that the Zariski closures of the images $\iota(H)$ and $(\iota \circ \psi_3)(H)$ in $\mathbb{P}^3$ intersect at a complex line K at infinity. Recall from Remark 9 that $\pi(\mathbb{U})$ is a conic at infinity and thus $1 \le |K \cap \pi(\mathbb{U})| \le 2$ by Bézout's theorem. We find that $\psi_2(K \cap \pi(\mathbb{U})) = K \cap \pi(\mathbb{U})$. Since H was chosen arbitrary, we deduce that $\psi_2(\mathfrak{q}) = \mathfrak{q}$ for all $\mathfrak{q} \in \pi(\mathbb{U})$. We have $\pi(L) = \mathfrak{q}$ for some $\mathfrak{q} \in \pi(\mathbb{U})$, and thus $\varphi(L) = L$ as asserted. ☐

Remark 12. Notice that $\mathbb{E}_{\mathbb{R}} = \varnothing$ and that the complex conjugate of a left or right generator in $\mathbb{E}$ is again left and right, respectively. The generators in $\mathbb{U}$ are all concurrent. Complex conjugate lines in $\mathbb{Y}$ intersect in a real point. A hyperplane section of $\mathbb{S}^3$ is Möbius equivalent to either $\mathbb{U}$, $\mathbb{E}$ or $\mathbb{Y}$. Since $\mathbb{U}$ is unlike $\mathbb{E}$ and $\mathbb{Y}$ a tangent hyperplane section, we could interpret Euclidean geometry as a limit case of both the hyperbolic and elliptic geometries.

Definition 13. *If $Z \subset \mathbb{R}^3$ is a Darboux cyclide, then we shall call its Möbius model $\mathbf{S}(Z)$ in $\mathbb{S}^3$ also a* Darboux cyclide. *Similarly for CH1 cyclide, EY cyclide, Blum cyclide and so on, and for attributes such as λ-circled, celestial, Bohemian, Cliffordian and great. We call a conic $C \subset \mathbb{S}^3$ a* (great/small) circle *if $\mathbf{R}(C)$ is a (great/small) circle in $S^3 \subset \mathbb{R}^4$.*

3. Associated pencils and absolutes

In Section 2, we considered elliptic and Euclidean geometries as subgroups of the Möbius transformations that preserve some fixed hyperplane section of the Möbius quadric. In this section, we characterize the intersection of Cliffordian and Bohemian surfaces with such hyperplane sections. In particular, we analyze how circles meet the elliptic or Euclidean absolute as they move in their respective pencils.

A *pencil* on a surface $X \subset \mathbb{S}^3$ is defined as an irreducible real hypersurface

$$F \subset X \times \mathbb{P}^1$$

such that the 1st and 2nd projections $\pi_1 \colon X \times \mathbb{P}^1 \dashrightarrow X$ and $\pi_2 \colon X \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ are dominant. If $i \in \mathbb{P}^1$ is reached by π_2, then the Zariski closure of $\pi_1(F \cap X \times \{i\}) \subset X$ is called a *member* of F and denoted by F_i. We call a complex point $p \in X$ a *base point* of F, if $p \in F_i$ for all $i \in \pi_2(F)$. We call F a *pencil of conics* if F_i is a complex irreducible conic for almost all $i \in \mathbb{P}^1$. We call F a *pencil of circles* if it is a pencil of conics such that F_i is a circle for infinitely many $i \in \mathbb{P}^1_{\mathbb{R}}$.

Example 14. Suppose that

$$F := \left\{(x_0 : x_1 : x_2 : x_3; i_0 : i_1) \,\middle|\, i_0\, x_3 = i_1\, x_0,\ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\right\}.$$

Thus $F \subset \mathbb{S}^2 \times \mathbb{P}^1$ is defined by the latitudinal circles on a sphere, where F_i is a circle if $i \in \mathbb{P}^1_{\mathbb{R}}$ such that $-1 \le i_1 / i_0 \le 1$ and otherwise F_i is a complex conic with at most one real point. By definition, F is a pencil of circles.

Remark 15. If $F \subset X \times \mathbb{P}^1$ is a pencil of conics on a celestial Darboux cyclide, then it follows from [23, Theorem 9] that F is the Zariski closure of the graph of a rational map $f \colon X \dashrightarrow \mathbb{P}^1$ whose fibers are complex conics. This implies that the first projection π_1 is birational and the second projection π_2 is surjective. The complex points where the map f is not defined correspond to the base points.

The following lemma is used in Lemma 30. We present an elementary proof based on Bézout's theorem to make our result accessible to a wider audience.

Lemma 16. *Suppose that $X \subset \mathbb{S}^3$ is a Darboux cyclide and $F \subset X \times \mathbb{P}^1$ is a pencil of circles.*

(a) *The member F_i is a complex conic that is not contained in a complex line for all $i \in \mathbb{P}^1$.*
(b) *If $L \subset X$ is a complex line such that $F_i \cap L \neq \varnothing$ for almost all $i \in \mathbb{P}^1$, then $\left|\{i \in \mathbb{P}^1 \mid p \in F_i\}\right| = 1$ for all complex $p \in L$.*
(c) *If $R \subset X$ is a complex line such that $F_i \cap R = \varnothing$ for almost all $i \in \mathbb{P}^1$, then there exists $k \in \mathbb{P}^1$ such that $R \subset F_k$ is a component.*

Proof. Let $q := (1:0:0:0:1)$ denote the center of the stereographic projection $\pi \colon \mathbb{S}^3 \dashrightarrow \mathbb{P}^3$ and recall from Remark 9 that π defines a biregular isomorphism between $X \setminus \mathbb{U}$ and $\pi(X \setminus \mathbb{U})$. We may assume up to Möbius equivalence that $q \in X_{\mathbb{R}}$ is a general point in X.

Claim 1. *The surface $\pi(X)$ is of degree three and does not contain a complex line through a general point. Moreover, $\pi(F_i)$ is a complex irreducible conic for almost all $i \in \mathbb{P}^1$.*

Since q is general, it is smooth in X so that $\deg \pi(X) = 3$. Moreover, q is not a base point of F or any other pencil of complex conics. A surface in $\mathbb{S}^3$ that contains infinitely many points and a complex line through a general point, must contain two complex conjugate lines through this point and thus be a sphere. As the inverse stereographic projection of a complex line is either a complex line or a complex irreducible conic that passes through q, we deduce that Claim 1 holds true.

Claim 2. *There exists a circle $C \subset X$ containing q and a set $\{H_i\}_{i\in\mathbb{P}^1}$ of complex hyperplane sections of $\mathbb{S}^3$ such that $H_i \cap X = F_i \cup C$ for all $i \in \mathbb{P}^1$.*

Let $I := \left\{i \in \mathbb{P}^1 \,\middle|\, \pi(F_i) \text{ is not contained in a complex line}\right\}$ so that $|\mathbb{P}^1 \setminus I| < \infty$ by Claim 1. Let $P_i \subset \mathbb{P}^3$ denote the complex plane containing $\pi(F_i)$ for all $i \in I$. Recall that $\deg \pi(X) = 3$ by Claim 1 and thus there exists by Bézout's theorem for all $i \in I$ a complex line $M_i \subset \pi(X)$ such that $P_i \cap \pi(X) = \pi(F_i) \cup M_i$. Since $\pi(X)$ contains by Claim 1 no continuous family of complex lines and F is a continuous family of complex curves, we deduce that $M := M_i = M_j$ for all $i, j \in I$. Moreover, since $\pi(F_i)$ and the plane P_i are for all $i \in I \cap \mathbb{P}^1_{\mathbb{R}}$ real and coplanar with the complex line M, it follows that M must be real as well. Let $C \subset X$ be the circle such that $\pi(C) = M$ so that $q \in C$. Let $\{H_i\}_{i\in\mathbb{P}^1}$ be the complex hyperplane sections of $\mathbb{S}^3$ that contain the circle C. We find that $\pi(H_i) = P_i$ for all $i \in I$, which implies that $H_i \cap X = F_i \cup C$ for all $i \in I$. Since the pencil F is defined as the zero set of algebraic equations and thus Zariski closed, we deduce that $\pi_2(F) = \mathbb{P}^1$ and $H_i \cap X = F_i \cup C$ for all $i \in \mathbb{P}^1 \setminus I$. This concludes the proof of Claim 1.

We are now ready to prove Assertions (a), (b) and (c).

(a). Since $\deg X = 4$, it follows from Claim 2 and Bézout's theorem that $\deg(F_i \cup C) = 4$ for all $i \in \mathbb{P}^1$ when counted with multiplicities. We remark that if the hyperplane section H_i is tangent to X along C, then $H_i \cap X = F_i = C$ has intersection multiplicity two. In any case, F_i is for all $i \in \mathbb{P}^1$ a complex curve of degree at most two. Now suppose by contradiction that there exists $c \in \mathbb{P}^1$ such that $\deg F_c = 1$. Let $\xi \colon X \dashrightarrow \mathbb{P}^3$ be the complex linear projection from a general complex point in F_c. Since $\mathbb{S}^3_{\mathbb{R}}$ does not contain lines, the complex conjugate line $\overline{F}_c$ is contained in X as well. Notice that $F_c, \overline{F}_c \nsubseteq \operatorname{Sing} X$, because otherwise $\xi(X)$ would be a reducible complex quadric such that $\xi(\overline{F}_c) \subset \operatorname{Sing}\xi(X)$. Thus the center of ξ lies in $X \setminus \operatorname{Sing} X$ so that $\deg\xi(X) = 3$. Since $X \cap H_c = F_c \cup C$ is a complex hyperplane section and the complex line F_c is projected to the complex point $\xi(F_c) \in \xi(C)$, we find that $\xi(C \cup F_c) \subset \xi(X)$ is a complex hyperplane section consisting of the complex irreducible conic $\xi(C)$. We arrived at a contradiction with Bézout's theorem, since a complex hyperplane section of $\xi(X)$ must be of odd degree $\deg\xi(X) = 3$ when counted with multiplicities. This concludes the proof of Assertion (a).

(b). Let $j \in \mathbb{P}^1$ be general so that $F_j \cap L \neq \varnothing$. Since $H_j \cap X = F_j \cup C$ by Claim 1, it follows from Bézout's theorem that $|H_j \cap L| = |F_j \cap L| = 1$. As F has no base point on $L \cap C$ by generality of q, we have $F_j \cap L \nsubseteq C$ which implies that $L \cap C = \varnothing$. Hence, for all $p \in L$ there exists a unique $i \in \mathbb{P}^1$ such that H_i contains p. We conclude the proof of Assertion (b) as $H_i \cap X = F_i \cup C$ and $L \cap C = \varnothing$, and thus $p \in F_i$.

(c). Let $j \in \mathbb{P}^1$ be general so that $F_j \cap R = \varnothing$. Because $|H_j \cap R| = 1$ by Bézout's theorem and $X \cap H_j = F_j \cup C$ by Claim 1, we find that $|R \cap C| = 1$. Hence, $\pi(R \cup C)$ spans a complex plane containing the line $\pi(C)$, which implies that there exists $k \in \mathbb{P}^1$ such that $R \cup C \subset H_k$. It follows from Claim 1 and Assertion (a) that $R \cup C \subset F_k \cup C$ so that $R \subset F_k$. □

Suppose that $A, B \subset S^3$ are circles such that $A \star B \subset S^3$ is a surface and observe that $\mathbf{P}(A) \cong \mathbb{P}^1$. The *left associated pencil* of the surface $A \star B$ is defined as the pencil of circles

$$F \subset \mathbf{P}(A \star B) \times \mathbf{P}(A)$$

such that $F_a = \varphi_a(\mathbf{P}(B))$ for all $a \in \mathbf{P}(A) \setminus \mathbb{E}$, where $\varphi_a \in \operatorname{LT}\mathbb{S}^3$ sends x to $a \mathbin{\widehat{\star}} x$. It follows from Proposition 10 (a) that for all $a \in \mathbf{P}(A)_{\mathbb{R}}$, we have

$$\mathbf{R}(F_a) = \{\mathbf{R}(a) \star b \,|\, b \in B\}.$$

We remark that $\operatorname{LT}\mathbb{S}^3 \subset \operatorname{Aut}_{\mathbb{E}}\mathbb{S}^3$ by Proposition 10 (b) and for all $a \in \mathbf{P}(A) \setminus \mathbb{E}$, we have $\left\{a \mathbin{\widehat{\star}} b \,\middle|\, b \in \mathbf{P}(B) \setminus \mathbb{E}\right\} \subseteq F_a$. The member F_a may be reducible for $a \in \mathbf{P}(A) \cap \mathbb{E}$.

Similarly, the *right associated pencil* of $A \star B$ is defined as the pencil of circles

$$G \subset \mathbf{P}(A \star B) \times \mathbf{S}(B)$$

such that $G_b = \varphi_b(\mathbf{P}(A))$ for all $b \in \mathbf{P}(B) \setminus \mathbb{E}$, where $\varphi_b \in \operatorname{RT}\mathbb{S}^3$ sends x to $x \mathbin{\widehat{\star}} b$.

Lemma 17. *Suppose that $A \star B$ is a surface for some circles $A, B \subset S^3$.*

(a) *If the left associated pencil $F \subset \mathbf{P}(A \star B) \times \mathbf{P}(A)$ has no base points on $\mathbb{E}$, then $\mathbf{P}(A \star B)$ contains complex conjugate left generators $L, \overline{L} \subset \mathbb{E}$ such that $|F_a \cap L| = |F_a \cap \overline{L}| = 1$ for almost all $a \in \mathbf{P}(A)$.*

(b) *If the right associated pencil $G \subset \mathbf{P}(A \star B) \times \mathbf{P}(B)$ has no base points on $\mathbb{E}$, then $\mathbf{P}(A \star B)$ contains complex conjugate right generators $R, \overline{R} \subset \mathbb{E}$ such that $|G_b \cap R| = |G_b \cap \overline{R}| = 1$ for almost all $b \in \mathbf{P}(B)$.*

Proof. Let us first prove (a). For infinitely many points $i, j \in \mathbf{P}(A)_{\mathbb{R}}$ the members F_i and F_j are circles and these circles are related by a left Clifford translation. We know from Bézout's theorem that F_i intersects $\mathbb{E}$ in two complex conjugate points, since $\mathbb{E}$ is a hyperplane section of $\mathbb{S}^3$. It follows from Proposition 10 (d) and Remark 12 that the complex conjugate left generators $L, \overline{L} \subset \mathbb{E}$ that pass through these points are left invariant. As F is Zariski closed and without base points on $\mathbb{E}$ we conclude that $L, \overline{L} \subset \mathbf{P}(A \star B)$.

The proof for (b) is analoguous. □

Let $A, B \subset \mathbb{R}^3$ be circles and notice that $\mathbf{S}(A) \cong \mathbb{P}^1$. The *associated pencil* of $A + B$ is defined as the pencil of circles $F \subset \mathbf{S}(A + B) \times \mathbf{S}(A)$ such that for almost all $a \in \mathbf{S}(A)$, there exists a unique $c \in \mathbf{C}(A)$ such that $F_a = \zeta_c(\mathbf{S}(B))$.

As a straightforward consequence of the definitions and Proposition 11 (a), we find that $\zeta_a(b) \in \mathbf{S}(A+B)$ for all $a \in A$ and $b \in \mathbf{S}(B)$. Hence, the circles $\{C_a\}_{a\in A}$ on the surface $A+B \subset \mathbb{R}^3$ that are defined by $C_a := \{a+b \mid b \in B\}$ correspond via $\mathbf{R}(\pi(\_))$ to a subset of $\{F_a\}_{a\in\mathbf{S}(A)}$.

Lemma 18. *Suppose that $A, B \subset \mathbb{R}^3$ are circles so that $A+B$ is a surface. If the associated pencil $F \subset \mathbf{S}(A+B) \times \mathbf{S}(A)$ of $A+B$ has no base points on $\mathbb{U}$, then $\mathbf{S}(A+B)$ contains complex conjugate generators $L, \overline{L} \subset \mathbb{U}$ such that $|F_a \cap L| = |F_a \cap \overline{L}| = 1$ for almost all $a \in \mathbf{S}(A)$.*

Proof. Let $i \in \mathbf{S}(A)$ be general. Since F has no base points on $\mathbb{U}$, we may assume without loss of generality that F_i does not meet the vertex of the tangent hyperplane section $\mathbb{U}$. Thus it follows from Bézout's theorem that

$$|F_i \cap \mathbb{U}| = \left|F_i \cap \left\{x \in \mathbb{P}^4 \,\middle|\, x_0 - x_4 = 0\right\}\right| = 2.$$

Recall from Proposition 11 (c) that the Euclidean translations of $\mathbb{S}^3$ leave the generators of the hyperplane section $\mathbb{U}$ invariant. Hence, there exist complex conjugate generators $L, \overline{L} \subset \mathbb{U}$ such that $|F_a \cap L| = |F_a \cap \overline{L}| = 1$ for almost all $a \in \mathbf{S}(A)$. As F is Zariski closed and without base points on $\mathbb{U}$, we conclude that $L, \overline{L} \subset \mathbf{S}(A+B)$. □

4. Divisor classes of curves on Darboux cyclides

We recall from [15] the possible sets of divisor classes of complex low degree curves on Darboux cyclides in $\mathbb{S}^3$ (recall Definition 13). Each entry in this classification translates into a diagram that visualizes how complex lines, complex isolated singularities and circles intersect.

A *smooth model* $\mathbf{O}(X)$ of a surface $X \subset \mathbb{P}^n$ is a nonsingular surface such that there exists a birational morphism $\varphi\colon \mathbf{O}(X) \to X$ that does not contract complex (-1)-curves. We refer to φ as a *desingularization*.

The smooth model $\mathbf{O}(X)$ is unique up to biregular isomorphisms and there exists a desingularization $\varphi\colon \mathbf{O}(X) \to X$ (see [10, Theorem 2.16]).

The *Néron–Severi lattice* $N(X)$ is an additive group defined by the divisor classes on $\mathbf{O}(X)$ up to numerical equivalence. This group comes with an unimodular intersection product $\cdot$ and a unimodular involution $\sigma_*\colon N(X) \to N(X)$ induced by the real structure $\sigma\colon X \to X$. We denote by $\operatorname{Aut} N(X)$ the group automorphisms that are compatible with both $\cdot$ and σ_*.

The *class* $[C]$ of a complex curve $C \subset X$ is defined as the divisor class of $\widetilde{C}$ in $N(X)$, where $\widetilde{C} \subset \mathbf{O}(X)$ is the union of complex curves in $\varphi^{-1}(C)$ that are not contracted to complex points by the morphism φ.

We consider the following subsets of $N(X)$:

- $B(X)$ denotes the set of divisor classes of complex irreducible curves $C \subset \mathbf{O}(X)$ such that $\varphi(C)$ is a complex point in X,
- $G(X)$ denotes the set of classes of complex irreducible conics in X, and
- $E(X)$ denotes the set of classes of complex lines in X.

We call $W \subset B(X)$ a *component* if it defines a maximal connected subgraph of the graph with vertex set $B(X)$ and edge set $\{\{a,b\} \mid a \cdot b > 0\}$. The latter subgraph is called the *graph of the component*.

We write $c \cdot W > 0$ for a subset $W \subset N(X)$, if there exists $w \in W$ such that $c \cdot w > 0$.

The *singular locus* of X is denoted by $\operatorname{Sing} X$.

The following proposition is an application of intersection theory on surfaces (see [7, Section V.1]). For its proof we assume some background in algebraic geometry, but its assertions are meant to be accessible to non-experts.

Proposition 19. *Suppose that $X \subset \mathbb{S}^3$ is a celestial Darboux cyclide. Let*

- *$\mathscr{W}(X)$ be the set of components in $B(X)$,*
- *$\mathscr{F}(X)$ be the set of pencils of circles on X,*
- *$\mathscr{G}(X) := \{g \in G(X) \mid \sigma_*(g) = g\}$, and*
- *$\mathscr{E}(X)$ be the set of complex lines in X.*

Suppose that $C, C' \subset X$ are complex lines and/or circles such that $C \neq C'$.

(a) *There exists a bijection $\Gamma\colon \mathscr{W}(X) \to \operatorname{Sing} X$ such that for all $W \in \mathscr{W}(X)$ the following two properties hold:*
  - *$\Gamma(W) \in \operatorname{Sing} X_{\mathbb{R}}$ if and only if $\sigma_*(W) = W$,*
  - *if $[C]\cdot W > 0$, then $\Gamma(W) \in C$.*

  The graph of a component in $B(X)$ is a Dynkin graph of type A_1, A_2 or A_3.

(b) *The map $\Lambda\colon \mathscr{F}(X) \to \mathscr{G}(X)$ that sends F to $[F_{(0:1)}]$ is a bijection that satisfies the following two properties for all $F, G \in \mathscr{F}(X)$:*
  - *$\Lambda(F)^2 = 0$, and*
  - *if $\Lambda(F)\cdot\Lambda(G) = 2$, then $|F_i \cap G_j| = 2$ for almost all $i, j \in \mathbb{P}^1$.*

  In particular, $[F_i] = [F_j]$ for all $i, j \in \mathbb{P}^1$.

(c) *The map $\mathscr{E}(X) \to E(X)$ that sends L to $[L]$ is bijective.*

(d) *We have that $C \cap C' \neq \varnothing$ if and only if either $[C]\cdot[C'] \neq 0$, or there exists $W \in \mathscr{W}(X)$ such that both $[C]\cdot W > 0$ and $[C']\cdot W > 0$.*

(e) *For all $p \in X$ and $F \in \mathscr{F}(X)$, the complex point p is a base point of F if and only if there exists $W \in \mathscr{W}(X)$ such that $\Lambda(F)\cdot W > 0$ and $\Gamma(W) = p$.*

Proof. Let $\varphi\colon \mathbf{O}(X) \to X$ be a desingularization.

Claim 1. *The smooth model $\mathbf{O}(X)$ is a weak del Pezzo surface of degree four and X is its anticanonical model with at most isolated singularities.*

This claim follows from [23, Proposition 1], where we followed the terminology at [6, Definition 8.1.18, Theorems 8.3.2 and 8.6.4].

Recall that we defined the class of a complex curve $U \subset X$ as the divisor class of the strict transform of U in the smooth model $\mathbf{O}(X)$. We denote the strict transforms of $C \subset X$ and $C' \subset X$ via φ by $D \subset \mathbf{O}(X)$ and $D' \subset \mathbf{O}(X)$, respectively. If $W \in \mathscr{W}(X)$, then we denote by C_W the union of complex curves in $\mathbf{O}(X)$ whose divisor class is in W.

Claim 2. *The morphism φ restricted to $\mathbf{O}(X) \setminus \varphi^{-1}(\operatorname{Sing} X)$ is an isomorphism, and $p \in \operatorname{Sing} X$ if and only if $p = \varphi(C_W)$ for some component W whose graph is a Dynkin graph of type A_1, A_2 or A_3.*

It follows from Claim 1 and [6, Proposition 8.1.10 and Theorem 8.2.27] that φ contracts (-2)-curves to isolated singularities and is an isomorphism outside the (-2)-curves. Thus $B(X)$ consist of the divisor classes of the (-2)-curves and C_W is a union of (-2)-curves with classes in W. We know from [6, Theorem 8.1.11 and Theorem 8.2.28] that the graph of the component W is a Dynkin graph. By [15, Corollary 5] this graph can only be of type A_1, A_2 or A_3.

Claim 3. *$D \cap U \neq \varnothing$ if and only if $[D]\cdot[U] > 0$ for all $U \in \{C_W \mid W \in \mathscr{W}(X)\} \cup \{D'\}$.*

Curves on rational surfaces are linearly equivalent if and only if the curves are numerically equivalent. Thus, it follows from [7, Theorem V.1.1 and Proposition V.1.4] that $[D]\cdot[U]$ is equal to number of intersections in $D \cap U$, when counted with multiplicity.

Claim 4. *$C \cap C' \setminus \operatorname{Sing} X \neq \varnothing$ if and only if $[C]\cdot[C'] > 0$.*

Notice that $[D]\cdot[D'] = [C]\cdot[C']$ by definition. If $C \cap C' \setminus \operatorname{Sing} X \neq \varnothing$, then $D \cap D' \neq \varnothing$ by Claim 2 and thus $[C]\cdot[C'] > 0$ by Claim 3. If $[C]\cdot[C'] > 0$, then $D \cap D' \neq \varnothing$ by Claim 3 and thus $C \cap C' \setminus \operatorname{Sing} X \neq \varnothing$ by Claim 2.

Claim 5. *$C \cap C' \cap \operatorname{Sing} X \neq \varnothing$ if and only if there exists $W \in \mathscr{W}(X)$ such that $[C] \cdot W > 0$ and $[C'] \cdot W > 0$.*

It follows from Claim 2 that $C \cap C' \cap \operatorname{Sing} X \neq \varnothing$ if and only if there exists $W \in \mathscr{W}(X)$ such that $C \cap C_W \neq \varnothing$ and $C' \cap C_W \neq \varnothing$. By definition, $[D] \cdot [C_W] > 0$ if and only if $[C] \cdot W > 0$, and thus Claim 5 follows from Claim 3.

Claim 6. *If C and C' meet transversally, then $|C \cap C'| \geq [C] \cdot [C']$.*

It follows from [7, Theorem V.1.1] that $|D \cap D'| = [D] \cdot [D']$ and thus Claim 6 follows from Claim 2. We are now ready to prove the Assertions (a), (b), (c), (d) and (e) of Proposition 19.

(a). We define $\Gamma(W) := \varphi(C_W)$ for all $W \in \mathscr{W}(X)$ and thus Γ is well-defined and bijective by Claim 2. Since $\sigma(C_W) = C_W$ if and only if $\sigma_*(W) = W$, the proof for this assertion is concluded by Claims 2 and 5.

(b). Complex curves on rational surfaces are linearly equivalent if and only if the complex curves are numerically equivalent and thus $[F_i] = [F_j]$ for all $i, j \in \mathbb{P}^1$. We may assume without loss of generality that $F_{(0:1)}$ is a circle and thus $[F_i] \in \mathscr{G}(X)$ for all $i \in \mathbb{P}^1$. It follows that the map Λ is well-defined and injective. We know from Claim 1 and [23, Theorem 9] that the strict transform of a conic in X to the smooth model $\mathbf{O}(X)$ belongs to a 1-dimensional base point free complete linear series of curves on $\mathbf{O}(X)$. This implies that any conic in X is the member of a pencil of conics on X. Hence, if $[U] \in \mathscr{G}(X)$ for some irreducible conic $U \subset X$, then $[U]^2 = 0$ and there exists a pencil $T \subset X \times \mathbb{P}^1$ of conics that has U as member. Since the members of T cover a Zariski open set of $X_{\mathbb{R}}$, we deduce that infinitely many members of T are circles, which implies that $T \in \mathscr{F}(X)$. We established that $\Lambda(T) = [U]$ so that Λ is surjective. Moreover, $[U]^2 = 0$ and thus $\Lambda(F)^2 = 0$. The members F_i and G_j meet transversally for general choice of $i, j \in \mathbb{P}^1$. Since complex conics in $\mathbb{S}^3$ intersect in at most two points, the second property follows from Claim 6.

(c). Complex lines $L \subset X$ do not move in a pencil and thus are in 1:1 correspondence with their classes $[L] \in E(X)$.

(d). Direct consequence of Claims 4 and 5.

(e). Let us additionally assume that C and C' are members of F. Since $\Lambda(F)^2 = [D] \cdot [D'] = |D \cap D'| = 0$ by Assertion (b) and Claim 3, the $\Rightarrow$ direction follows from Claims 2, 4 and 5. The $\Leftarrow$ direction follows the second property at Assertion (a) and Λ being well-defined. $\square$

In this article, $N(X) \cong \langle \ell_0, \ell_1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle_{\mathbb{Z}}$, where the nonzero intersections between the generators are $\ell_0 \cdot \ell_1 = 1$ and $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = \varepsilon_4^2 = -1$. We use the following shorthand notation; those are going to be elements in $B(X) \cup G(X) \cup E(X)$:

$$
\begin{aligned}
&b_1 := \varepsilon_1 - \varepsilon_3, && b_{ij} := \ell_0 - \varepsilon_i - \varepsilon_j && b_0 := \ell_0 + \ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4,\\
&b_2 := \varepsilon_2 - \varepsilon_4, && b'_{ij} := \ell_1 - \varepsilon_i - \varepsilon_j, &&
\end{aligned}
$$

$$
\begin{aligned}
&g_0 := \ell_0, && g_2 := 2\ell_0 + \ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4, && g_{ij} := \ell_0 + \ell_1 - \varepsilon_i - \varepsilon_j,\\
&g_1 := \ell_1, && g_3 := \ell_0 + 2\ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4, &&
\end{aligned}
$$

$$
e_i = \varepsilon_i, \qquad e_{ij} = \ell_i - \varepsilon_j, \qquad e'_i = b_0 + \varepsilon_i.
$$

For convenience, we included at [16] a table and graphs that encode the pairwise intersection numbers of the above elements.

We consider the following unimodular involutions $\sigma_*\colon N(X) \to N(X)$ that are induced by the real structure $\sigma\colon X \to X$:

$$\begin{aligned}
2A_1 &: \sigma_*(\ell_0) = \ell_0,\ \sigma_*(\ell_1) = \ell_1,\ \sigma_*(\varepsilon_1) = \varepsilon_2,\ \sigma_*(\varepsilon_3) = \varepsilon_4,\\
3A_1 &: \sigma_*(\ell_0) = \ell_1,\ \sigma_*(\varepsilon_1) = \varepsilon_2,\ \sigma_*(\varepsilon_3) = \varepsilon_4,\\
D_4 &: \sigma_*(\ell_0) = g_3,\ \sigma_*(\ell_1) = \ell_1,\ \sigma_*(\varepsilon_i) = \ell_1 - \varepsilon_i \text{ for } 1 \le i \le 4.
\end{aligned}$$

Recall from Proposition 19 (a) that the graph of a component $W \subset B(X)$ is a Dynkin graph of type A_1, A_2 or A_3. The corresponding isolated double point is called a *node*, *cusp* or *tacnode*, respectively. If $\sigma_*(W) \neq W$, then type$(W) \in \{A_1, A_2, A_3\}$ denotes the type of W. If $\sigma_*(W) = W$, then type$(W) \in \{\underline{A_1}, \underline{A_2}, \underline{A_3}\}$ denotes the underlined type of W. If $\{W_1, \ldots, W_n\}$ is the set of components in $B(X)$, then we denote the *singular type* SingType X as a formal sum type$(W_1) + \cdots +$ type(W_n).

A Darboux cyclide $X \subset \mathbb{S}^3$ is called a *S1 cyclide* or *S2 cyclide*, if $\mathbf{R}(X)$ is smooth and homeomorphic to a sphere or the disjoint union of two spheres, respectively.

The following theorem follows from [15, Theorem 4 and Corollary 5].

Theorem A. *If $X \subset \mathbb{S}^3$ is a λ-circled Darboux cyclide such that $\lambda \ge 2$, then*

$$N(X) \cong \langle \ell_0, \ell_1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle_{\mathbb{Z}},$$

and σ_*, SingType X*, $B(X)$, $E(X)$, $G(X)$, λ are up to* Aut $N(X)$ *defined by a row in Tables 5 and 6, together with the name of X.*

Remark 20. The $G(X)$, $E(X)$ and SingType X are, up to Aut $N(X)$, uniquely determined by $B(X)$ together with σ_* (see [15, Theorem 4 and Corollary 5]). Notice that the 3th, 5th, and 6th columns in Table 5 and the ordering/underlines in Table 6 are immediate corollaries of Theorem A.

Table 5. See Theorem A. A class is send by the unimodular involution σ_* to itself if underlined.

cyclide	σ_*	components in $B(X)$	SingType X	$\lvert E(X)\rvert$	$\lvert G(X)\rvert$	λ
Blum	$2A_1$	$\varnothing$	$\varnothing$	16	10	6
Perseus	$2A_1$	$\{b_1\}$, $\{b_2\}$	$2A_1$	8	7	5
ring	$2A_1$	$\{b_{13}\}$, $\{b_{24}\}$, $\{b'_{14}\}$, $\{b'_{23}\}$	$4A_1$	4	4	4
EH1	$2A_1$	$\{b_{12}\}$	$\underline{A_1}$	12	8	4
CH1	$2A_1$	$\{b_{13}\}$, $\{b_{24}\}$, $\{\underline{b'_{12}}\}$	$2A_1 + \underline{A_1}$	6	5	3
HP	$2A_1$	$\{\underline{b_{12}}, \underline{b'_{34}}\}$	$\underline{A_2}$	8	6	2
EY	$2A_1$	$\{\underline{b_{12}}, b_1, b_2\}$	$\underline{A_3}$	4	5	3
CY	$2A_1$	$\{\underline{b_{12}}, b_1, b_2\}$, $\{b'_{13}\}$, $\{b'_{24}\}$	$\underline{A_3} + 2A_1$	2	2	2
EO	$2A_1$	$\{\underline{b_{12}}\}$, $\{\underline{b_{34}}\}$	$2\underline{A_1}$	8	7	3
CO	$2A_1$	$\{\underline{b_{12}}\}$, $\{\underline{b_{34}}\}$, $\{b'_{13}\}$, $\{b'_{24}\}$	$2\underline{A_1} + 2A_1$	4	4	2
EE/EH2	$3A_1$	$\{\underline{b_0}\}$	$\underline{A_1}$	12	8	2
EP	$3A_1$	$\{b_{13}, b'_{24}\}$	$\underline{A_2}$	8	6	2
S1	$3A_1$	$\varnothing$	$\varnothing$	16	10	2
S2	D_4	$\varnothing$	$\varnothing$	16	10	2

Table 6. See Theorem A. A class is send by the unimodular involution σ_* to itself if underlined, and otherwise to its left or right neighbor in the listing, if its position is even or odd, respectively. The dashed row dividers indicate when σ_* is defined by $2A_1, 3A_1$ or D_4.

cyclide	$B(X), E(X), G(X)$
Blum	$\{\}, \{e_1, e_2, e_3, e_4, e_{01}, e_{02}, e_{03}, e_{04}, e_{11}, e_{12}, e_{13}, e_{14}, e'_4, e'_3, e'_2, e'_1\},$ $\{\underline{g_0}, \underline{g_1}, \underline{g_{12}}, \underline{g_{34}}, \underline{g_2}, \underline{g_3}, g_{13}, g_{24}, g_{14}, g_{23}\}$
Perseus	$\{b_1, b_2\}, \{e_3, e_4, e_{01}, e_{02}, e_{11}, e_{12}, e'_4, e'_3\}, \{g_{13}, g_{24}, \underline{g_0}, \underline{g_1}, \underline{g_{12}}, \underline{g_2}, \underline{g_3}\}$
ring	$\{b_{13}, b_{24}, b'_{14}, b'_{23}\}, \{e_1, e_2, e_3, e_4\}, \{\underline{g_0}, \underline{g_1}, \underline{g_{12}}, \underline{g_{34}}\}$
EH1	$\{\underline{b_{12}}\}, \{e_1, e_2, e_3, e_4, e_{03}, e_{04}, e_{11}, e_{12}, e_{13}, e_{14}, e'_2, e'_1\},$ $\{\underline{g_0}, \underline{g_1}, \underline{g_{34}}, \underline{g_3}, g_{13}, g_{24}, g_{14}, g_{23}\}$
CH1	$\{b_{13}, b_{24}, \underline{b'_{12}}\}, \{e_1, e_2, e_3, e_4, e_{13}, e_{14}\}, \{g_{14}, g_{23}, \underline{g_0}, \underline{g_1}, \underline{g_{34}}\}$
HP	$\{\underline{b_{12}}, \underline{b'_{34}}\}, \{e_1, e_2, e_3, e_4, e_{03}, e_{04}, e_{11}, e_{12}\}, \{\underline{g_0}, \underline{g_1}, g_{13}, g_{24}, g_{14}, g_{23}\}$
EY	$\{\underline{b_{12}}, b_1, b_2\}, \{e_3, e_4, e_{11}, e_{12}\}, \{g_{13}, g_{24}, \underline{g_0}, \underline{g_1}, \underline{g_3}\}$
CY	$\{\underline{b_{12}}, b_1, b_2, b'_{13}, b'_{24}\}, \{e_3, e_4\}, \{\underline{g_0}, \underline{g_1}\}$
EO	$\{\underline{b_{12}}, \underline{b_{34}}\}, \{e_1, e_2, e_3, e_4, e_{11}, e_{12}, e_{13}, e_{14}\}, \{g_{13}, g_{24}, g_{14}, g_{23}, \underline{g_0}, \underline{g_1}, \underline{g_3}\}$
CO	$\{\underline{b_{12}}, \underline{b_{34}}, b'_{13}, b'_{24}\}, \{e_1, e_2, e_3, e_4\}, \{\underline{g_0}, \underline{g_1}, g_{14}, g_{23}\}$
EE/EH2	$\{\underline{b_0}\}, \{e_1, e_2, e_3, e_4, e_{01}, e_{12}, e_{02}, e_{11}, e_{03}, e_{14}, e_{04}, e_{13}\},$ $\{g_0, g_1, \underline{g_{12}}, \underline{g_{34}}, g_{13}, g_{24}, g_{14}, g_{23}\}$
EP	$\{b_{13}, b'_{24}\}, \{e_1, e_2, e_3, e_4, e_{02}, e_{11}, e_{04}, e_{13}\}, \{g_0, g_1, \underline{g_{12}}, \underline{g_{34}}, g_{14}, g_{23}\}$
S1	$\{\}, \{e_1, e_2, e_3, e_4, e_{01}, e_{12}, e_{02}, e_{11}, e_{03}, e_{14}, e_{04}, e_{13}, e'_4, e'_3, e'_1, e'_2\},$ $\{g_0, g_1, \underline{g_{12}}, \underline{g_{34}}, g_{13}, g_{24}, g_{14}, g_{23}, g_2, g_3\}$
S2	$\{\}, \{e_1, e_{11}, e_2, e_{12}, e_3, e_{13}, e_4, e_{14}, e_{01}, e'_1, e_{02}, e'_2, e_{03}, e'_3, e_{04}, e'_4\},$ $\{g_0, g_3, g_{12}, g_{34}, g_{13}, g_{24}, g_{14}, g_{23}, \underline{g_1}, \underline{g_2}\}$

EY cyclide CY cyclide

Figure 3. Incidences between complex lines and isolated singularities on a Darboux cyclide X (see Example 21). Each complex line is represented as a line segment and labeled with its corresponding class in $E(X)$. A real or non-real isolated singularity is represented as a disc with a solid and dashed border, respectively. Each singularity is labeled with the sum of classes in the corresponding component in $B(X)$.

Example 21. The diagrams in Figures 3, 4 and 10 show the incidences between complex lines and isolated singularities in EY cyclide, CY cyclide, Perseus cyclide, CH1 cyclide, ring cyclide, CO cyclide, and EO cyclide. In case the Darboux cyclides are inversions of quadratic surfaces, it is straightforward to compute such incidences via elementary computations. For example, an EY cyclide $X \subset \mathbb{S}^3$ is for some $\lambda > 0$, Möbius equivalent to

$$X' := \{x \in \mathbb{P}^4 \mid x_1^2 + \lambda^2 x_2^2 = (x_0 - x_4)^2,\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2\}.$$

If $L \subset X'$ is a complex line, then it must be contained in $\mathbb{U} = \{x \in \mathbb{S}^3 \mid x_0 = x_4\}$ and thus there exist $\alpha, \beta \in \{1, -1\}$ such that

$$L = \left\{x \in \mathbb{P}^4 \,\middle|\, x_0 = x_4,\ x_1 = \alpha\, \mathrm{i}\, \lambda\, x_2 = \tfrac{\beta\lambda}{\sqrt{1-\lambda^2}} x_3\right\}.$$

For Darboux cyclides $X \subset \mathbb{S}^3$ such that $X_\mathbb{R}$ is smooth this approach is less feasible. Instead, we obtained the diagrams by applying Proposition 19 to Theorem A as follows. Since the approach is the same for each case, we suppose again that $X \subset \mathbb{S}^3$ is an EY cyclide. We apply Theorem A and find that $\operatorname{SingType} X = \underline{A_3}$, $B(X) = \{b_1, b_2, \underline{b_{12}}\}$ and $E(X) = \{e_{11}, e_{12}, e_3, e_4\}$ (see Tables 5 and 6). A component $W \subset B(X)$ corresponds to an isolated singularity by Proposition 19 (a) and a class in $E(X)$ corresponds to a complex line in X by Proposition 19 (c). If $\sigma_*(W) = W$, then the isolated singularity is real. Each line segment in the diagram for the EY cyclide in Figure 3 represents a complex line in X. Two line segments intersect at a disc if and only if the corresponding complex lines meet at a complex point $p \in X$. If p is real or non-real, then the disc has solid and dashed border, respectively. If p is an isolated singularity of X, then the disc is labeled with the sum of the classes in the corresponding component $W \subset B(X)$. In the EY cyclide case, $W = \{b_1, b_2, b_{12}\}$ and thus the label equals $b_1 + b_2 + b_{12} = b_{34}$. We use Proposition 19 (a) and (d) to determine whether complex lines and/or isolated singularities intersect. If $\sigma_*(a) = b$ and $a \cdot b = 1$ for some $a, b \in E(X)$, then the corresponding complex conjugate lines meet in a real point. The diagrams for the remaining cyclides are obtained analogously, and are automatically verified at [16, `cyclides`]. Notice that a non-real singularity in an CY cyclide meets only one complex line.

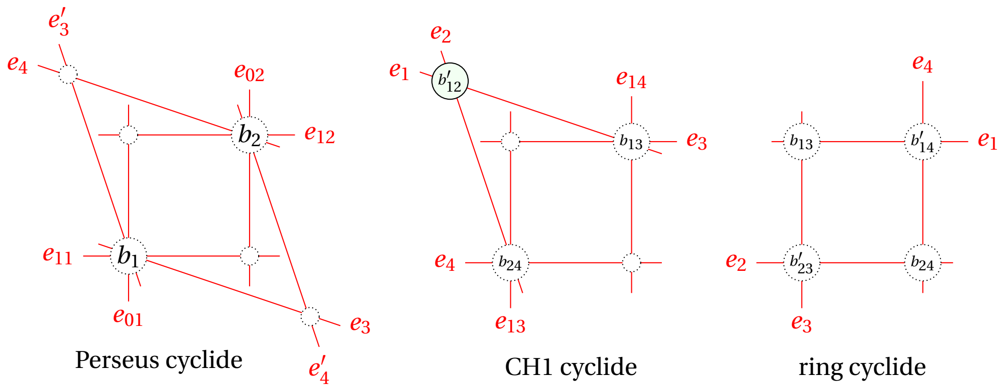


Figure 4. Incidences between complex lines and isolated singularities (see the caption of Figure 3).

Example 22. Suppose that $Z \subset \mathbb{R}^3$ is a smooth torus of revolution. The astronomer Yvon Villarceau observed that Z contains through a general point a latitudinal circle, a longitudinal circle and two cospherical *Villarceau circles* [30, p. 1848]. Let us identify the classes of these circles and identify those circles that are members of a pencil with base points. For this purpose, we suppose that $\Gamma\colon \mathscr{W}(X) \to \operatorname{Sing} X$ and $\Lambda\colon \mathscr{F}(X) \to \mathscr{G}(X)$ are the bijections defined at Proposition 19, where $X := \mathbf{S}(Z)$. Let $F, G \in \mathscr{F}(X)$ be the two pencils of Villarceau circles, and let $F', G' \in \mathscr{F}(X)$ be the pencils of latitudinal and longitudinal circles, respectively. Since $X_\mathbb{R}$ is smooth and X is covered by four pencils of circles, it follows from Proposition 19 (a) and (b) that $\sigma_*(W) \neq W$ for all $W \in \mathscr{W}(X)$ and $|\mathscr{G}(X)| = 4$. Thus Theorem A implies that the corresponding type entries in the $\operatorname{SingType} X$ column of Table 5 are not underlined and the corresponding value in the λ columns is equal to 4. We find that X is a ring cyclide so that $\mathscr{G}(X) = \{g_0, g_1, g_{12}, g_{34}\}$ up to

$\operatorname{Aut} N(X)$ (see the corresponding row in Table 6). Notice that $\alpha\cdot\beta = 2$ for $\alpha,\beta \in \mathscr{G}(X)$ if and only if $\{\alpha,\beta\} = \{g_{12}, g_{34}\}$. It now follows from Proposition 19 (b) that without loss of generality $\big(\Lambda(F), \Lambda(G), \Lambda(F'), \Lambda(G')\big) = (g_{12}, g_{34}, g_0, g_1)$. Since $g_0\cdot\{b'_{14}\} > 0$ and $g_1\cdot\{b_{13}\} > 0$, we deduce from Proposition 19 (a) and (e) and that F' and G' each have complex conjugate base points in $\operatorname{Sing} X$.

The following Lemmas 23 and 24 are needed in Section 5.

Lemma 23. *Suppose that $X \subset \mathbb{S}^3$ is a celestial Darboux cyclide.*

(a) *X is covered by a pencil of circles with complex conjugate base points if and only if X is either a Perseus cyclide, ring cyclide, CH1 cyclide, CY cyclide or CO cyclide.*
(b) *If X is covered by at least two pencils of circles with complex conjugate base points, then X is a ring cyclide.*
(c) *If X is a CY cyclide or CO cyclide, then $X \cap \mathbb{E}$ does not contain complex lines.*

Proof. Suppose that $\Gamma\colon \mathscr{W}(X) \to \operatorname{Sing} X$ and $\Lambda\colon \mathscr{F}(X) \to \mathscr{G}(X)$ are the bijections defined at Proposition 19 and let $\mathscr{M}(X) := \{W \in \mathscr{W}(X) \mid \sigma_*(W) \neq W\}$.

Claim 1. *Each of the following items hold up to* $\operatorname{Aut} N(X)$*:*

- *If X is a Perseus cyclide, then $\mathscr{G}(X) = \{g_0, g_1, g_{12}, g_2, g_3\}$,*
$$\mathscr{M}(X) = \{\{b_1\},\{b_2\}\}, \qquad g_{12}\cdot\{b_1\} > 0, \qquad g_{12}\cdot\{b_2\} > 0,$$
and $g\cdot b = 0$ for all $g \in \{g_0, g_1, g_2, g_3\}$ and $b \in \{b_1, b_2\}$.
- *If X is a ring cyclide, then $\mathscr{G}(X) = \{g_0, g_1, g_{12}, g_{34}\}$,*
$$\mathscr{M}(X) = \{\{b_{13}\},\{b_{24}\},\{b'_{14}\},\{b'_{23}\}\},$$
$$g_0\cdot\{b'_{14}\} > 0, \qquad g_0\cdot\{b'_{23}\} > 0, \qquad g_1\cdot\{b_{13}\} > 0, \qquad g_1\cdot\{b_{24}\} > 0,$$
and $g\cdot b = 0$ for all $g \in \{g_{12}, g_{34}\}$ and $b \in \{b_{13}, b_{24}, b'_{14}, b'_{23}\}$.
- *If X is a CH1 cyclide, then $\mathscr{G}(X) = \{g_0, g_1, g_{34}\}$,*
$$\mathscr{M}(X) = \{\{b_{13}\},\{b_{24}\}\}, \qquad g_1\cdot\{b_{13}\} > 0, \qquad g_1\cdot\{b_{24}\} > 0,$$
and $g\cdot b = 0$ for all $g \in \{g_0, g_{34}\}$ and $b \in \{b_{13}, b_{24}\}$.
- *If X is a CY cyclide or CO cyclide, then $\mathscr{G}(X) = \{g_0, g_1\}$,*
$$\mathscr{M}(X) = \{\{b'_{13}\},\{b'_{24}\}\}, \qquad g_0\cdot\{b'_{13}\} > 0, \qquad g_0\cdot\{b'_{24}\} > 0,$$
and $g_1\cdot b'_{13} = g_1\cdot b'_{24} = 0$.

This claim follows from Theorem A (see Tables 5 and 6 for $\mathscr{M}(X)$ and $\mathscr{G}(X)$, respectively). See [16, `cyclides`] for a table with entries $g\cdot b$ for all $g \in G(X)$ and $b \in B(X)$.

Claim 2. *The pencil $F \in \mathscr{F}(X)$ has complex conjugate base points if and only if $\Lambda(F)\cdot W > 0$ for some $W \in \mathscr{M}(X)$.*

This claim follows from Proposition 19 (a) and (b).

Claim 3. *If $|\operatorname{Sing} X| - |\operatorname{Sing} X_{\mathbb{R}}| > 0$, then X is either a Perseus cyclide, ring cyclide, CH1 cyclide, CY cyclide or CO cyclide.*

This claim follows from Theorem A (see the SingType X column in Table 5).

(a). It follows from Proposition 19 (a) and (e) that a non-real base point of a pencil $F \in \mathscr{F}(X)$ is contained in $\operatorname{Sing} X$. Hence, the $\Rightarrow$ direction follows from Claim 3. The $\Leftarrow$ direction follows from Proposition 19 (b) together with Claims 1 and 2.

(b). Suppose that the pencil $F \in \mathscr{F}(X)$ has complex conjugate base points. It follows from Claims 1 and 2 that if X is a Perseus cyclide, CH1 cyclide, CY cyclide or CO cyclide, then $\Lambda(F)$ equals g_{12}, g_1, g_0 and g_0, respectively. Moreover, if X is a ring cyclide, then $\Lambda(F) \in \{g_0, g_1\}$ (see also Example 22). Hence, this assertion follows from Assertion (a) and the injectivity of Λ.

(c). Recall from Example 21 that the incidences between complex lines and real isolated singularities in a CY cyclide and CO cyclide are illustrated in the diagrams of Figure 3 (right) and Figure 10 (left), respectively. We observe that each line segment in these two diagrams meet a disc with solid border. Thus each complex line in X meets some real isolated singularity. Since $\mathbb{E}_{\mathbb{R}} = \varnothing$, we conclude that $X \cap \mathbb{E}$ does not contain complex lines. $\square$

Lemma 24. *Suppose that $X \subset \mathbb{S}^3$ is either a Perseus cyclide, CH1 cyclide or ring cyclide, and suppose that $C \subset X$ is a hyperplane section.*

(a) *If X is a ring cyclide and $\operatorname{Sing} X \subset C$, then C consists of four complex lines.*
(b) *If $R, \overline{R} \subset C$ are complex conjugate lines such that $|R \cap \overline{R}| = 0$, then there exist complex conjugate lines L and $\overline{L}$ such that $C = L \cup \overline{L} \cup R \cup \overline{R}$ and $|L \cap R| = |L \cap \overline{R}| = |\overline{L} \cap R| = |\overline{L} \cap \overline{R}| = 1$.*

Proof. (a). Recall from Example 21 that the incidences of complex lines and isolated singularities in X are depicted in the rightmost diagram of Figure 4. By assumption, the singular points are contained in the hyperplane section C. A complex line in X that meets a hyperplane in $\mathbb{P}^4$ in more than one complex point must be contained in this hyperplane. Hence, C consists by Bézout's theorem of four complex lines.

(b). By Proposition 19 (c), two complex lines in X are complex conjugate if and only if their classes in $E(X)$ are related via σ_*. By assumption, X is either a Perseus cyclide, CH1 cyclide or ring cyclide. Hence, σ_* is of type $2A_1$ by Theorem A (see the second column of Table 5). In particular, $\sigma_*(e_1) = e_2$, $\sigma_*(e_3) = e_4$, $\sigma_*(e_3') = e_4'$, $\sigma_*(e_{01}) = e_{02}$, $\sigma_*(e_{11}) = e_{12}$ and $\sigma_*(e_{13}) = e_{14}$.

First, suppose that X is a Perseus cyclide. Let us consider the incidences between the complex lines in X as depicted in the leftmost diagram of Figure 4. Notice that R and $\overline{R}$ are presented by line segments that are labeled with $[R]$ and $\sigma_*([R])$, respectively. Since $|R \cap \overline{R}| = 0$, we observe that $([R], [\overline{R}])$ is equal to either (e_3, e_4), (e_3', e_4'), (e_{11}, e_{12}), or (e_{01}, e_{02}). If $([R], [\overline{R}]) = (e_3, e_4)$, then the complex conjugate lines L and $\overline{L}$ such that $([L], [\overline{L}]) = (e_3', e_4')$, each meet $R \cup \overline{R}$ (and thus the hyperplane C) in two complex points. A complex line in X that meets a hyperplane in $\mathbb{P}^4$ in more than one complex point must be contained in this hyperplane and thus $L, \overline{L} \subset C$. Therefore, $C = L \cup \overline{L} \cup R \cup \overline{R}$ by Bézout's theorem and $|L \cap R| = |L \cap \overline{R}| = |\overline{L} \cap R| = |\overline{L} \cap \overline{R}| = 1$. The remaining three cases for $([R], [\overline{R}])$ are symmetric, and thus Assertion (b) holds for the Perseus cyclide.

If X is a CH1 cyclide or ring cyclide, then Assertion (b) is shown analogously using the corresponding diagrams in Figure 4, except that $([R], [L]) \in \{(e_3, e_{13}), (e_{13}, e_3)\}$ and $([R], [L]) \in \{(e_1, e_3), (e_3, e_1)\}$, respectively, where $[\overline{R}] = \sigma_*([R])$ and $[\overline{L}] = \sigma_*([L])$. In particular, if X is a CH1 cyclide, then $([R], [\overline{R}]) \neq (e_1, e_2)$, since $|R \cap \overline{R}| = 0$ and the line segments labeled with e_1 and e_2 in the middle diagram of Figure 4 represent complex conjugate lines that meet at an isolated singularity. $\square$

5. Cliffordian Darboux cyclides

We develop a necessary condition for a Darboux cyclide X to be Cliffordian in terms of the sets $B(X)$, $E(X)$ and $G(X)$ in Table 6.

Suppose that $X \subset \mathbb{S}^3$ is a Darboux cyclide. For $a, b \in N(X)$, we set $a \odot b := 1$ if either $a \cdot b = 1$ or if $a \neq b$ and there exists a component $W \subset B(X)$ such that both $a \cdot W > 0$ and $b \cdot W > 0$; in all other cases we set $a \odot b := 0$. Notice that if $L, L' \subset X$ are different complex lines, then $|L \cap L'| = [L] \odot [L']$ by Proposition 19 (d) and thus the operator $\odot$ provides an algebraic criterion for complex lines to intersect.

A *Clifford quartet* is defined as a subset $\{a,b,c,d\}\subset E(X)$ of cardinality four such that $\sigma_*(a)=b$, $\sigma_*(c)=d$, $a\odot b=c\odot d=0$ and $a\odot c=c\odot b=b\odot d=d\odot a=1$.

Example 25. If X is a ring cyclide, then $E(X)=\{e_1,e_2,e_3,e_4\}$ by Theorem A. Since $|E(X)|=4$, $\sigma_*(e_1)=e_2$, $\sigma_*(e_3)=e_4$, $e_1\odot e_2=e_3\odot e_4=0$ and $e_1\odot e_3=e_3\odot e_2=e_2\odot e_4=1$, we find that $E(X)$ forms a Clifford quartet. Recall from Example 21 that the diagram for the ring cyclide in Figure 4 represents each class $a\in E(X)$ in terms of a line segment, and $a\odot b=1$ for $a,b\in E(X)$ if and only if the two corresponding line segments in the diagram meet at a disc. Hence, we can use such diagrams, together with the specification of σ_*, to recognize Clifford quartets.

Recall from Remark 8 that $\mathbf{P}(S^3)=\mathbb{S}^3$ and thus $\mathbf{P}(A\star B)\subset\mathbb{S}^3$ for all $A,B\subset S^3$.

Lemma 26. *If $\mathbf{P}(A\star B)$ is a Darboux cyclide for some circles $A,B\in S^3$, then $\mathbf{P}(A\star B)\cap\mathbb{E}$ consists of two left generators and two right generators whose classes form a Clifford quartet.*

Proof. Let $F\subset\mathbf{P}(A\star B)\times\mathbf{P}(A)$ and $G\subset\mathbf{P}(A\star B)\times\mathbf{P}(B)$ be the left and right associated pencils of $A\star B$, respectively. We set $X:=\mathbf{P}(A\star B)$.

If neither F nor G has base points in $\mathbb{E}$, then it follows from Lemma 17 and Bézout's theorem that $X\cap\mathbb{E}=\{x\in X\mid x_0=0\}$ consist of two left generators and two right generators. By Proposition 19 (d), the classes of these generators form a Clifford quartet and thus the proof is concluded for this case.

In the remainder of the proof we assume that F has a base point in $\mathbb{E}$. Since $\mathbb{E}_{\mathbb{R}}=\varnothing$, we find that F has two complex conjugate base points in $\mathbb{E}$. Recall from Proposition 19 (a) and (e) that each base point corresponds to a complex isolated singularity of X.

First suppose that G has base points in $\mathbb{E}$ as well. These base points must be complex conjugate and thus X is a ring cyclide by Lemma 23 (b). It follows from Lemma 24 (a) that the hyperplane section $\mathbb{E}\cap X$ consists of four complex lines. Recall from Example 25 that $E(X)$ defines a Clifford quartet and thus we concluded the proof.

Finally, suppose that G does not have base points in $\mathbb{E}$. In this case, the hyperplane section $X\cap\mathbb{E}$ contains two right generators R and $\overline{R}$ by Lemma 17. We apply Lemma 23 (a) and (c) and find that X is either a Perseus cyclide, ring cyclide or CH1 cyclide. The main assertion now follows from Lemma 24 (b). □

Now we introduce an algebraic necessary condition for a Darboux cyclide to be Cliffordian.

Definition 27. *Suppose that $X\subset\mathbb{S}^3$ is a celestial Darboux cyclide. We call (A,a,g,U) a* Clifford data *if*

- *$A=\{a,b,c,d\}$ is a Clifford quartet for X with distinguished element a,*
- *$g\in G(X)$ such that $\sigma_*(g)=g$ and $g\cdot a\neq 0$, and*
- *$U=\{e\in E(X)\mid e\cdot a=1$ and $e\odot b=e\odot c=e\odot d=0\}$.*

We call the Clifford data (A,a,g,U) a certificate *if $g\cdot u\neq 0$ for all $u\in U$. We say that X satisfies the* Clifford criterion *if there exist at least one certificate.*

Example 28. It follows from Theorem A that $(\{e_1,e_2,e_3,e_4\},e_1,g_{12},\varnothing)$ is a certificate for a ring cyclide $X\subset\mathbb{S}^3$, and thus a ring cyclide satifies the Clifford criterion. Similarly, a Perseus cyclide and CH1 cyclide satisfy the Clifford criterion with certificates $(\{e_{01},e_{02},e_{11},e_{12}\},e_{01},g_1,\varnothing)$ and $(\{e_3,e_4,e_{13},e_{14}\},e_3,g_{34},\varnothing)$, respectively. In contrast, the Clifford data $(\{e_1,e_2,e'_3,e'_4\},e_1,g_{12},\{e_{01},e_{11},e'_2\})$ for a Blum cyclide is not a certificate, since $g_{12}\cdot e_{11}=0$ (see Figure 5). In fact, the Blum cyclide does not satisfy the Clifford criterion. See [16, `cyclides`] for a software implementation that computes for each case in Table 6 all possible Clifford data, and checks whether they are certificates. For the Blum cyclide see alternatively Figure 11.

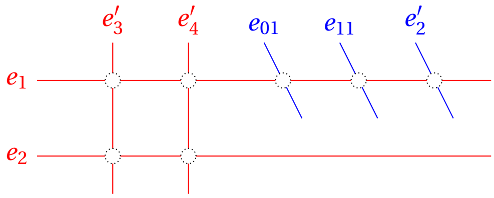


Figure 5. Incidences between 7 of the 16 lines in a Blum cyclide.

Remark 29. The following lemma can be seen as a generalization of [4, p. 7.94]. Donald Coxeter refers to [9, Chapter X] and Felix Klein attributes these insights to William Kingdon Clifford (1845–1879 CE). Clifford passed away at an early age and his theories in elliptic geometry were only partially published. Klein saw it as a duty to workout and disseminate these theories [9, p. 238]. To my mind, Clifford taught us that by combining most elementary curves we gain essential insights into the geometry of space.

Lemma 30. *If $X = \mathbf{P}(A \star B) \subset \mathbb{S}^3$ is a Darboux cyclide for some circles $A, B \in S^3$, then X satisfies the Clifford criterion.*

Proof. Let $F \subset \mathbf{P}(A \star B) \times \mathbf{P}(A)$ and $G \subset \mathbf{P}(A \star B) \times \mathbf{P}(B)$ be the left and right associated pencils of $A \star B$, respectively.

First suppose that F or G has base points on $\mathbb{E}$. These base points must be complex conjugate as $\mathbb{E}_{\mathbb{R}} = \varnothing$. Lemmas 26 and 23 (c) imply that X is not a CY cyclide or CO cyclide. Thus, it follows from Lemma 23 (a) that X is either a Perseus cyclide, ring cyclide or CH1 cyclide and the main assertion holds for these three cases by Example 28.

In the remainder of the proof we assume that neither F nor G has base points on $\mathbb{E}$.

Notice that $X \cap \mathbb{E} = \{x \in X \mid x_0 = 0\}$ defines a hyperplane section. Hence, the intersection $\mathbf{P}(A) \cap \mathbb{E}$ consists by Bézout's theorem of the complex conjugate points $\{\mathfrak{a}, \overline{\mathfrak{a}}\}$. We obtain for all complex $\alpha \in \mathbf{P}(A) \setminus \{\mathfrak{a}, \overline{\mathfrak{a}}\}$ the complex left Clifford translation $\varphi_\alpha \in \mathrm{LT}\,\mathbb{S}^3$ such that $\varphi_\alpha(x) = \alpha \mathbin{\widehat{\star}} x$ for all $x \in \mathbb{S}^3 \setminus \mathbb{E}$. By definition, $\varphi_\alpha(\mathbf{P}(B))$ is a member of the pencil F for all $\alpha \in \mathbf{P}(A) \setminus \{\mathfrak{a}, \overline{\mathfrak{a}}\}$. We know from Proposition 10 (b) that $\varphi_\alpha \in \mathrm{Aut}_{\mathbb{E}}\,\mathbb{S}^3$ and thus $\varphi_\alpha(\mathbf{P}(B))$ is an irreducible complex conic for all $\alpha \in \mathbf{P}(A) \setminus \{\mathfrak{a}, \overline{\mathfrak{a}}\}$.

We know from Lemma 26 that $X \cap \mathbb{E}$ consists of two left generators $L, \overline{L} \subset \mathbb{E}$ and two right generators $R, \overline{R} \subset \mathbb{E}$ intersecting in four complex points $\mathfrak{p}, \overline{\mathfrak{p}}, \mathfrak{q}, \overline{\mathfrak{q}} \in \mathbb{E}$ (see Figure 6). It follows from Lemmas 16 (b) and 17 (a) that $\left|\left\{i \in \mathbb{P}^1 \;\middle|\; p \in F_i\right\}\right| = 1$ for all $p \in L \cup \overline{L}$. Since $\mathbb{E} \subset \mathbb{S}^3$ is a hyperplane section and $|F_i \cap (L \cup \overline{L})| = 2$, we deduce from Bézout's theorem that $F_i \cap R = \varnothing$ for almost all $i \in \mathbb{P}^1$. Thus, we know from Lemma 16 (c) that the unique member of F that contains $\mathfrak{p} \in L$ is a complex reducible conic with R as component. Similarly, the unique member of F that contains $\mathfrak{q} \in L$ has $\overline{R}$ as component. Since the complex left Clifford translations of $\mathbf{P}(B)$ are irreducible, it follows that $\varphi_\alpha(\mathbf{P}(B)) \cap \{\mathfrak{p}, \mathfrak{q}\} = \varnothing$ for all $\alpha \in \mathbf{P}(A) \setminus \{\mathfrak{a}, \overline{\mathfrak{a}}\}$. We established the following complex continuous map:

$$\xi\colon \mathbf{P}(A) \setminus \{\mathfrak{a}, \overline{\mathfrak{a}}\} \to L \setminus \{\mathfrak{p}, \mathfrak{q}\}, \quad \alpha \mapsto \varphi_\alpha\big(\mathbf{P}(B)\big) \cap L.$$

In fact, ξ is an complex isomorphism as each point on L is reached by exactly one member of F by Lemma 16 (b). In other words, the complex left Clifford translations of the circle $\mathbf{P}(B)$ trace out $L \setminus \{\mathfrak{p}, \mathfrak{q}\}$.

Suppose that $U \subset E(X)$ is the set of classes of complex lines $M \subset X$ such that $[M] \cdot [L] = 1$ and $[M] \odot [\overline{L}] = [M] \odot [R] = [M] \odot [\overline{R}] = 0$. By Proposition 19 (d), we have $M \cap L = \{m\}$ and $m \in L \setminus \{\mathfrak{p}, \mathfrak{q}\}$. We established that there exists $\alpha \in \mathbf{P}(A)$ such that $m \in F_\alpha$ and thus $F_\alpha \cap M \neq \varnothing$. Suppose by contradiction that $[F_\alpha] \cdot [M] = 0$. By Proposition 19 (d), there exists a component $W \subset B(X)$ such that $[F_\alpha] \cdot W > 0$ and $[M] \cdot W > 0$. It follows from Proposition 19 (a) and (e) that m is a base

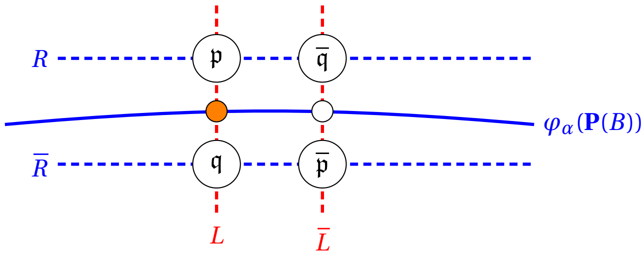


Figure 6. The incidences between L, $\overline{L}$, R, $\overline{R}$ and $\varphi_\alpha(\mathbf{P}(B))$ for some $\alpha \in \mathbf{P}(A) \setminus \{\mathfrak{a}, \overline{\mathfrak{a}}\}$.

point of F. We arrived at a contradiction as F does not have base points on $\mathbb{E}$. Therefore, we require that $[F_\alpha] \cdot [M] \neq 0$ for all $[M] \in U$. We conclude from Proposition 19 (b), (c) and (d) that $\left(\left\{[L], [\overline{L}], [R], [\overline{R}]\right\}, [L], [F_\alpha], U\right)$ is a certificate and thus X satisfies the Clifford criterion. □

Proposition 31. *A Cliffordian Darboux cyclide $X \subset \mathbb{S}^3$ is either a Perseus cyclide, ring cyclide or CH1 cyclide.*

Proof. The Darboux cyclide X satisfies the Clifford criterion by Lemma 30. We apply Theorem A and consider the 14 triples $\big(B(X), E(X), G(X)\big)$ in Table 6. For each such triple we go through all possible Clifford quartets in $E(X)$. For each such Clifford quartet A we consider all possible Clifford data (A, a, g, U). We verify that only a Perseus cyclide, ring cyclide or CH1 cyclide admits a Clifford data (A, a, g, U) that is a certificate. We used [16, `cyclides`] to do the verification automatically. In particular, we find that a Clifford quartet exists only if X is a Blum cyclide, Perseus cyclide, ring cyclide, EH1 cyclide, CH1 cyclide, HP cyclide, or S1 cyclide. □

Example 32. We show that the surfaces Z_{01}, Z_{23} and Z_{45} defined at Section 1 are a ring cyclide, Perseus cyclide and CH1 cyclide, respectively. Moreover, we show that Z_{06} and Z_{78} are Cliffordian surfaces of degree 8. The required computations are done automatically at [16, `cyclides`]. See [14, `orbital`] for an alternative implementation of these methods. Suppose that $0 \le i \le 8$. Let $M_i :=$`Mi` be the corresponding 5×5 matrix in Table 7.

Table 7. 5×5 matrices that represent elements in $\operatorname{Aut}\mathbb{S}^3$.

```
M0 = [( 1, 0,0, 0, 0), ( 0, 1,0,0,  0), (0,0, 1, 0,0), ( 0,0,0,1, 0), (0, 0,0, 0, 1)]
M1 = [( 5, 0,0, 0, 0), ( 0, 5,0,0,  0), (0,0, 4,-3,0), ( 0,0,3,4, 0), (0, 0,0, 0, 5)]
M2 = [( 3, 0,0,-2,-2), ( 0, 1,0,0,  0), (0,0, 1, 0,0), (-2,0,0,1, 2), (2, 0,0,-2,-1)]
M3 = [( 3, 0,0, 2,-1), ( 0, 2,0,0,  0), (0,0, 2, 0,0), ( 2,0,0,2,-2), (1, 0,0, 2, 1)]
M4 = [( 3,-2,0, 0,-1), (-2, 2,0,0,  2), (0,0, 2, 0,0), ( 0,0,0,2, 0), (1,-2,0, 0, 1)]
M5 = [( 3, 2,0, 0,-1), ( 2, 2,0,0, -2), (0,0, 2, 0,0), ( 0,0,0,2, 0), (1, 2,0, 0, 1)]
M6 = [(17,12,0, 0,-9), (12, 8,0,0,-12), (0,0, 8, 0,0), ( 0,0,0,8, 0), (9,12,0, 0,-1)]
M7 = [( 3,-2,0, 0,-1), ( 2,-2,0,0, -2), (0,0,-2, 0,0), ( 0,0,0,2, 0), (1,-2,0, 0, 1)]
M8 = [( 3, 2,0, 0,-1), ( 2, 2,0,0, -2), (0,0, 2, 0,0), ( 0,0,0,2, 0), (1, 2,0, 0, 1)]
```

Let J be the diagonal matrix with $(-1, 1, 1, 1, 1)$ on its diagonal. We verify that there exists $\lambda \in \mathbb{Q} \setminus \{0\}$ such that $M_i^\top \cdot J \cdot M_i = \lambda J$, and thus M_i defines a Möbius transformation $\varphi_i : \mathbb{S}^3 \to \mathbb{S}^3$. The curve parametrization $C_i(t)$ in Table 2 is related to the matrix M_i as follows, where $\psi(t) := (1 : \cos(t) : \sin(t) : 0 : 0)$:

$$\mathbf{P}(\{C_i(t) \,|\, 0 \le t \le 2\pi\}) = \left\{(\varphi_i \circ \psi)(t) \,\middle|\, 0 \le t \le 2\pi\right\}.$$

Since $\psi(t)$ parametrizes a great circle, we observe that $C_i(t)$ parametrizes a circle as well. Let $\vec{c} := (1, 0, 0, 0, 0)$ and notice that $\mathbf{R}((1:0:0:0:0))$ is the center of S^3. If $i \in \{0, 1\}$, then we verify that there exists $\lambda \in \mathbb{Q} \setminus \{0\}$ such that $M_i \cdot \vec{c} = \lambda\, \vec{c}$. Hence, $C_0(t)$ and $C_1(t)$ parametrize great circles. For all $(i, j) \in \{(0,1), (2,3), (4,5), (0,6), (7,8)\}$, we implicitize the surface

$$X_{ij} := \mathbf{P}\left(\left\{C_i(\alpha) \star C_j(\beta) \,\middle|\, 0 \le \alpha, \beta < 2\pi\right\}\right) \subset \mathbb{S}^3$$

and find the following (we refer to [16, `cyclides`] for the details):

$$(\deg X_{01}, \deg X_{23}, \deg X_{45}, \deg X_{06}, \deg X_{78}) = (4,4,4,8,8) \quad \text{and}$$
$$(|\operatorname{Sing} X_{01}|, |\operatorname{Sing} X_{23}|, |\operatorname{Sing} X_{45}|) = (4,2,3).$$

Since $X_{ij} = \mathbf{S}(Z_{ij})$ by definition, it follows from Proposition 31 and Theorem A (see the SingType X column in Table 5) that Z_{01}, Z_{23} and Z_{45} are a ring cyclide, Perseus cyclide and CH1 cyclide, respectively. As $\deg X_{06} = \deg X_{78} = 8$, we find that Z_{06} and Z_{78} are Cliffordian surfaces of Möbius degree 8. Since $C_0(t)$ parametrizes a great circle, it follows that the surfaces Z_{01} and Z_{06} are great.

6. Bohemian Darboux cyclides

We show that a Bohemian Darboux cyclide is the pointwise sum of a line and a circle in $\mathbb{R}^3$, namely a CY or EY as in Figure 1, and thus not the pointwise sum of two circles in $\mathbb{R}^3$.

Remark 33. Before we apply the methods established in the previous sections, let us sketch an alternative proof strategy for the claim that a Darboux cyclide is not the pointwise sum of two circles. Indeed, if $Z \subset \mathbb{R}^3$ is the pointwise sum of two circles, then there exist infinitely many parallel plane sections of Z that consist of two circles that belong to the same pencil (see for example Figure 2). As we continuously vary such a plane section, the two circles can be deformed into a single circle. The points along which the coplanar circles intersect trace out an arc in the singular locus of Z. A real singularity in a Darboux cyclide must be isolated and thus Z is not a Darboux cyclide. Depending on the background of the reader this or a similar strategy may be more appropriate.

Lemma 34. *Suppose that $X \subset \mathbb{S}^3$ is a Darboux cyclide that contains two circles through a general point that do not meet in two points. If some pair of complex conjugate lines in X intersect, then these complex lines meet at an isolated singularity.*

Proof. Let $\mathcal{G}(X) := \{g \in G(X) \,|\, \sigma_*(g) = g\}$ and suppose that $L, \overline{L} \subset X$ are complex conjugate lines such that $L \cap \overline{L} \neq \varnothing$. We know from Proposition 19 (b) that there exist different $f, f' \in \mathcal{G}(X)$ such that $f \cdot f' \neq 2$. By Proposition 19 (c), both $[L]$ and $\sigma_*([L]) = [\overline{L}]$ belong to $E(X)$. We apply Theorem A and verify for each of the 14 cases in Table 6 that the following statement holds: If there exist different $f, f' \in \mathcal{G}(X)$ such that $f \cdot f' \neq 2$, then $e \cdot \sigma_*(e) = 0$ for all $e \in E(X)$. See [16, `cyclides`] for an automatic verification of this statement. It follows from Proposition 19 (d) that there exists a component $W \subset B(X)$ such that $[L] \cdot W > 0$ and $\sigma_*([L]) \cdot W > 0$. By Proposition 19 (a) such a component W corresponds to an isolated singularity in $L \cap \overline{L}$ and thus we concluded the proof. □

Proposition 35. *If $A, B \subset \mathbb{R}^3$ are generalized circles such that $\mathbf{S}(A+B)$ is a Darboux cyclide, then $A+B$ is either a CY or EY.*

Proof. We first consider the quadratic case:

Claim 1. *If $\deg(A+B) \le 2$, then $A+B$ is either a CY or EY and either A or B is a line.*

Since $\deg \mathbf{S}(A+B) = 4$, we find that $\deg(A+B) = 2$. We go through the well-known classification of quadratic surfaces up to Euclidean similarity (see [15, Proposition 4] and Figure 1) and conclude that Claim 1 holds.

Now let us assume by contradiction that $\deg A = \deg B = 2$. Notice that $A+B$ is not a plane by Claim 1 and thus the circles A and B are not coplanar.

Let $C_v := \{v\} + B$ and $D_v := A + \{v\}$ for $v \in \mathbb{R}^3$.

Claim 2. $|C_a \cap D_b| = 1$ *for almost all* $a \in A$ *and* $b \in B$.

Let $H_a \subset \mathbb{R}^3$ denote the spanning plane of C_a. Since a and b are general, there exists $q \in A \setminus \{a\}$ such that the circle D_b meets H_a transversally at the two points $a+b \in A$ and $q+b$. As we translate the circle D_b along C_a, the incidence points $a+b$ and $q+b$ trace out the coplanar circles $C_a \subset H_a$ and $C_q \subset H_a$, respectively. Since $|C_a \cap C_q| < \infty$ and b is general, we deduce that $q + b \notin C_a$. It follows that $C_a \cap D_b = \{a+b\}$ and thus Claim 2 holds true.

Claim 3. $|C_i \cap C_j| = 0$ *for almost all* $i, j \in A$.

The spanning planes of C_i and C_j are parallel, but not equal. This implies the assertion of Claim 3.

Let $F \subset \mathbf{S}(A+B) \times \mathbf{S}(A)$ and $G \subset \mathbf{S}(B+A) \times \mathbf{S}(B)$ be the associated pencils of $A+B$ and $B+A$, respectively, where $A+B = B+A$. It follows from Claim 2 that $|F_a \cap G_b| = 1$ for almost all $a \in \mathbf{S}(A)$ and $b \in \mathbf{S}(B)$. We deduce from Claim 3 that a base point of F or G must lie in $\mathbb{U}$. We make a case distinction on the base points of F and G.

First, we suppose that either F or G has no base points. We know from Lemma 18 that X contains complex conjugate lines that meet at the vertex of $\mathbb{U}$. General members of F and G meet in one point and thus X has an isolated singularity at this vertex by Lemma 34. Hence, $\deg \pi(X) = \deg(A+B) = 2$ with $\deg A = \deg B = 2$. We arrived at a contradiction with Claim 1.

Next, we suppose that F has a real base point in $\mathbb{U}_{\mathbb{R}}$. In this case $\deg \pi(X) = \deg(A+B) = 2$ with $\deg A = \deg B = 2$, since $\mathbb{U}_{\mathbb{R}} \subset \operatorname{Sing} X$ by Proposition 19 (a) and (e). We arrived at a contradiction with Claim 1.

Finally, suppose by contradiction that both F and G have non-real base points in $\mathbb{U}$. Then X must be a ring cyclide by Lemma 23 (b) and $\operatorname{Sing} X \subset \mathbb{U}$ by Proposition 19 (a) and (e). It follows from Lemma 24 (a) that the hyperplane section $X \cap \mathbb{U}$ consists of four lines. We arrived at a contradiction with the diagram in Figure 4 (see Example 21), as all lines in $\mathbb{U}$ should be concurrent.

We arrived at a contradiction at all three cases and thus we established either A or B is a line. Therefore, $A+B$ is covered with lines, so that either F or G has a base point in $\mathbb{U}_{\mathbb{R}}$. Hence, by Proposition 19 (a) and (e) the center of stereographic projection π is in $\operatorname{Sing} X$ so that $\deg(A+B) \leq 2$. The proof is now concluded by Claim 1. □

7. Great Darboux cyclides

In this section, we show that a Darboux cyclide is Möbius equivalent to a great celestial surface if and only if this cyclide is either a Blum cyclide, Perseus cyclide, ring cyclide, EO cyclide or CO cyclide. Moreover, we show that great ring cyclides are Cliffordian and that great Perseus cyclides are not Cliffordian. The hyperbolic and Euclidean analogues of great Darboux cyclides are considered as well.

Remark 36. We prove Theorem 1 (c) by applying also methods from [3, 21, 28] as formulated at Lemmas 37, 40 and 41 and Proposition 39 below. Remark 38 below sketches how these methods can be used to classify Darboux cyclides up to Möbius equivalence. Since Theorem 1 (c) can be read off from this classification, this is an alternative proof strategy avoiding the use of Theorem A and Proposition 19.

We call a Darboux cyclide $X \subset \mathbb{S}^3$ *elliptic* or δ*-elliptic* for some $\delta \in \{1,2\}$, if there exists a Möbius transformation $\varphi \in \operatorname{Aut}\mathbb{S}^3$ such that $(\tau \circ \varphi)(X)$ is a ruled quadratic surface in $\mathbb{P}^3$ that is covered by $\delta > 0$ pencils of lines. The δ*-hyperbolic* and δ*-Euclidean* Darboux cyclides are defined analogously, but with the central projection τ replaced with the vertical projection ν and stereographic projection π, respectively (recall Remark 9).

Notice that an elliptic Darboux cyclide is Möbius equivalent to a great Darboux cyclide, since great circles are centrally projected to lines.

The following classical result is essentially [3, Chapter VII, Theorem 20, p. 296], but we followed the proof of [27, Theorem 5.30 in the updated arXiv version].

Lemma 37. *If $X \subset \mathbb{S}^3$ is a surface such that* $\deg X \neq 2$ *and $F, G \subset X \times \mathbb{P}^1$ are pencils of circles such that $|F_i \cap G_j| = 2$ for almost all $i, j \in \mathbb{P}^1_{\mathbb{R}}$, then X is a Darboux cyclide that is either elliptic, hyperbolic or Euclidean. Moreover, if F is base point free, then X is either 2-elliptic or 2-hyperbolic.*

Proof. Let $u, v \in \mathbb{P}^1_{\mathbb{R}}$ be general. Two planes in $\mathbb{P}^4$ have non-empty intersection and thus the planes spanned by the irreducible conics F_u and F_v meet at a point $p \in \mathbb{P}^4$. As $\deg X \neq 2$, we deduce that p is the unique intersection point between these two planes. Since $|F_u \cap G_j| = 2$, the spanning planes of F_u and G_j intersect along a line. This implies that p is the intersection point of the two lines containing the pairs $F_u \cap G_j$ and $F_v \cap G_j$, respectively. We established that the spanning plane of G_j contains p. Repeating the same argument with F and G interchanged shows that the complex spanning plane of F_i contains the point p as well.

We claim that there exists a Möbius transformation $\varphi \in \operatorname{Aut}\mathbb{S}^3$ such that $\varphi(p)$ coincides with the projection center of either τ, ν or π. Indeed, $p \in \mathbb{P}^4$ corresponds via the hyperquadric $\mathbb{S}^3$ uniquely to its polar hyperplane section $H_p \subset \mathbb{S}^3$. Since H_p is Möbius equivalent to either $\mathbb{E}$, $\mathbb{Y}$ or $\mathbb{U}$, there exists $\varphi \in \operatorname{Aut}\mathbb{S}^3$ such that $\varphi(p)$ is equal to the projection centers $(1:0:0:0:0)$, $(0:0:0:0:1)$ and $(1:0:0:0:1)$, respectively.

First suppose that $\varphi(p) = (1:0:0:0:0)$. In this case, the general members F_i and G_j are 2:1 projected to complex lines via the map $\tau \circ \varphi$. Since $\deg X \neq 2$ by assumption, we deduce that $\deg(\tau \circ \varphi)(X) = 2$ and thus X is an elliptic Darboux cyclide. If F is base point free, then two general members of F are disjoint. This implies that the 2:1 projections of two general members of F are disjoint lines in the quadric $(\tau \circ \varphi)(X)$. Hence, the ruled quadric $(\tau \circ \varphi)(X)$ must be smooth and thus doubly ruled. This implies that X is 2-elliptic.

If $\varphi(p) = (0:0:0:0:1)$, then X must be a hyperbolic Darboux cyclide and if F is base point free, then X is 2-hyperbolic. The proof is analogous as in the elliptic case.

Finally, suppose that $\varphi(p) = (1:0:0:0:1)$. In this case, F and G have a common base point at p. Thus, the general members F_i and G_j are via the map $\pi \circ \varphi$ birationally projected to complex lines. Since $\deg X \neq 2$, we find that X must be an Euclidean Darboux cyclide.

We considered all three cases and thus the proof is concluded. □

We remark that a celestial Darboux cyclide that is neither a CO cyclide nor a CY cyclide satisfies the hypothesis of Lemma 37 by Theorem A and Proposition 19.

Let $\operatorname{Aut}_H \mathbb{P}^3 := \{\varphi \in \operatorname{Aut}\mathbb{P}^3 \,|\, \varphi(H) = H\}$, where $\operatorname{Aut}\mathbb{P}^3$ denotes the real projective transformations of $\mathbb{P}^3$ and $H \in \{\pi(\mathbb{U}), \tau(\mathbb{E}), \nu(\mathbb{Y})\}$ (see Remark 9). Recall from Section 2 the related definition of $\operatorname{Aut}_W \mathbb{S}^3 \cap \operatorname{Aut}\mathbb{S}^3$ for $W \in \{\mathbb{U}, \mathbb{E}, \mathbb{Y}\}$.

Remark 38. We sketch how the methods in [21] can recover the classification of Darboux cyclides up to $\operatorname{Aut}\mathbb{S}^3$ as stated in [28]. If $X \subset \mathbb{S}^3$ is a Darboux cyclide, then it follows from [21, Theorem 3] that up to $\operatorname{Aut}\mathbb{S}^3$ either $\pi(X)$, $\tau(X)$ or $\nu(X)$ is a quadratic surface $Y \subset \mathbb{P}^3$. This implies that X is the intersection of $\mathbb{S}^3$ with a quadratic hypercone in $\mathbb{P}^4$ over Y. The Möbius transformations $\operatorname{Aut}_W \mathbb{S}^3 \cap \operatorname{Aut}\mathbb{S}^3$ for $W \in \{\mathbb{U}, \mathbb{E}, \mathbb{Y}\}$ induce the projective transformations $\operatorname{Aut}_H \mathbb{P}^3$ such that $H \in \{\pi(\mathbb{U}), \tau(\mathbb{E}), \nu(\mathbb{Y})\}$, respectively. The classification of X up to $\operatorname{Aut}\mathbb{S}^3$ can therefore be reduced to the classification of $Y \subset \mathbb{P}^3$ up to $\operatorname{Aut}_H \mathbb{P}^3$ for $H \in \{\pi(\mathbb{U}), \tau(\mathbb{E}), \nu(\mathbb{Y})\}$.

- If $Y = \pi(X)$, then $\mathrm{Aut}_{\pi(\mathbb{U})}\,\mathbb{P}^3$ is a projectivization of the Euclidean similarities of $\mathbb{R}^3$ and thus the classification of Y is classically known (see for example [15, Proposition 4]).
- If $Y = \tau(X)$, then an application of orthogonal diagonalization in linear algebra shows that Y is up to $\mathrm{Aut}_{\tau(\mathbb{E})}\,\mathbb{P}^3$ the zero set of a diagonal quadratic form.
- The case when $Y = \nu(\mathbb{Y})$ is less known and treated by Proposition 39 below. Mikhail Skopenkov explained me how this proposition is an application of Uhlig's [29, Theorem 1]. See [28, Section 6] by Takeuchi for a related approach.

We can use [21, Theorem 3] to recover from Y the number λ such that X is λ-circled (see Lemma 40 below).

The *signature* of a real symmetric matrix M is denoted by $\mathrm{sgn}(M) = (p,n)$, where p and n denote the number of positive and negative eigenvalues of M, respectively. Let $\mathrm{diag}(M_1,\ldots,M_r)$ denote the block diagonal matrix that has the square matrices $M_1,\ldots,M_r$ on its diagonal. If the square matrix M_i is equal to the 1×1 matrix $[m_i]$ for all $1 \le i \le r$, then we write $\mathrm{diag}(m_1,\ldots,m_r)$ instead.

Proposition 39. *If $Y \subset \mathbb{P}^3$ is a quadratic surface, then there exist $\varphi \in \mathrm{Aut}_{\nu(\mathbb{Y})}\,\mathbb{P}^3$ and $a,b,c,d \in \mathbb{R}$ such that Y is the zero set of one of the following quadratic forms:*

- $q_1 := a\,x_0^2 + b\,x_1^2 + c\,x_2^2 + d\,x_3^2$,
- $q_2 := a\,x_0^2 + x_0\,x_1 + (1-a)\,x_1^2 + b\,x_2^2 + c\,x_3^2$,
- $q_3 := a\,x_0^2 + x_0\,x_1 - a\,x_1^2 + b\,x_2^2 + c\,x_3^2$, *or*
- $q_4 := a\,x_0^2 - a\,x_1^2 + x_0x_2 + x_1x_2 - a\,x_2^2 + b\,x_3^2$.

Proof. Let J and M be the symmetric 4×4 matrices such that the quadratic surfaces $\nu(\mathbb{Y})$ and Y are the zero set of quadratic forms $\vec{x}^\top\cdot J\cdot\vec{x}$ and $\vec{x}^\top\cdot M\cdot\vec{x}$, respectively, where $\vec{x} = (x_0,x_1,x_2,x_3)^\top$. As a direct consequence of the definitions, we find that the main assertion is equivalent to the following statement: *There exists a matrix V, coefficients $a,b,c,d \in \mathbb{R}$ and $i \in \{1,2,3,4\}$ such that $V^\top\cdot J\cdot V = J$ and*

$$(V\cdot\vec{x})^\top\cdot M\cdot(V\cdot\vec{x}) = \vec{x}^\top\cdot(V^\top\cdot M\cdot V)\cdot\vec{x} = q_i.$$

The specialization of Uhlig's [29, Theorem 1] to dimension 4 states that there exists a matrix R and $\epsilon_i \in \{1,-1\}$ and $\alpha_i,\beta_i \in \mathbb{R}$ for $1 \le i \le 4$ and $1 \le r \le 4$ such that

$$K := R^\top\cdot J\cdot R = \mathrm{diag}(\epsilon_1\cdot E_1,\ldots,\epsilon_r\cdot E_r) \;\text{ and }\; N := R^\top\cdot M\cdot R = \mathrm{diag}(\epsilon_1\cdot F_1,\ldots,\epsilon_r\cdot F_r),$$

where (E_i,F_i) is for all $1 \le i \le r$ one of the following five pairs of matrices:

$$([\alpha_i],[1]),\qquad \left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&\alpha_i\\\alpha_i&1\end{bmatrix}\right),\qquad \left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}\beta_i&\alpha_i\\\alpha_i&-\beta_i\end{bmatrix}\right),$$

$$\left(\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&\alpha_i\\0&\alpha_i&1\\\alpha_i&1&0\end{bmatrix}\right),\qquad \left(\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&\alpha_i\\0&0&\alpha_i&1\\0&\alpha_i&1&0\\\alpha_i&1&0&0\end{bmatrix}\right).$$

Moreover, we require that if $\epsilon_i = -1$, then

$$(E_i,F_i) \neq \left(\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}\beta_i&\alpha_i\\\alpha_i&-\beta_i\end{bmatrix}\right).$$

We have $\mathrm{sgn}(K) = \mathrm{sgn}(J)$ as the signature is invariant under real congruence transformations and thus

$$\mathrm{sgn}(K) = \mathrm{sgn}(\epsilon_1\cdot E_1,\ldots,\epsilon_r\cdot E_r) = \mathrm{sgn}(\epsilon_1\cdot E_1)+\cdots+\mathrm{sgn}(\epsilon_r\cdot E_r) = (3,1),$$
$$\text{where }\; \mathrm{sgn}(E_i) \in \{(1,0),(1,1),(2,1),(2,2)\} \;\text{ and }\; \mathrm{sgn}(-E_i) \in \{(0,1),(1,1),(1,2),(2,2)\}.$$

It follows that the matrix pair (K,N) is equal to one of the following five pairs:

$$\left(\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}\alpha_1&0&0&0\\0&\alpha_2&0&0\\0&0&\alpha_3&0\\0&0&0&\alpha_4\end{bmatrix}\right),\quad \left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\begin{bmatrix}\alpha_1&0&0&0\\0&\alpha_2&0&0\\0&0&0&\alpha_3\\0&0&\alpha_3&1\end{bmatrix}\right),$$
$$\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\begin{bmatrix}\alpha_1&0&0&0\\0&\alpha_2&0&0\\0&0&\beta_3&\alpha_3\\0&0&\alpha_3&-\beta_3\end{bmatrix}\right),\quad \left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1\\0&0&-1&0\end{bmatrix},\begin{bmatrix}\alpha_1&0&0&0\\0&\alpha_2&0&0\\0&0&0&-\alpha_3\\0&0&-\alpha_3&-1\end{bmatrix}\right),$$
$$\left(\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix},\begin{bmatrix}\alpha_1&0&0&0\\0&0&0&\alpha_2\\0&0&\alpha_2&1\\0&\alpha_2&1&0\end{bmatrix}\right).$$

For each of these five cases for (K,N), we do the following computations as implemented at [16, `cyclides`]. We compute the orthogonal diagonalization of K and obtain an orthogonal matrix U such that $U^\top\cdot K\cdot U=J$. We verify that there exists $1\le i\le 4$ such that $q_i=\vec{x}^\top\cdot(U^T\cdot N\cdot U)\cdot\vec{x}$. It follows that if $V=R\cdot U$, then $V^\top\cdot J\cdot V^\top=J$ and $q_i=\vec{x}^\top\cdot(V^T\cdot M\cdot V)\cdot\vec{x}$ as was to be shown. □

We remark that if the ideal of $X\subset\mathbb{S}^3$ is generated by the quadratic forms q_3 and $-x_0^2+x_1^2+x_2^2+x_3^2+x_4^2$, then X is Möbius equivalent to the surface given by [28, Equation (3.2)].

Lemma 40. *Let* $X:=\{x\in\mathbb{P}^4\,|\,\vec{x}^\top\cdot Q\cdot\vec{x}=\vec{x}^\top\cdot J\cdot\vec{x}=0\}$*, where* $J:=\operatorname{diag}(-1,1,1,1,1)$*,* Q *is a symmetric* 5×5 *matrix and* $\vec{x}:=(x_0,\ldots,x_4)^\top$*. If* $X\subset\mathbb{S}^3$ *is a* λ*-circled Darboux cyclide and* $\{t\in\mathbb{R}\,|\,\det(Q-t\cdot J)=0\}=\{t_1,\ldots,t_r\}$*, then*

$$\lambda=\sum_{i=1}^{r}\chi\big(\operatorname{sgn}(Q-t_i\cdot J)\big),$$

where the function $\chi\colon\mathbb{Z}^2_{\ge 0}\to\{0,1,2\}$ *is defined as follows:*

$$\chi(\alpha,\beta):=\begin{cases}2 & \textit{if}\ \{\alpha,\beta\}=\{2\},\\ 1 & \textit{if}\ \{\alpha,\beta\}=\{1,2\},\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. This assertion is a consequence of [21, Theorem 3 and Remark 8]. We outline the argument for convenience of the reader, but refer to [21, Sections 2.3 and 2.4] for the details. By assumption, X is the base locus of the pencil of hyperquadrics in $\mathbb{P}^4$ that are generated by the hyperquadrics $\{x\in\mathbb{P}^4\,|\,\vec{x}^\top\cdot Q\cdot\vec{x}=0\}$ and $\mathbb{S}^3$. For each point $p\notin\mathbb{S}^3$ there exists a unique hyperquadric $H\subset\mathbb{P}^4$ such that $X=\mathbb{S}^3\cap H$ and $p\in H$ (see [21, Section 2.3]). If we choose p in a plane spanned by a circle $C\subset X$, then $\{p\}\cup C\subset H$ and thus H must be a cone by Bézout's theorem. This implies that H is equal to the hypercone $H_i:=\{x\in\mathbb{P}^4\,|\,\vec{x}^\top\cdot(Q-t_i\cdot J)\cdot\vec{x}=0\}$ for some $1\le i\le r$ and the 2-planes in H_i intersect X in conics. Moreover, for each pencil of circles F on X there exists a unique $1\le j\le r$ and a unique pencil G of 2-planes on H_j such that the circles that belong to F span infinitely many 2-planes that belong to G. As X is a Darboux cyclide by assumption, we deduce that $\operatorname{Sing}H_i$ consists of a single vertex v_i. Suppose that $\rho_i\colon\mathbb{P}^4\dashrightarrow\mathbb{P}^3$ is a linear projection with center v_i and let λ_i denote the number of pencils of 2-planes on H_i. We deduce that $\rho_i(X)$ is a quadratic surface with signature $\operatorname{sgn}(Q-t_i\cdot J)$ that is covered by exactly λ_i pencils of lines. It is now straightforward to see that $\lambda_i=\chi\big(\operatorname{sgn}(Q-t_i\cdot J)\big)$ and thus $\lambda=\lambda_1+\cdots+\lambda_r$ as was to be shown. □

Lemma 41. *If* $X\subset\mathbb{S}^3$ *is a* λ*-circled hyperbolic celestial Darboux cyclide such that* $\operatorname{Sing}X_{\mathbb{R}}=\varnothing$ *and* X *is not elliptic, then* $\lambda=2$.

Proof. Since X is hyperbolic, we may assume without loss of generality that $\nu(X)$ is a ruled quadric. Notice that the hyperbolic transformation of $\mathbb{S}^3$ commute via the vertical projection ν with the projective transformations in $\operatorname{Aut}_{\nu(\mathbb{Y})}\mathbb{P}^3$. Hence, it follows from Proposition 39 that there exists $\varphi\in\operatorname{Aut}\mathbb{S}^3$ and $a,b,c,d\in\mathbb{R}$ such that $(\nu\circ\varphi)(X)$ is the zero set of the quadratic form q_i for some $1 \leq i \leq 4$. This implies that the ideal of $\varphi(X)$ is equal to $\langle q_i, s\rangle$, where $s := -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$. We consider the following Jacobian matrix of the generators of the ideal $\langle q_i, s\rangle$:

$$F_i(x) := \begin{bmatrix} \partial_0 q_i & \partial_1 q_i & \partial_2 q_i & \partial_3 q_i & \partial_4 q_i \\ \partial_0 s & \partial_1 s & \partial_2 s & \partial_3 s & \partial_4 s \end{bmatrix} = \begin{bmatrix} \partial_0 q_i & \partial_1 q_i & \partial_2 q_i & \partial_3 q_i & 0 \\ -2x_0 & 2x_1 & 2x_2 & 2x_3 & 2x_4 \end{bmatrix},$$

where $\partial_j(\cdot)$ denotes the partial derivative with respect to the variable x_j. Notice that $p \in \operatorname{Sing} X$ if and only if $p \in X$ and the rank of the matrix $F_i(p)$ is not equal to 2. A direct computation shows that $F_k(x)$ has for all $k \in \{2,4\}$ rank one at the point $(1:-1:0:0:0)$. We refer to [16, `cyclides`] for the details of the matrix computations in this proof. Notice that

$$q_1 + a\cdot s = (b+a)\,x_1^2 + (c+a)\,x_2^2 + (d+a)\,x_3^2 + a\,x_4^2$$

and thus $\langle q_1, s\rangle$ is the ideal of an elliptic Darboux cyclide. Since $\operatorname{Sing} X_{\mathbb{R}} = \varnothing$ and X is not elliptic by assumption, we find that $i = 3$ is the only remaining option. Let $J := \operatorname{diag}(-1,1,1,1,1)$ and let Q be the symmetric matrix associated to q_3. In order to recover the number λ from the matrix Q, we apply Lemma 40. A direct computation shows that

$$\left\{t \in \mathbb{R} \,\middle|\, \det(Q - t\cdot J) = 0\right\} = \{0, b, c\}.$$

Moreover, we find that the nonzero eigenvalues of $Q - t\cdot J$ for $t \in \{0,b,c\}$ are as follows, where $u(t) := \frac{1}{2}\sqrt{1+4(a+t)^2}$:

$$\{u(0), -u(0), b, c\}, \quad \{u(b), -u(b), -b, -b+c\}, \quad \{u(c), -u(c), -c, b-c\}.$$

The next step is to use these eigenvalues to recover the following triple of signatures:

$$\Lambda := \big(\operatorname{sgn}(Q - 0\cdot J),\ \operatorname{sgn}(Q - b\cdot J),\ \operatorname{sgn}(Q - c\cdot J)\big).$$

We assume without loss of generality that $b \geq c$ and observe that $u(t) > 0$. We make a case distinction:

- If $b, c > 0$ or $b, c < 0$, then $\Lambda \in \{\big((3,1),(1,3),(2,2)\big),\ \big((1,3),(2,2),(3,1)\big)\}$.
- If $b > 0$ and $c \leq 0$, then $\Lambda \in \{\big((2,2),(1,3),(3,1)\big),\ \big((2,1),(1,3),(2,1)\big)\}$.
- If $b = 0$ and $c \leq 0$, then $\Lambda \in \{\big((1,2),(1,2),(3,1)\big),\ \big((1,1),(1,1),(1,1)\big)\}$.
- If $b = c \neq 0$, then $\Lambda \in \{\big((3,1),(1,2),(1,2)\big),\ \big((1,3),(2,1),(2,1)\big)\}$.

It follows from Lemma 40 and $|\{0,b,c\}| \leq 3$ that $\lambda \leq \chi(\Lambda_1) + \chi(\Lambda_2) + \chi(\Lambda_3)$, where $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$. We conclude that $\lambda = 2$ as was to be shown. ☐

Lemma 42. *Suppose that $X \subset \mathbb{S}^3$ is a celestial Darboux cyclide.*

(a) *If X is elliptic, then* $|\operatorname{Sing} X_{\mathbb{R}}| \in \{0,2\}$ *and* $\operatorname{Sing} X_{\mathbb{R}} = (\operatorname{Sing} X) \setminus \mathbb{E}$.
(b) *If X is 2-elliptic, then complex conjugate lines in X do not intersect.*
(c) *The surface X is 1-elliptic if and only if* $|\operatorname{Sing} X_{\mathbb{R}}| = 2$ *and X is covered by a pencil of circles with two real base points.*

Proof. (a). We may assume up to Möbius equivalence that $\tau(X)$ is a ruled quadric. Notice that the central projection τ is a 2:1 covering that defines, with respect to the complex analytic topology, locally a complex isomorphism outside the ramification locus $\mathbb{E}$. This implies that $\tau((\operatorname{Sing} X) \setminus \mathbb{E}) \subset \operatorname{Sing} \tau(X)$. If $|\operatorname{Sing} X_{\mathbb{R}}| = 0$, then $\tau(X)$ must be smooth and thus $\operatorname{Sing} X \subset \mathbb{E}$ so that $\operatorname{Sing} X_{\mathbb{R}} = (\operatorname{Sing} X) \setminus \mathbb{E} = \varnothing$. Now suppose that $|\operatorname{Sing} X_{\mathbb{R}}| > 0$. In this case, $\tau(X)$ must be singular and thus $(\operatorname{Sing} X) \setminus \mathbb{E}$ consist of two real antipodal points that are send via τ to the vertex of the quadratic cone $\tau(X)$. It follows that $|\operatorname{Sing} X_{\mathbb{R}}| = 2$ and $\operatorname{Sing} X_{\mathbb{R}} = (\operatorname{Sing} X) \setminus \mathbb{E}$ as asserted.

(b). We may assume up to Möbius equivalence that $\tau(X)$ is a doubly ruled quadric. Suppose by contradiction that there exist complex conjugate lines $L, \overline{L} \subset X$ that intersect at some point $\mathfrak{p}$. Complex conjugate lines in the doubly ruled quadric $\tau(X)$ do not intersect and thus $\tau(L) = \tau(\overline{L})$

so that $\mathfrak{p}$ is contained in the ramification locus $\mathbb{E}$. We arrived at a contradiction since $\mathfrak{p} \in X_\mathbb{R}$ and $\mathbb{E}_\mathbb{R} = \varnothing$.

(c). First, we show the $\Rightarrow$ direction. We may assume without loss of generality that $\tau(X)$ is a quadratic cone and thus $\operatorname{Sing}\tau(X)_\mathbb{R} = \{v\}$ for some point v. We deduce that there exist antipodal points $\mathfrak{p}, \mathfrak{q} \in \operatorname{Sing} X_\mathbb{R}$ such that $\tau(\mathfrak{p}) = \tau(\mathfrak{q}) = v$. Moreover, there exists a pencil of circles $F \subset X \times \mathbb{P}^1$ with base points $\mathfrak{p}$ and $\mathfrak{q}$, and its members are 2:1 centrally projected to lines in $\tau(X)$ that meet at v. We know from Assertion (a) that $|\operatorname{Sing} X_\mathbb{R}| = 2$.

Next, we show the $\Leftarrow$ direction. We may assume up to Möbius equivalence that the base points $\mathfrak{p}, \mathfrak{q} \in X_\mathbb{R}$ are antipodal so that the circles in the pencil are 2:1 projected by τ to lines in $\tau(X)$ that pass through the point $\tau(\mathfrak{p}) = \tau(\mathfrak{q})$. Hence, $\tau(X)$ is a quadratic cone with vertex $\tau(\mathfrak{p})$, which implies that X is 1-elliptic. □

Proposition 43. *A celestial Darboux cyclide $X \subset \mathbb{S}^3$ is elliptic if and only if X is either a Blum cyclide, Perseus cyclide, ring cyclide, EO cyclide, or CO cyclide.*

Proof. We first show the $\Rightarrow$ direction.

If $\operatorname{Sing} X_\mathbb{R} \neq \varnothing$, then $|\operatorname{Sing} X_\mathbb{R}| = 2$ by Lemma 42 (a) and thus X is either a EO cyclide or CO cyclide by Theorem A (see Table 5).

Now suppose that $\operatorname{Sing} X_\mathbb{R} = \varnothing$. By Theorem A (see Table 5), X is either a S1, S2, Blum, Perseus or ring cyclide. If X is a S1 cyclide, then we know from Theorem A (see Table 6) that $e_{01}, e_{12} \in E(X)$, $\sigma_*(e_{01}) = e_{12}$ and $e_{01} \cdot e_{12} = 1$. Similarly, if X is a S2 cyclide, then $e_1, e_{11} \in E(X)$, $\sigma_*(e_1) = e_{11}$ and $e_1 \cdot e_{11} = 1$. We apply Proposition 19 (c) and (d) and find that S1 cyclides and S2 cyclides contain two complex conjugate lines that intersect. Such cyclides are not elliptic by Lemma 42 (b) and (c). Thus if $X_\mathbb{R}$ is smooth, then X must be either a Blum, Perseus or ring cyclide.

We proceed by showing the $\Leftarrow$ direction.

First, suppose that X is either an EO cyclide or CO cyclide. We know from Theorem A that $|\operatorname{Sing} X_\mathbb{R}| = 2$, $b_{12}, b_{34} \in B(X)$, $g_1 \in G(X)$, $\sigma_*(g_1) = g_1$, $\sigma_*(b_{12}) = b_{12}$, $\sigma_*(b_{34}) = b_{34}$ and $g_1 \cdot b_{12} = g_1 \cdot b_{34} = 1$. Hence, X is by Proposition 19 (b) and (e) covered by a pencil of circles with two real base points. It follows from Lemma 42 (c) that X is elliptic.

Finally, we assume that X is either a Blum, Perseus or ring cyclide. By definition, $\operatorname{Sing} X_\mathbb{R} = \varnothing$ and X is λ-circled with $\lambda \geq 4$. It follows that $\deg \pi(X) > 2$ and thus X is not Euclidean. Hence, X is by Lemma 37 either hyperbolic, elliptic or both. We conclude from Lemma 41 that X must be elliptic. □

Remark 44. I do not know how to show that all the Blum, Perseus and ring cyclides are elliptic without using Proposition 39 and Lemmas 40 and 41. We include an alternative proof for the ring cyclide case as it provides additional geometric insight and does not rely on this proposition and two lemmas:

A ring cyclide $X \subset \mathbb{S}^3$ is 2-elliptic and not 2-hyperbolic.

This proof goes as follows. By Theorem A (see Table 6) there exist

$$g_{12}, g_{34} \in \{g \in G(X) \,|\, \sigma_*(g) = g\}$$

such that $g_{12} \cdot g_{34} = 2$ and $g_{12} \cdot b = 0$ for all $b \in B(X)$, where $B(X) = \{b_{13}, b_{24}, b'_{14}, b'_{23}\}$. Hence, we know from Proposition 19 (b) and (e) that there exist pencils $F, G \subset X \times \mathbb{P}^1$ such that $|F_i \cap G_j| = 2$ for general $i, j \in \mathbb{P}^1$ and F is base point free. It follows from Lemma 37 that X is either 2-elliptic, 2-hyperbolic or both.

Now suppose by contradiction that X is 2-hyperbolic. We may assume without loss of generality that $\nu(X) \subset \mathbb{P}^3$ is a doubly ruled quadric. The restriction of the 2:1 covering ν to X defines outside the ramification locus $\mathbb{Y} \subset \mathbb{S}^3$ a local complex analytic isomorphism on each of the two sheets. Since $\nu(X)$ is smooth, it follows that $\operatorname{Sing} X \subset \mathbb{Y}$. Recall from Example 21 (see the rightmost diagram of Figure 4) that there exist skew complex lines $L, \overline{L} \subset X$ such that $[L] = e_1$, $[\overline{L}] = e_2$, $|L \cap \overline{L}| = 0$ and $|L \cap \operatorname{Sing} X| = |\overline{L} \cap \operatorname{Sing} X| = 2$. Since $\mathbb{Y} \cap X$ is a hyperplane section of X and $|L \cap \mathbb{Y}| = |\overline{L} \cap \mathbb{Y}| = 2$, it follows from Bézout's theorem that $L, \overline{L} \subset \mathbb{Y}$. Notice that L and $\overline{L}$ are complex conjugate lines as $\sigma_*([L]) = [\overline{L}]$. We arrived at a contradiction, because $\mathbb{Y}_{\mathbb{R}} \cong S^2$ and thus complex conjugate lines in $\mathbb{Y}$ are not disjoint (see Remark 12). This concludes the alternative proof.

Proposition 45. *If $X \subset \mathbb{S}^3$ is a great ring cyclide, then there exist great circles $A, B \subset S^3$ such that $X = \mathbf{P}(A \star B)$.*

Proof. We fix a point $e \in X_{\mathbb{R}}$. First, suppose that $\mathbf{R}(e)$ equals the identity quaternion in S^3.

We know from Theorem A that $|\operatorname{Sing} X_{\mathbb{R}}| = 0$ and $|\operatorname{Sing} X| = 4$. It follows from Lemma 42 (c) and (a) that $\tau(X)$ is a doubly ruled quadric and $\operatorname{Sing} X \subset \mathbb{E}$. We apply Lemma 24 (a) and find that $X \cap \mathbb{E}$ consist of two left generators L, $\overline{L}$ and two right generators R, $\overline{R}$. As $\tau(X)$ is doubly ruled, there exist two great circles $A, B \subset \mathbf{R}(X)$ such that $e \in \mathbf{P}(A) \cap \mathbf{P}(B)$. We may assume without loss of generality that the line $\tau(\mathbf{P}(A))$ does not belong to the pencil of lines containing $\tau(R)$ and $\tau(\overline{R})$. Hence, each circle in the pencil containing $\mathbf{P}(A)$ meets both right generators $R, \overline{R} \subset \mathbb{E}$.

We assume by contradiction that $X \neq \mathbf{P}(A \star B)$. Let $F \subset \mathbf{P}(A \star B) \times \mathbf{P}(B)$ and $G \subset \mathbf{P}(A \star B) \times \mathbf{P}(A)$ be the right and left associated pencils of $A \star B$. Notice that $\mathbf{R}(c)$ with $c := (1:0:0:0:0)$ is the center of S^3 and that $p \mathbin{\widehat{\star}} c = c \mathbin{\widehat{\star}} p$ for all $p \in \mathbb{S}^3$. It follows that the left or right Clifford translation of a great circle in $\mathbb{S}^3$ is again great. Thus, both F and G have infinitely many members that are great circles on $\mathbf{P}(A \star B)$. These great circles are centrally projected to lines on the doubly ruled quadric $\tau(\mathbf{P}(A \star B))$. This implies that both F and G are base point free. It now follows from Lemma 17 (b) that $|F_b \cap R'| = |F_b \cap \overline{R'}| = 1$ for almost all $b \in B$ and some complex conjugate right generators $R', \overline{R'} \subset \mathbb{E}$. Since $F_e = \mathbf{P}(A)$ and $|\mathbf{P}(A) \cap R| = |\mathbf{P}(A) \cap \overline{R}| = 1$, we deduce from Lemma 16 (b) that $R' = R$ and $\overline{R'} = \overline{R}$. Notice that $F_e = \mathbf{P}(A)$ and $G_e = \mathbf{P}(B)$ and thus $\mathbf{P}(A), \mathbf{P}(B) \subset X \cap \mathbf{P}(A \star B)$. We fix some general point $b \in B$. Let $C \subset X$ be the great circle that passes through b and belongs to the same pencil on X as $\mathbf{P}(A)$. Recall that $|C \cap R| = |C \cap \overline{R}| = 1$ as is illustrated in Figure 7. By assumption, F does not cover X and thus C is not a member of F. We arrived at a contradiction as the lines $\tau(F_b)$ and $\tau(C)$ span a plane so that the complex lines $\tau(R)$ and $\tau(\overline{R})$ cannot be skew. We establised that $X = \mathbf{P}(A \star B)$ for great circles $A, B \subset \mathbf{R}(X)$.

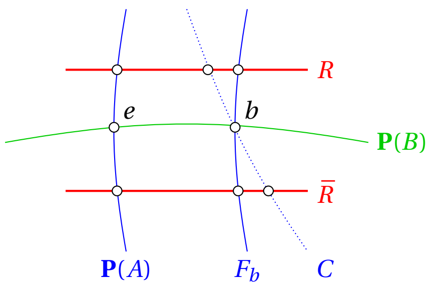


Figure 7. See the proof of Proposition 45. The incidences between the great circles $\mathbf{P}(A)$, $\mathbf{P}(B)$, C, F_b and the right generators $R, \overline{R} \subset \mathbb{E}$, under the assumption that $\mathbf{R}(e)$ is the identity quaternion in S^3.

Finally, suppose that $\mathbf{R}(e)$ is not equal to the identity quaternion in S^3. There exists a right Clifford translation $\varphi \in \operatorname{RT}\mathbb{S}^3$ such that $\mathbf{R}(\varphi(e))$ is equal to the identity quaternion. Recall from Proposition 10 (d) that φ leaves the right generators of $\mathbb{E}$ invariant. The right Clifford translation of a great circle is again great and thus $\varphi(X) = \mathbf{P}(A \star B)$ for some great circles $A, B \subset \varphi(X)$. The unit quaternions S^3 form a group and thus there exists $r \in S^3$ such that $\mathbf{R}(\varphi(p)) = \mathbf{R}(p) \star r$ for all $p \in \mathbb{S}^3_{\mathbb{R}}$. Therefore, $X = \mathbf{P}(A \star B')$, where $B' := \{ b \star r^{-1} \mid b \in B \}$. This concludes the proof as $X = \mathbf{P}(A \star B')$ for some great circles $A, B' \subset S^3$. $\square$

Proposition 46. *If $X \subset \mathbb{S}^3$ is a great Perseus cyclide, then X is not Cliffordian, $\tau(X)$ is a doubly ruled quadric,* $\operatorname{Sing} X \subset \mathbb{E}$ *and* $X \cap \mathbb{E}$ *does not contain lines.*

Proof. Recall from Example 21 and the leftmost diagram in Figure 4 that the incidences between all the complex lines and complex conjugate isolated singularities are as depicted in Figure 8, where $L, \overline{L}, R, \overline{R}, M, \overline{M}, T, \overline{T} \subset X$ denote the complex lines and $\mathfrak{p}, \overline{\mathfrak{p}} \in \operatorname{Sing} X$ are the complex conjugate isolated singularities. We know from Lemma 42 (c) and (a) that $\tau(X)$ is a doubly ruled quadric and $\operatorname{Sing} X = \{\mathfrak{p}, \overline{\mathfrak{p}}\} \subset \mathbb{E}$.

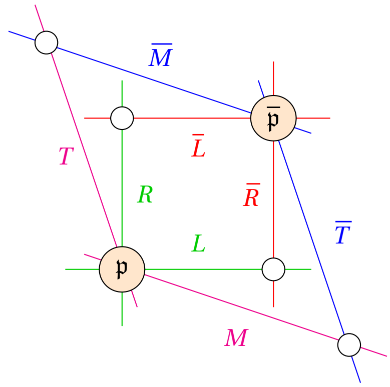


Figure 8. Incidences between complex conjugate lines and isolated singularities in a great Perseus cyclide, where $\mathfrak{p}, \overline{\mathfrak{p}} \in \mathbb{E}$.

We claim that none of the complex lines in X are contained in $\mathbb{E}$. First suppose by contradiction that $L \subset \mathbb{E}$. In this case the complex conjugate line $\overline{L}$ must also be contained in $\mathbb{E}$. Therefore, $L, \overline{L}, R, \overline{R} \subset \mathbb{E}$ by Bézout's theorem. It follows again from Bézout's theorem that $T, \overline{T}, M, \overline{M} \not\subseteq \mathbb{E}$. We arrived at a contradiction, since $\tau(X)$ contains three complex lines $\tau(R)$, $\tau(L)$ and $\tau(T)$ through the complex point $\tau(\mathfrak{p})$ instead of two. We established that $L \not\subseteq \mathbb{E}$, and by using the same arguments we find that $\overline{L}, R, \overline{R}, M, \overline{M}, T, \overline{T} \not\subseteq \mathbb{E}$ as well. Since $X \cap \mathbb{E}$ does not contain complex lines, we conclude from Lemma 26 that X is not Cliffordian. $\square$

We proceed in Examples 47 and 48 to provide implicit equations for some great celestial Darboux cyclides. This section is concluded with Remarks 49 to 51, namely an analysis of the geometries of great celestial Darboux cyclides by using the introduced methods. The reader may opt to jump directly to Section 8 at this point.

Example 47 (Great EO/CO cylides). We consider the following surface

$$X := \tau^{-1}\left(\left\{y \in \mathbb{P}^3 \,\middle|\, \alpha\, y_0^2 + \beta\, y_1^2 - y_2^2 = 0\right\}\right) = \left\{x \in \mathbb{S}^3 \,\middle|\, \alpha\, x_1^2 + \beta\, x_2^2 - x_3^2 = 0\right\},$$

for some $\alpha, \beta \in \mathbb{R}_{>0}$. Notice that $\left\{y \in \mathbb{P}^3 \,\middle|\, \alpha\, y_0^2 + \beta\, y_1^2 - y_2^2 = 0\right\}$ is a ruled quadric and thus X is great. Suppose that $X' \subset \mathbb{S}^3$ is a CO cyclide or EO cyclide. We claim that there exist $\alpha, \beta \in \mathbb{R}_{>0}$ such that X' is Möbius equivalent to X and $\alpha = \beta$ if and only if X is a CO cyclide. We may assume up to Möbius equivalence that the center p of π lies in $\operatorname{Sing} X_{\mathbb{R}}$. The Möbius transformations that leave p invariant correspond via $\mathbf{R}(\pi(\_))$ to Euclidean similarities in $\mathbb{R}^3$. Hence, it follows from the classical result [15, Proposition 4] that there exists $\varphi \in \operatorname{Aut} \mathbb{S}^3$ and $\alpha, \beta \in \mathbb{R}_{>0}$ such that $\varphi(p) = p$ and $(\pi \circ \varphi)(X') = \left\{y \in \mathbb{P}^3 \,\middle|\, \alpha\, y_1^2 + \beta\, y_2^2 - y_3^2\right\}$. Moreover, $\alpha = \beta$ if and only if X' is a CO cyclide. Since $\pi(x) = (x_0 - x_4 : x_1 : x_2 : x_3)$, we deduce that $\varphi(X') = \left\{x \in \mathbb{S}^3 \,\middle|\, \alpha\, x_1^2 + \beta\, x_2^2 - x_3^2 = 0\right\}$ as was to be shown.

Example 48 (Great ring/Persues/Blum cyclide). We consider the following surface

$$X := \tau^{-1}\left(\left\{y \in \mathbb{P}^3 \,\middle|\, \alpha\, y_0^2 + y_1^2 - y_2^2 - \beta\, y_3^2 = 0\right\}\right) = \left\{x \in \mathbb{S}^3 \,\middle|\, \alpha\, x_1^2 + x_2^2 - x_3^2 - \beta\, x_4^2 = 0\right\},$$

for some $\alpha, \beta \in \mathbb{R}_{>0}$. We claim that $X \subset \mathbb{S}^3$ is a great ring cyclide, great Persues cyclide or great Blum cyclide if (α,β) is equal to $(1,1)$, $(1,2)$ and $(2,2)$, respectively. Since $\tau(X)$ is a doubly ruled quadric it follows that X is great. To identify X, let us first compute $\operatorname{Sing} X$. The Jacobian matrix of the generators of the ideal $\langle \alpha\, x_1^2 + x_2^2 - x_3^2 - \beta\, x_4^2, -x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 \rangle$ is up to scaling of the rows as follows:

$$\begin{bmatrix} 0 & \alpha\, x_1 & x_2 & -x_3 & -\beta\, x_4 \\ -x_0 & x_1 & x_2 & x_3 & x_4 \end{bmatrix}.$$

If $(\alpha,\beta) = (1,1)$, then the Jacobian matrix has rank one at the four complex points $(0:1:\pm\mathrm{i}:0:0)$ and $(0:0:0:1:\pm\mathrm{i})$ in $X \cap \mathbb{E}$. Hence, X is a ring cyclide by Theorem A (see the SingType X column in Table 5). If $(\alpha,\beta) = (1,2)$, then the Jacobian matrix has rank one at the complex points $(0:1:\pm\mathrm{i}:0:0)$ in $X \cap \mathbb{E}$. Hence, X is a Perseus cyclide by Theorem A. If $(\alpha,\beta) = (2,2)$, then the Jacobian matrix has rank two at all complex points in X so that $\operatorname{Sing} X = \varnothing$. Hence, X is a Blum cyclide by Theorem A and Proposition 43. In Figure 9 we depicted its stereographic projection $\mathbf{R}(\pi(X)) = \left\{z \in \mathbb{R}^3 \,\middle|\, (x^2+y^2+z^2)^2 - 6\,x^2 - 4\,y^2 + 1\right\}$.

Figure 9. Stereographic projection of a great Blum cyclide (see Example 48).

Remark 49 (Great CO cyclide). Suppose that $X \subset \mathbb{S}^3$ is a great CO cyclide. Recall from Example 21 that we encoded the corresponding row in Table 6 in terms of the diagram in Figure 10 (left), where $G(X) = \{g_0, \underline{g_1}, g_{14}, g_{23}\}$. By Proposition 19 (a), the components $\{b_{34}\}$ and $\{b_{12}\}$ correspond to real antipodal singularities of X and are centrally projected to the vertex of the quadratic cone $\tau(X)$. The great circles in X have class g_1 and form a pencil with real antipodal base points in $\operatorname{Sing} X$. The components $\{b'_{13}\}$ and $\{b'_{24}\}$ correspond to the complex isolated singularities that lie in the ramification locus $\mathbb{E}$. The central projection of these complex singular points are smooth complex branching points in $\tau(X)$.

Remark 50 (Great Blum cyclide). Suppose that X is the great Blum cyclide in Example 48 with $(\alpha,\beta) = (2,2)$. Our goal is to identify, up to $\operatorname{Aut} N(X)$, the classes of great and small circles, and pairs $([L],[L'])$ of classes such that $L, L' \subset X$ are complex lines such that $\tau(L) = \tau(L')$. As a byproduct, we recover the compact diagram in Figure 11 from which we can read off the Clifford quartets and the incidences between the complex lines in Blum cyclides. By Theorem A, we may assume up to $\operatorname{Aut} N(X)$ that $B(X)$, $E(X)$ and $G(X)$ are as in Table 6. Since $\tau(X)$ is a doubly ruled quadric by Lemma 42 (c), there exist great circles $C, C' \subset X$ such that $|C \cap C'| = 2$. By Proposition 19 (b), we may assume without loss of generality that $([C],[C']) = (g_{12}, g_{34})$ as $g_{12}, g_{34} \in \left\{g \in G(X) \,\middle|\, \sigma_*(g) = g\right\}$ and $g_{12} \cdot g_{34} = 2$. If $D, D' \subset X$ are antipodal small circles, then $|D \cap D' \cap \mathbb{E}| = 2$ and thus $([D],[D']) \in \{(g_0, g_3), (g_1, g_2)\}$ by Proposition 19 (b). Since the ramification locus $X \cap \mathbb{E}$ of the central projection τ does not contain complex lines, it follows that for all complex lines $L \subset X$, there exists a complex line $L' \subset X$ such that $\tau(L) = \tau(L')$.

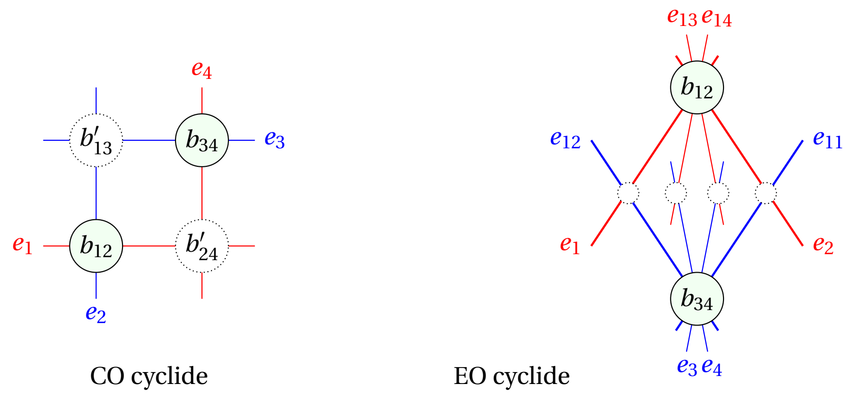


Figure 10. Incidences between complex lines and isolated singularities on Darboux cyclides (see Example 21 or the caption of Figure 3). If the Darboux cyclide is great, then a red line is centrally projected to a blue line (see Remark 49).

The ramification locus $\mathbb{E}$ is a hyperplane section, which implies that $|L \cap L' \cap \mathbb{E}| = 1$ and thus $|L \cap L'| = [L] \cdot [L'] = 1$ by Proposition 19 (d). Let $a := g_{12} \cdot ([L] + [L'])$ and $b := g_{34} \cdot ([L] + [L'])$. Notice that $\tau(L)$ belongs to one of the two rulings of $\tau(X)$ and thus (a, b) is equal to either $(0, 2)$ or $(2, 0)$. If $(a, b) = (0, 2)$, then $([L], [L']) \in \{(e_1, e'_2), (e_2, e'_1), (e_{13}, e_{04}), (e_{14}, e_{03})\}$ and if $(a, b) = (2, 0)$, then $([L], [L']) \in \{(e_{01}, e_{12}), (e_{02}, e_{11}), (e'_3, e_4), (e'_4, e_3)\}$. See Figure 11 for a diagrammatic representation of these pairs such that two line segments represent complex lines in the same pencil on the doubly ruled quadric $\tau(X)$ if and only if the line segments are both horizontal or both vertical. The details are left to the reader.

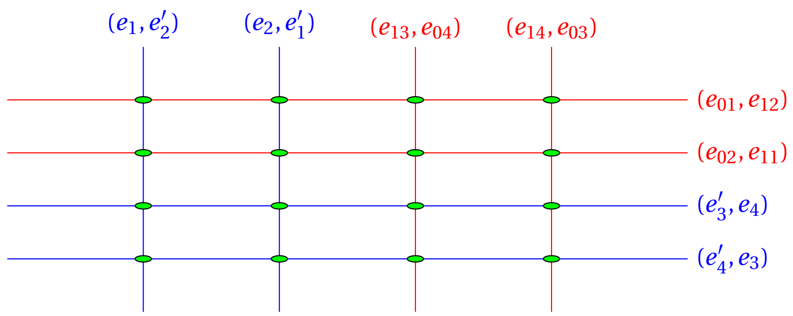


Figure 11. Each line segment is labeled with $([L], [L'])$, where L and L' are complex lines in a great Darboux cyclide $X \subset \mathbb{S}^3$ such that $\tau(L) = \tau(L')$. For each such label, we have $|L \cap L'| = 1$. Two line segments labeled with $([L], [L'])$ and $([M], [M'])$ meet at a green disc if and only if $\{[L] \cdot [M], [L] \cdot [M']\} = \{0, 1\}$ and $\{[L'] \cdot [M], [L'] \cdot [M']\} = \{0, 1\}$. The four Clifford quartets are up to $\operatorname{Aut} N(X)$ given by $\{e_1, e_2, e'_3, e'_4\}$, $\{e'_2, e'_1, e_3, e_4\}$, $\{e_{13}, e_{14}, e_{01}, e_{02}\}$ and $\{e_{04}, e_{03}, e_{12}, e_{11}\}$.

Remark 51 (Great Perseus cyclide). Let us describe the geometry of a great Perseus cyclide $X \subset \mathbb{S}^3$, by identifying the classes of great circles, small circles and complex lines, and the components corresponding to base points. By Theorem A, we may assume up to $\operatorname{Aut} N(X)$ that

$$\{g \in G(X) \,|\, \sigma_*(g) = g\} = \{g_0, g_1, g_{12}, g_2, g_3\}.$$

By Proposition 19 (c), there exist complex lines $L, \overline{L}, R, \overline{R}, M, \overline{M}, T, \overline{T} \subset X$ such that

$$[L] = e_{11},\ [\overline{L}] = e_{12},\ [R] = e_{01},\ [\overline{R}] = e_{02},\ [M] = e_3,\ [\overline{M}] = e_4,\ [T] = e'_3,\ [\overline{T}] = e'_4.$$

By Proposition 19 (a), the complex conjugate isolated singularities $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ in $\operatorname{Sing} X$ correspond to the components $\{b_1\}$ and $\{b_2\}$, respectively. The incidences between the complex lines and isolated singularities are illustrated in Figure 8 (see also Figure 4).

We know from Proposition 46 that $\tau(X)$ is a doubly ruled quadric, $\mathfrak{p}, \overline{\mathfrak{p}} \in \mathbb{E}$ and $L, \overline{L}, R, \overline{R}, M, \overline{M}, T, \overline{T} \nsubseteq \mathbb{E}$. It follows that either $\tau(L) = \tau(R)$, $\tau(L) = \tau(M)$ or $\tau(L) = \tau(T)$.

First, suppose by contradiction that $\tau(L) = \tau(R)$. In this case $\tau(\overline{L}) = \tau(\overline{R})$. This is a contradiction, since $\tau(L)$ intersects its complex conjugate line $\tau(\overline{L})$ and thus $\tau(X)$ must be an ellipsoid instead of being a doubly ruled quadric.

Next, we suppose that $\tau(L) = \tau(M)$. In this case, the complex conjugate lines $\tau(L) = \tau(M)$ and $\tau(\overline{L}) = \tau(\overline{M})$ belong to the first pencil of lines on the doubly ruled quadric $\tau(X)$. The complex conjugate lines, $\tau(R) = \tau(T)$ and $\tau(\overline{R}) = \tau(\overline{T})$ belong to the second pencil of lines on $\tau(X)$. By Proposition 19, a circle with class g_1 meets each of the lines in $\{L, \overline{L}, M, \overline{M}\}$ and belongs to a base point free pencil. Similarly, a circle with class g_2 meets each of the lines in $\{R, \overline{R}, T, \overline{T}\}$ and belongs to a base point free pencil. From this we establish that g_1 and g_2 correspond to pencils of great circles that are centrally projected the first and second ruling of the quadric $\tau(X)$, respectively. A circle $C'' \subset X$ such that $[C'']$ equals either g_0 or g_3 meets each of the lines in $\{L, \overline{L}, T, \overline{T}\}$ and $\{R, \overline{R}, M, \overline{M}\}$, respectively. Therefore, each circle $C \subset X$ such that $[C] = g_0$ is a small circle whose antipodal points form a small circle $C' \subset X$ such that $[C'] = g_3$ and $\tau(C) = \tau(C')$. A circle with class g_{12} passes through the complex conjugate isolated singularities $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. Hence, each circle $C \subset X$ such that $[C] = g_{12}$ is a small circle whose antipodal points form a small circle $C' \subset X$ such that $[C'] = g_{12}$ and $\tau(C) = \tau(C')$.

The case $\tau(L) = \tau(T)$ is analoguous to the previous case: $\{g_0, g_3\}$ are classes of great circles, $\{g_1, g_2\}$ are the classes of antipodal little circles, and g_{12} is the class of a small circle that meets $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. The details are left to the reader.

8. Combining the results

In order to prove Theorem 1 we use the following theorem from [15, Theorem 1].

Theorem B. *If $X \subset \mathbb{S}^3$ is a λ-circled surface of degree d such that $\lambda \geq 2$, then either X is either a Darboux cyclide or $(\lambda, d) \in \{(\infty, 2), (2, 8)\}$.*

Lemma 52. *If $X \subset \mathbb{S}^3$ is a surface such that $\mathbf{R}(X)$ is a 2-dimensional sphere, then X is not Cliffordian.*

Proof. Suppose by contradiction that $X = \mathbf{P}(A \star B)$ for some circles $A, B \subset S^3$. The left Clifford translations correspond to isoclinic rotations of S^3. Thus the infinitesimal left Clifford translations of points on the circle B define a nowhere vanishing vector field on $\mathbf{R}(X) \subset S^3$. We arrived at a contradiction, since a 2-dimensional sphere does not admit such a vector field by the hairy ball theorem. □

Proof of Theorem 1. Since $Z \subset \mathbb{R}^3$ is λ-circled and of Möbius degree d such that $(d, \lambda) \neq (8, 2)$, it follows from Theorem B that $d \in \{2, 4\}$. If $d = 2$, then Z is either a plane or a 2-dimensional sphere.

(a). A CY or EY is always Bohemian as it can be obtained by translating a circle along a line (see Figure 1). A plane is the translation of a line along a line. Hence the proof for this assertion is concluded by Proposition 35.

(b). By Lemma 52 we have $d = 4$ and thus the first part follows from Proposition 31, where $X = \mathbf{S}(Z)$. It follows from Propositions 43 and 45 that ring cyclides are Möbius equivalent to Cliffordian surfaces.

(c). Direct consequence of Proposition 43, where $X = \mathbf{S}(Z)$. □

Corollaries 3 and 4 are direct consequences of Theorem 1.

Proof of Corollary 5. The central projection of a surface $Z \subseteq S^n$ that is covered by two pencils of great circles is a doubly ruled quadric. Therefore, $n = 3$ and Z has no real singularities. In particular, the stereographic projection $\mu(Z)$ is not a CO cyclide or EO cyclide. Thus the proof is concluded by Theorem 1 (c). □

Proof of Corollary 6. Direct consequence of Propositions 45 and 46. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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