ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus

Mathématique

Marco Schlichting

Corrigendum to “Symplectic and orthogonal K-groups of the integers”

Volume 362 (2024), p. 307-308

Online since: 2 May 2024

https://doi.org/10.5802/crmath.606

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2024, Vol. 362, p. 307-308
https://doi.org/10.5802/crmath.606

Erratum / *Erratum*
Number Theory / *Théorie des nombres*

Corrigendum to "Symplectic and orthogonal K-groups of the integers"

Corrigendum à « K-groupes symplectiques et orthogonaux de l'anneau des entiers »

Marco Schlichting [a]

[a] Marco Schlichting, Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK

E-mail: m.schlichting@warwick.ac.uk

Abstract. The action of the duality functor on the odd torsion of $K_n(\mathbb{Z})$ was stated incorrectly in [3], in half of the cases, and lead to incorrect formulas for the odd primary torsion of $\pi_n B\mathrm{Sp}(\mathbb{Z})^+$ and $\pi_n BO_{\infty,\infty}(\mathbb{Z})^+$.

Résumé. L'action du foncteur de dualité sur la torsion impaire de $K_n(\mathbb{Z})$ était énoncé incorrectement dans [3], dans la moitié des cas, et a conduit à des formules incorrectes pour la torsion primaire impaire de $\pi_n B\mathrm{Sp}(\mathbb{Z})^+$ et $\pi_n BO_{\infty,\infty}(\mathbb{Z})^+$.

Manuscript received 6 November 2023, accepted 7 January 2024.

Let R be a ring of integers in a number field and ℓ an odd prime. The formula [3, (2.3)] for the ℓ-primary torsion subgroups of $K_n(R)$ is false. The correct formulas extracted from [4, Proof of Theorem 70] are for $i > 1$

$$K_{2i}(R)\{\ell\} = K_{2i}(R')\{\ell\} = H^2_{et}(R', (K_{2i+2})_\ell)$$

where the right hand term is the inverse limit $\lim_\nu H^2_{et}(R', (K_{2i+2})/\ell^\nu)$. In particular, the duality acts by $(-1)^{i+1}$ on this group. In odd degree for $i > 1$ we have

$$K_{2i-1}(R)\{\ell\} = K_{2i-1}(R')\{\ell\} = H^0_{et}(R', (K_{2i})/\ell^\infty)$$

where the right hand term is the direct limit $\mathrm{colim}_\nu H^0_{et}(R', (K_{2i})/\ell^\nu)$. In particular, the duality acts by $(-1)^i$ on this group. These formulas are not new; see [4, Theorem 70 and proof thereof]; see also [1, Lemma 3.4.4]. Together with [3, Lemma 2.1] this yields now a correction of [3, Theorem 2.2] in the cases $n \equiv 0, 2 \mod 4$.

Theorem 1. *Let R be a ring of integers in a number field, and $\ell \in \mathbb{Z}$ an odd prime. Then for all $n \geq 1$ we have isomorphisms*

$$GW_n(R)\{\ell\} \cong K\mathrm{Sp}_n(R)\{\ell\} \cong KQ_n(R)\{\ell\} \cong \begin{cases} K_n(R)\{\ell\} & n \equiv 2,3 \mod 4 \\ 0 & n \equiv 0,1 \mod 4. \end{cases}$$

[3, Section 3] is unaffected by this error and thus, we obtain the following table of homotopy groups correcting [3, Théorème 0.1 and Theorem 1.1] where the 2-primary part comes from [2, 4.7.2] and [3, Theorem 3.3] and the odd primary part from Theorem 1. See also [1, Theorem 3.2.1].

Theorem 2. *The homotopy groups of the spaces* $B\,\mathrm{Sp}(\mathbb{Z})^+$ *and* $BO_{\infty,\infty}(\mathbb{Z})^+$ *for* $n \geq 1$ *are given in the following table*

$n \mod 8$	0	1	2	3	4	5	6	7
$\pi_n B\,\mathrm{Sp}(\mathbb{Z})^+$	0	0	$\mathbb{Z} \oplus K_n(\mathbb{Z})_{odd}$	$K_n(\mathbb{Z})$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus K_n(\mathbb{Z})_{odd}$	$K_n(\mathbb{Z})$
$\pi_n BO_{\infty,\infty}(\mathbb{Z})^+$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2 \oplus K_n(\mathbb{Z})_{odd}$	$\mathbb{Z}/8 \oplus K_n(\mathbb{Z})_{odd}$	$\mathbb{Z}$	0	$K_n(\mathbb{Z})_{odd}$	$K_n(\mathbb{Z})$

Finally, this leads to a correction of [3, Remark 1.3] and the following table for $n > 0$. Denote by B_k the k-th Bernoulli number [4, Example 24] and let d_n denote the denominator of $\frac{1}{n+1}B_{(n+1)/4}$ for $n = 3 \mod 4$. By [4, Introduction, Lemma 27] we have $K_n(\mathbb{Z}) = \mathbb{Z}/2d_n$ for $n = 3 \mod 8$ and $K_n(\mathbb{Z}) = \mathbb{Z}/d_n$ for $n = 7 \mod 8$. Similarly, denote by c_k the numerator of $B_k/4k$. Then $K_{4k-2}(\mathbb{Z})$ is a finite group of order c_k when k is even and of order $2c_k$ when k is odd. This group is conjectured to be cyclic.

$n \mod 8$	0	1	2	3	4	5	6	7
$\pi_n B\,\mathrm{Sp}(\mathbb{Z})^+$	0	0	$\mathbb{Z} \oplus \lvert\mathbb{Z}/c_k\rvert$	$\mathbb{Z}/2d_n$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \lvert\mathbb{Z}/c_k\rvert$	$\mathbb{Z}/d_n$
$\pi_n BO_{\infty,\infty}(\mathbb{Z})^+$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2 \oplus \lvert\mathbb{Z}/c_k\rvert$	$\mathbb{Z}/d_n$	$\mathbb{Z}$	0	$\lvert\mathbb{Z}/c_k\rvert$	$\mathbb{Z}/d_n$

where $|\mathbb{Z}/m|$ denotes a finite group of order n conjectured to be cyclic and $n = 4k - 2$.

Full proof of the claims in [3] are now available in [1].

References

[1] B. Calmès, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, W. Steimle, "Hermitian K-theory for stable ∞-categories, Part III: Grothendieck-Witt groups of rings", https://arxiv.org/abs/2009.07225, 2020.

[2] M. Karoubi, "Bott periodicity in topological, algebraic and Hermitian K-theory", in *Handbook of K-theory. Vol. 1, 2*, Springer, 2005, p. 111-137.

[3] M. Schlichting, "Symplectic and orthogonal K-groups of the integers", *C. R. Math. Acad. Sci. Paris* **357** (2019), no. 8, p. 686-690.

[4] C. Weibel, "Algebraic K-theory of rings of integers in local and global fields", in *Handbook of K-theory. Vol. 1, 2*, Springer, 2005, p. 139-190.