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Exponential decay of harmonic 1-forms for wild harmonic bundles on curves

Décroissance exponentielle des 1-formes harmoniques pour de fibrés harmoniques sauvages sur une courbe

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Abstract. We give a slightly reorganized exposition of the proof of the exponential decay result of T. Mochizuki for harmonic 1-forms for wild harmonic bundles on a disc.

Résumé. Nous donnons une démonstration légèrement modifiée du résultat de décroissance exponentielle des 1-formes harmoniques pour un fibré harmonique sauvage sur un disque, dû à T. Mochizuki.

Keywords. Harmonic bundle, harmonic section, irregular singularity.

Mots-clés. fibré harmonique, section harmonique, singularité irrégulière.

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1. Introduction

One of the key analytic results needed in the proof of the Hard Lefschetz theorem for polarized wild pure twistor D-modules over a smooth projective curve [3, Corollary 18.1.2] is the exponential decay property of harmonic 1-forms. The purpose of the present text is to provide an alternative proof of the relevant exponential decay estimates, along the method developped in [6, Chapter 2]. This method relies on the repeated application of the Weitzenböck formula and Stokes' theorem. Moreover, the procedure takes all irregular parts of the same order in one step rather than one by one, which perhaps makes the presentation simpler. On the other hand, the proof relies on strong analytic results proved in [3, Section 7]. Analysis of tame harmonic bundles was pioneered in [5]. The first analytical study of wild harmonic bundles was performed in [4].

2. Statement of exponential energy decay

We consider an unramified wild harmonic bundle $(E, \bar\partial_E, \theta, h)$ over the unit disc $\Delta = \{|z| < 1\} \subset \mathbf{C}$, equipped with the Euclidean metric $\mathrm{d}z \cdot \mathrm{d}\bar z$. We will denote by ∇^+ the connection in E compatible with $(\bar\partial_E, h)$, by $\Phi = \theta + \theta_h^\dagger$ the self-adjoint part of the associated integrable connection, and by

$$\Delta = (\bar\partial_E + \theta) \circ (\bar\partial_E + \theta)_h^* + (\bar\partial_E + \theta)_h^* \circ (\bar\partial_E + \theta)$$

the associated Laplace operator. An L^2-section τ of $E \otimes \Omega^1$ is said to be *harmonic* if $\Delta(\tau) = 0$. There exists a finite set $\mathrm{Irr}(\theta) \subset \mathbf{C}[z^{-1}]$ and a holomorphic decomposition

$$E = \oplus_{\mathfrak{a} \in \mathrm{Irr}(\theta)} E_{\mathfrak{a}} \tag{1}$$

into eigenbundles with respect to the various irregular parts of θ, i.e. such that

$$\theta|_{E_{\mathfrak{a}}} - \mathrm{d}\mathfrak{a}\,\mathrm{Id}_{E_{\mathfrak{a}}}$$

have at most a first-order pole. Here we assume that $\mathrm{rk}(E_{\mathfrak{a}}) \geq 1$ for every $\mathfrak{a} \in \mathrm{Irr}(\theta)$. For any section τ of $E \otimes \Omega^1$ we write

$$\tau = \sum_{\mathfrak{a} \in \mathrm{Irr}(\theta)} \tau_{\mathfrak{a}}$$

the corresponding decomposition. For any $\mathfrak{a} \in \mathrm{Irr}(\theta)$, we set

$$\mathrm{ord}(\mathfrak{a}) = \min\{d \in \mathbf{Z} \mid z^{-d}\tilde{\mathfrak{a}} \text{ has a removable singularity at } 0\}$$

for any lift $\tilde{\mathfrak{a}}$ of $\mathfrak{a}$ to a meromorphic function on Δ. We have $\mathrm{ord}(\mathfrak{a}) \leq 0$ and $\mathrm{ord}(\mathfrak{a}) = 0$ if and only if $\mathfrak{a} = 0$. For an integer $j \leq 0$, let us set

$$\tau_j = \sum_{\mathrm{ord}(\mathfrak{a}) = j} \tau_{\mathfrak{a}}.$$

We set

$$\tau_{<j} = \sum_{j' < j} \tau_{j'}, \qquad \tau_{>j} = \sum_{j' > j} \tau_{j'},$$

and

$$m = \min\{j \mid \tau_j \neq 0\}.$$

We have the decomposition

$$\tau = \sum_{j=m}^{0} \tau_j,$$

which is analogous to the decomposition of the endomorphism bundle of E that appears in [2].

For any section τ of $E \otimes \Omega^1$ we introduce the norm

$$\|\tau\|^2_{H^1(\Delta)} = \int_\Delta |\tau|_h^2 + |\nabla^+ \tau|_h^2 + |\Phi \otimes \tau|_h^2 \,\mathrm{d}z\,\mathrm{d}\bar z$$

and the resulting Hilbert space $H^1(\Delta)$ as the completion of smooth compactly supported sections with respect to the norm $\|\cdot\|_{H^1(\Delta)}$. Following [6], given any $\tau \in H^1(\Delta)$ and $r \in (0,1)$ we now define the *energy functional* on $H^1(\Delta)$ by

$$E(\tau, r) = \int_{|z| \leq r} |\nabla^+ \tau|_h^2 + |\Phi \otimes \tau|_h^2 \,\mathrm{d}z\,\mathrm{d}\bar z.$$

It is obvious that $E(\tau, r)$ is non-negative, monotonically increasing in r and satisfies

$$E(\tau, r) \leq \|\tau|_{\Delta(r)}\|^2_{H^1},$$

where $\Delta(r) = \{|z| < r\} \subset \Delta$.

Theorem 1. *Let τ satisfy $\Delta(\tau) = 0$ and $\tau, \nabla^+\tau, \Phi \otimes \tau \in L^2_{h, \mathrm{d}z \cdot \mathrm{d}\bar z}$. For every $j < 0$ there exist $R_j \in (0,1)$ and $C_j, \varepsilon_j > 0$ only depending on $(E, \bar\partial_E, \theta, h)$ such that for each $0 < R < R_j$ we have*

$$E(\tau_j, R) \leq C_j \exp\left(-\varepsilon_j R^j\right) E(\tau, R_j).$$

We now turn our attention to the case $j = 0$. Let us denote by $E_{0,\lambda}$ the generalized λ-eigenspace of $\operatorname{res}_{z=0}\theta$ on E_0 (the summand in (1) corresponding to irregular part $\mathfrak{a} = 0$), and set

$$E_{0,\text{sing}} = \oplus_{\lambda\neq 0} E_{0,\lambda}.$$

We then have

$$E_0|_{z=0} = E_{0,\text{sing}} \oplus E_{0,0}.$$

This direct sum extends holomorphically to an open neighborhood of $z = 0$. Let us write

$$\tau_0 = \tau_{0,\text{sing}} + \tau_{0,0}$$

with respect to this decomposition. We are ready to state the energy estimate in the case $j = 0$.

Theorem 2. *There exists $R_0 \in (0,1)$ and $C_0, \varepsilon_0 > 0$ only depending on $(E, \bar\partial_E, \theta, h)$ such that for each $0 < R < R_0$ we have*

$$E(\tau_{0,\text{sing}}, R) \leq C_0 E(\tau, R_0) R^{\varepsilon_0}.$$

Remark 3.

(1) These results are similar to [3, Proposition 8.4.5], up to three differences. First, in [3] a pointwise result is proved rather than an energy bound. Second, in [3] one takes the Poincaré metric instead of the Euclidean one. This latter point, however, makes no difference, because the two metrics are mutually bounded by polynomial factors in $|z|$. Third, in [3] L^2-conditions on $\nabla^+\tau, \Phi\otimes\tau$ are not assumed. For applications to the Hard Lefschetz theorem in [3, Section 18.2], this point does not make any difference either. Indeed, for such applications one has to deal there with certain L^2-cohomology classes (see [3, Equation (360)]), which presumes precisely L^2-conditions on $\nabla^+\tau, \Phi\otimes\tau$.

(2) In the case of ramified wild harmonic bundles, similar estimates hold and can be deduced from the unramified version.

(3) Similar results hold for sections of associated vector bundles, for instance endomorphism-valued 1-forms.

3. Energy estimates

From now on, unless otherwise specified, τ will stand for a harmonic section of $E\otimes\Omega^1$.

3.1. *Preliminaries*

We make use of the following results.

Theorem 4 (Consequence of [3, Theorem 7.2.1]). *Let τ, ω be arbitrary sections of $E\otimes\Omega^1$. There exist $A_1' > 0, \varepsilon_1' > 0$ only depending on $(E, \bar\partial_E, \theta, h)$ such that for any $j' \neq j$ the following pointwise estimate holds:*

$$|h(\omega_{j'}, \tau_j)| \leq A_1' \exp\left(-\varepsilon_1' |z|^{\min(j,j')}\right) |\omega_{j'}|_h |\tau_j|_h.$$

The actual form in which we will apply this result is its combination with the inequality between the geometric and quadratic means:

$$|h(\omega_{j'}, \tau_j)| \leq \frac{1}{2} A_1' \exp\left(-\varepsilon_1' |z|^{\min(j,j')}\right) \left(c|\omega_{j'}|_h^2 + c^{-1}|\tau_j|_h^2\right),$$

where $c > 0$ is an arbitrary constant.

Theorem 5 ([1, Theorem 5.4.]). *We have the Weitzenböck formula*

$$\Delta = (\nabla^+)^*\nabla^+ + (\Phi\otimes)^*\Phi\otimes.$$

A straightforward modification of the argument of [6, Claim 2.10] (see also [3, Lemma 8.4.8]) then gives

Lemma 6. *The subspace of smooth* 1*-forms that are compactly supported in* $\Delta \setminus \{0\}$ *is dense in* $H^1(\Delta)$.

3.2. *Proof of Theorem 1*

By virtue of Lemma 6, we may assume that τ is smooth and compactly supported in $\Delta\setminus\{0\}$. Stokes' theorem gives

$$E(\tau_j, R) = \int_{|z|<R} h\big((\nabla^+)^*\nabla^+\tau_j + (\Phi\otimes)^*\Phi\otimes\tau_j, \tau_j\big) + \int_{|z|=R} h(\nabla^+_{\partial_r}\tau_j, \tau_j) \tag{2}$$

where $\partial_r = \frac{\partial}{\partial r}$ is the unit normal vector of $|z| = R$. It follows from Theorem 5 that

$$(\nabla^+)^*\nabla^+\tau + (\Phi\otimes)^*\Phi\otimes\tau = 0,$$

equivalently

$$(\nabla^+)^*\nabla^+\tau_j + (\Phi\otimes)^*\Phi\otimes\tau_j = -\sum_{j'\neq j} (\nabla^+)^*\nabla^+\tau_{j'} - (\Phi\otimes)^*\Phi\otimes\tau_{j'}.$$

Plugging this identity to (2) and applying Stokes' theorem in the converse direction we find

$$E(\tau_j, R) = -\sum_{j'\neq j}\int_{|z|<R} h\big(\nabla^+\tau_{j'}, \nabla^+\tau_j\big) + h\big(\Phi\otimes\tau_{j'}, \Phi\otimes\tau_j\big) + \int_{|z|=R} h(\nabla^+_{\partial_r}\tau, \tau_j).$$

Applying Theorem 4 to this identity produces the estimate

$$E(\tau_j, R) \leq D\exp(-\varepsilon_1' R^j) E(\tau, R) + \int_{|z|=R} h(\nabla^+_{\partial_r}\tau_j, \tau_j) R\mathrm{d}\varphi + \sum_{j'\neq j}\int_{|z|=R} h(\nabla^+_{\partial_r}\tau_{j'}, \tau_j) R\mathrm{d}\varphi \tag{3}$$

for suitable $D, \varepsilon_1 > 0$.

Now, we are given that there exist $R_j, c > 0$ only depending on (E, θ) such that for any (not necessarily harmonic) section ω of E^j the following pointwise estimate holds for all $|z| < R_j$:

$$|\Phi\otimes\omega(z)|_h > c|z|^{j-1}|\omega(z)|_h.$$

Indeed, for any $\mathfrak{a} \in \mathrm{Irr}(\theta)$ of order $j < 0$, we have $\mathrm{ord}(\mathrm{d}\mathfrak{a}) = j - 1$, and we may choose any $c > 0$ such that $2c$ is smaller than the minimum of the absolute values of the leading-order terms of all irregular parts $\mathfrak{a}$ of order j appearing in (1). Importantly, notice that we may decrease $c > 0$ to make it arbitrarily small (without having to modify R_j). It then follows that for every $0 < r < R_j$ we have

$$\begin{aligned}\frac{\mathrm{d}E(\tau_j, r)}{\mathrm{d}r} &= \int_{|z|=r} |\nabla^+\tau_j|_h^2 + |\Phi\otimes\tau_j|_h^2 r\mathrm{d}\varphi \\ &\geq cr^{j-1}\int_{|z|=r} \frac{1}{cr^{j-1}}|\nabla^+\tau_j|_h^2 + cr^{j-1}|\tau_j|_h^2 r\mathrm{d}\varphi.\end{aligned} \tag{4}$$

Similarly, we have

$$\begin{aligned}\frac{\mathrm{d}E(\tau, r)}{\mathrm{d}r} &= \int_{|z|=r} |\nabla^+\tau|_h^2 + |\Phi\otimes\tau|_h^2 r\mathrm{d}\varphi \\ &\geq \frac{cr^{j-1}}{m}\sum_{j'\neq j}\int_{|z|=r} \frac{1}{cr^{j-1}}|\nabla^+\tau_{j'}|_h^2 + cr^{j-1}|\tau_j|_h^2 r\mathrm{d}\varphi.\end{aligned} \tag{5}$$

On the other hand, combining (3) with the inequality

$$2h(a, b) \leq \frac{1}{cr^{j-1}}|a|_h^2 + cr^{j-1}|b|_h^2,$$

we find

$$E(\tau_j, r) \le D\exp(-\varepsilon_1' r^j) E(\tau, r) + \frac{1}{2}\int_{|z|=r} \frac{1}{cr^{j-1}} |\nabla^+ \tau_j|_h^2 + cr^{j-1}|\tau_j|_h^2 r\,\mathrm{d}\varphi$$
$$+ \frac{D}{2}\exp(-\varepsilon_1' r^j) \sum_{j'\neq j} \int_{|z|=r} \frac{1}{cr^{j-1}} \left|\nabla^+ \tau_{j'}\right|_h^2 + cr^{j-1}|\tau_j|_h^2 r\,\mathrm{d}\varphi. \quad (6)$$

Let us introduce the modified energy functional

$$\widetilde{E}(\tau_j, R) = E(\tau_j, R) + F\exp(-\varepsilon_1' R^j) E(\tau, R),$$

where $F > 0$ is a constant to be chosen later. Then, using estimates (4), (5) and (6), for every $0 < r < R_j$ we infer

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{E}(\tau_j, r)}{\mathrm{d}r} &\ge \frac{cr^{j-1}}{m} E(\tau_j, r) - \frac{cr^{j-1}}{m} D\exp(-\varepsilon_1' r^j) E(\tau, r) \\
&\quad - \frac{cr^{j-1}}{m}\frac{D}{2}\exp(-\varepsilon_1' r^j) \sum_{j'\neq j} \int_{|z|=r} \frac{1}{cr^{j-1}} \left|\nabla^+ \tau_{j'}\right|_h^2 + cr^{j-1}|\tau_j|_h^2 r\,\mathrm{d}\varphi \\
&\quad - Fj\varepsilon_1' r^{j-1}\exp(-\varepsilon_1' r^j) E(\tau, r) + F\exp(-\varepsilon_1' r^j)\frac{\mathrm{d}E(\tau, r)}{\mathrm{d}r} \\
&\ge \frac{cr^{j-1}}{m} E(\tau_j, r) - (cD + Fj\varepsilon_1') r^{j-1}\exp(-\varepsilon_1' r^j) E(\tau, R_j) \\
&\quad + \left(F - \frac{D}{2}\right)\exp(-\varepsilon_1' r^j)\frac{\mathrm{d}E(\tau, r)}{\mathrm{d}r}.
\end{aligned}$$

Let us now choose F so that

$$F = \max\left(\frac{-cD}{\frac{c}{m} + j\varepsilon_1'}, \frac{D}{2}\right).$$

Notice that if $c > 0$ is chosen sufficiently small, then $\frac{c}{m} + j\varepsilon_1' < 0$ because $j\varepsilon_1' < 0$. Then, we have $-(cD + Fj\varepsilon_1') \ge \frac{c}{m}F$ and the above bound implies

$$\frac{\mathrm{d}\widetilde{E}(\tau_j, r)}{\mathrm{d}r} \ge \frac{cr^{j-1}}{m}\widetilde{E}(\tau_j, r).$$

In different terms, this states that

$$\frac{\mathrm{d}\ln\widetilde{E}(\tau_j, r)}{\mathrm{d}r} \ge \frac{cr^{j-1}}{m}$$

for all $r < R_j$. For any $0 < R < R_j$, integrating this inequality from R to R_j with respect to r shows that

$$\ln\widetilde{E}(\tau_j, R_j) - \ln\widetilde{E}(\tau_j, R) \ge \frac{c}{mj}(R_j^j - R^j).$$

Taking exponential of both sides gives the bound

$$K\exp\left(\frac{c}{mj}R^j\right)\widetilde{E}(\tau_j, R_j) \ge \widetilde{E}(\tau_j, R)$$

for

$$K = \exp\left(-\frac{c}{mj}R_j^j\right).$$

Finally, this estimate combined with

$$E(\tau_j, R) = \widetilde{E}(\tau_j, R) - F\exp(-\varepsilon_1' R^j) E(\tau, R_j) < \widetilde{E}(\tau_j, R)$$

gives the desired bound.

3.3. *Proof of Theorem 2*

Similarly to (6), using that $\min(j,0) = j$ for all $j < 0$, we now find

$$E(\tau_{0,\mathrm{sing}}, r) \leq D\exp(-\varepsilon_1' r^{-1}) E(\tau, r) + \frac{1}{2}\int_{|z|=r} \frac{r}{c}|\nabla^+ \tau_{0,\mathrm{sing}}|_h^2 + \frac{c}{r}|\tau_{0,\mathrm{sing}}|_h^2 r\mathrm{d}\varphi$$
$$+ \frac{D}{2}\exp(-\varepsilon_1' r^{-1}) \sum_{j'\neq 0}\int_{|z|=r} \frac{r}{c}\left|\nabla^+ \tau_{j'}\right|_h^2 + \frac{c}{r}|\tau_{0,\mathrm{sing}}|_h^2 r\mathrm{d}\varphi.$$

It then follows that

$$\begin{aligned}\frac{\mathrm{d}E(\tau_{0,\mathrm{sing}}, r)}{\mathrm{d}r} &\geq \frac{c}{rm}E(\tau_{0,\mathrm{sing}}, r) - \frac{c}{rm}D\exp(-\varepsilon_1' r^{-1})E(\tau, r)\\ &\quad - \frac{c}{rm}\frac{D}{2}\exp(-\varepsilon_1' r^{-1}) \sum_{j'\neq 0}\int_{|z|=r} \frac{r}{c}\left|\nabla^+ \tau_{j'}\right|_h^2 + \frac{c}{r}|\tau_{0,\mathrm{sing}}|_h^2 r\mathrm{d}\varphi\\ &\geq \frac{c}{rm}E(\tau_{0,\mathrm{sing}}, r) - \frac{c}{rm}D\exp(-\varepsilon_1' r^{-1})E(\tau, r)\\ &\quad - \frac{D}{2}\exp(-\varepsilon_1' r^{-1})\frac{\mathrm{d}E(\tau, r)}{\mathrm{d}r}.\end{aligned}$$

We apply the same argument as in Theorem 1, this time with the modified energy functional

$$\widetilde{E}(\tau_{0,\mathrm{sing}}, r) = E(\tau_{0,\mathrm{sing}}, r) + F\exp(-\varepsilon_1' r^{-1})E(\tau, r)$$

for some $F > 0$. We obtain

$$\frac{\mathrm{d}\widetilde{E}(\tau_{0,\mathrm{sing}}, r)}{\mathrm{d}r} \geq \frac{c}{rm}E(\tau_{0,\mathrm{sing}}, r) + \left(\frac{\varepsilon_1'}{r^2}F - \frac{c}{rm}D\right)\exp(-\varepsilon_1' r^{-1})E(\tau, r)$$
$$+ \left(F - \frac{D}{2}\right)\exp(-\varepsilon_1' r^{-1})\frac{\mathrm{d}E(\tau, r)}{\mathrm{d}r}.$$

We infer that if $F > D/2$ is fixed sufficiently large then we have

$$\frac{\mathrm{d}\widetilde{E}(\tau_{0,\mathrm{sing}}, r)}{\mathrm{d}r} \geq \frac{c}{rm}\widetilde{E}(\tau_{0,\mathrm{sing}}, r),$$

i.e.

$$\frac{\mathrm{d}\ln\widetilde{E}(\tau_{0,\mathrm{sing}}, r)}{\mathrm{d}r} \geq \frac{c}{rm}.$$

Integrating from R to R_0 and exponentiating this estimate yields

$$\widetilde{E}(\tau_{0,\mathrm{sing}}, R) \leq \widetilde{E}(\tau_{0,\mathrm{sing}}, R_0)R_0^{-c/m}R^{c/m}.$$

for every $0 < R < R_0$. Finally, comparing to the definition of $\widetilde{E}$ shows that

$$\begin{aligned}E(\tau_{0,\mathrm{sing}}, R) &\leq \left(E(\tau_{0,\mathrm{sing}}, R_0) + F\exp(-\varepsilon_1' R_0^{-1})E(\tau, R_0)\right)R_0^{-c/m}R^{c/m}\\ &\leq C_0 E(\tau, R_0)R^{c/m}.\end{aligned}$$

This concludes the proof.

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