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The radial limits and boundary uniqueness

Limites radiales et unicité frontière

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Abstract. The paper sheds a new light on the fundamental theorems of complex analysis due to P. Fatou, F. and M. Riesz, N. N. Lusin, I. I. Privalov, and A. Beurling. Only classical tools available at the times of Fatou are used. The proofs are very simple and in some cases – almost trivial.

Résumé. L'article apporte un nouvel éclairage sur les théorèmes fondamentaux de l'analyse complexe dus à P. Fatou, F. et M. Riesz, N. N. Lusin, I. I. Privalov et A. Beurling. Seuls les outils classiques disponibles à l'époque de Fatou sont utilisés. Les preuves sont très simples et dans certains cas, presque triviales.

Keywords. Radial limits, Boundary uniqueness, Fatou's Theorem, Lusin's theorem.

Mots-clés. Limites radiales, unicité frontière, théorème de Fatou, théorème de Lusin.

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1. Introduction and the main result

Let U and T be the open unit disk and the unit circle in $\mathbb{C}$, respectively. For a function f defined on U we denote by $f(e^{i\theta})$ the radial limit of f at $e^{i\theta}$ if the limit exists. The following classical theorems of P. Fatou [4] (cf. [3, Theorem 2.1]) and of F. and M. Riesz [11] (cf. [3, Theorem 2.5]) are among the most fundamental results of complex analysis.

Theorem A (P. Fatou, 1906). *Let f be analytic and bounded on U. Then for almost all $e^{i\theta}$ on T the non-tangential limit $f(e^{i\theta})$ exists.*

Theorem B (F. and M. Riesz, 1916). *Let f be analytic and bounded on U such that $f(e^{i\theta}) = 0$ on a set E of positive measure on T. Then f is identically zero.*

For univalent functions the analogous results are due to A. Beurling [1] (cf. [3, Theorem 3.5]).

Theorem C (A. Beurling, 1940). *Let f be univalent on U. Then: (i) at every point $e^{i\theta}$ of T, except possibly a set of zero (logarithmic) capacity, the radial limit $f(e^{i\theta})$ exists; (ii) $f(e^{i\theta}) = \lim_{r\to 1} f(r e^{i\theta})$ cannot be constant on any positive capacity set on T.*[1]

[1] The referee has kindly noted that Beurling [1] announces (ii) in [1] before stating Théorème II and proves it at the end of the proof of that theorem.

Theorem A presents the property of almost everywhere existence of the radial limits of bounded analytic functions, while Theorem B is the boundary uniqueness property of the same functions. Both these properties (theorems) have been presented and extended in many books and papers. Under the lights of the influential original works of Fatou and F. and M. Riesz, quite naturally, Theorem A and Theorem B have been universally regarded as two fundamental, but different properties of analytic functions. For instance the well known paper [2] by L. Carleson, emphasizing the difference between Theorem A and Theorem B, begins with the following sentences:

"For a large number of classes C of functions $f(z)$ regular in the unit circle, we have very complete knowledge concerning the existence of a boundary function

$$F(\theta) = \lim_{r\to 1} f(r\,\mathrm{e}^{\mathrm{i}\theta}),$$

the classical result being that of Fatou. However, very little is known about the properties of this boundary function $F(\theta)$, and in particular about the sets E associated with the class C, having the property that $f(z)$ vanishes identically if $F(\theta) = 0$ on E. Let us call a closed set of this kind a set of uniqueness for the class C. If E is not a set of uniqueness, we speak of a set of multiplicity. Our whole knowledge in this direction seems to be contained in a classical result of F. and M. Riesz: E is a set of uniqueness for the class of bounded functions if and only if it has positive Lebesgue measure."

The present paper sheds a new light on the theory and shows that, in fact, the boundary uniqueness property is a direct corollary, or a particular case, of the property of the existence of the radial limit. Indeed, *the radial limit cannot be constant on a set E precisely because the radial limit exist on E (or a. e. on E),* as the following theorem shows.

Theorem 1. *Let f be univalent (respectively, zero free and bounded analytic) on U such that $f(\mathrm{e}^{\mathrm{i}\theta}) = 0$ on a subset E of T. Then f generates a univalent (respectively, bounded analytic) function g on U such that g has no radial limit on E. Furthermore, if $f \neq 0$ is bounded analytic with zeros on U such that $f(\mathrm{e}^{\mathrm{i}\theta}) = 0$ on E, then f generates a bounded analytic g on U which has no radial limit on E except perhaps a subset of E of measure zero.*

Remark 2. If f in Theorem 1 is zero free, analytic and bounded, the relation between f and g is especially simple and given explicitly by the equation $g(z) = \mathrm{e}^{-\mathrm{i}\log\log f(z)}$, as below proof implies. Also, as noted in [3, p. 57], with no loss of generality it is enough to prove Theorem C just for *bounded* univalent functions and we use the same reduction in our proof of the univalent case of Theorem 1.

1.1. *Elementary proof of Theorem 1 for the case of zero free, bounded analytic f*

Let f be zero free, analytic and bounded on U such that $f(\mathrm{e}^{\mathrm{i}\theta}) = 0$ on E. We may assume f is bounded by 1.

Then $f(z) = \mathrm{e}^{h(z)}$, where h is analytic, $\Re h(z) < 0$ on U and $h(\mathrm{e}^{\mathrm{i}\theta}) = \infty$ on E. We have an analytic $\log h(z) = \log|h(z)| + \mathrm{i}\arg h(z)$ on U with $\pi/2 < \arg h(z) < 3\pi/2$. Then $g(z) = \mathrm{e}^{-\mathrm{i}\log h(z)} = \mathrm{e}^{\arg h(z)}(\cos\log|h(z)| - \mathrm{i}\sin\log|h(z)|)$ is analytic and bounded by $\mathrm{e}^{3\pi/2}$. On each radius ending on E the oscillation of g exceeds $\mathrm{e}^{\pi/2}$. The proof is over. □

The proof of the remaining part of Theorem 1 is also simple. Note that already just proved part of Theorem 1 mplies a theorem of Lusin [6] (formulated below).

Obviously Theorem 1 implies that the part (ii) of Theorem C is a corollary of part (i) of Theorem C. Similarly, Theorem 1 derives Theorem B from Theorem A. Thus it gives the first unified proof of boundary uniqueness for univalent or merely analytic functions.

Our approach extremely simplifies the case of univalent functions and below new proof of part (ii) of Theorem C should be compared with its prior proof; cf. the proof of part (ii) of Theorem C in [3, pp. 61–64], or in [1].

Lusin has proved the following important converse of Theorem A (see [7] or [6]).

Theorem D (N.N. Lusin, 1919). *Let E be a subset of T of measure zero. Then there exists an analytic and bounded function f on U such that the radial limit $f(\mathrm{e}^{\mathrm{i}\theta})$ does not exist at each $\mathrm{e}^{\mathrm{i}\theta} \in E$.*

The following classical result is due to Privalov [8] (cf. [9, p. 295] or [12, p. 276]).

Theorem E (I.I. Privalov, 1919). *Let E be a subset of T of measure zero. Then there exists a zero free, analytic and bounded (by 1) function f on U such that $f(z)$ tends to 0 as z approaches, in an arbitrary manner (in particular, radially), any point of E.*

Since the function f existing by Theorem E is zero free, analytic and bounded, and $f(\mathrm{e}^{\mathrm{i}\theta}) = 0$ on E, the zero free case of Theorem 1 implies that Theorem D is nothing else but *an obvious corollary* of Theorem E.[2] Since also Privalov's proof of Theorem E is a simple construction, we arrive to the first elementary self-contained proof of Theorem D, which is in a sharp contrast to its (very complex) original proof. We present the proof of Theorem E.

1.2. *Proof of Theorem E*

Denote $E_1 = \{x \in [0, 2\pi] : \mathrm{e}^{\mathrm{i}x} \in E\}$. Let $\{G_n\}$ be a sequence of open sets on $[0, 2\pi]$ such that $E_1 \subset G_{n+1} \subset G_n$ and the measure of G_n is less than $\frac{1}{n^4}$. Let f_n be the characteristic function of G_n. Denote $f(x) = \sum_{n=1}^{\infty} n^2 f_n(x)$. Obviously $f \geq 0$ and $f \geq n^2 f_n \geq 0$ for all n. Since

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{2\pi} n^2 f_n(x)\,\mathrm{d}x < \sum_{n=1}^{\infty} n^2 \,\frac{1}{n^4} < \infty,$$

f is summable.

Let $U(r\,\mathrm{e}^{\mathrm{i}\theta})$ and $U_n(r\,\mathrm{e}^{\mathrm{i}\theta})$ be the Poisson integrals of f and $n^2 f_n$, respectively. Then $U(r\,\mathrm{e}^{\mathrm{i}\theta}) \geq U_n(r\,\mathrm{e}^{\mathrm{i}\theta})$. Obviously, $U_n(r\,\mathrm{e}^{\mathrm{i}\theta})$ continuously extends to G_n and $U_n(\mathrm{e}^{\mathrm{i}x}) = n^2$ for $x \in G_n$. For a fixed $\mathrm{e}^{\mathrm{i}x} \in E$, we have $x \in G_n$ (for all n). Thus, for $r\,\mathrm{e}^{\mathrm{i}\theta} \to \mathrm{e}^{\mathrm{i}x}$, we have $\liminf U(r\,\mathrm{e}^{\mathrm{i}\theta}) \geq \liminf U_n(r\,\mathrm{e}^{\mathrm{i}\theta}) = \lim U_n(r\,\mathrm{e}^{\mathrm{i}\theta}) = n^2$, which implies that $\lim U(r\,\mathrm{e}^{\mathrm{i}\theta}) = \infty$.

Let $V(r\,\mathrm{e}^{\mathrm{i}\theta})$ be a conjugate harmonic function of $U(r\,\mathrm{e}^{\mathrm{i}\theta})$. The bounded analytic function $(1 + U + iV)^{-1}$ (or $\mathrm{e}^{-U-\mathrm{i}V}$) has the required properties. The proof is over. □

Lusin and Privalov have been collaborating for many years in topics involving Theorem D and Theorem E, and Theorem D even appears again in their joint paper [7] (essentially with same original difficult proof of 1919), but they did not notice that Theorem D immediately follows from Theorem E. Perhaps it is hard to overestimate the importance of Theorem D as a converse to Theorem A, but unfortunately it is much less known than Theorem A. Our elementary proof of Theorem D is easily accessible even for the students. This will help to make Theorem D more popular.[3]

[2] If in the proof in Subsection 1.1 the starting function f is taken to be the function existing by Theorem E, then the proof in Subsection 1.1 just becomes a proof of Theorem D.

[3] Theorem A is proved in the standard textbooks on complex analysis, but, as a rule, none of them (say, W. Rudin's comprehensive "Real and Complex Analysis") even mentions on the existence of Theorem D. Including Theorem D (with its new proof) in textbooks will complete the presentation of the classical theory as it shows that the conclusion of Theorem A is precise (cannot be improved).

2. A conformal mapping approach for Theorem 1

In this section, we use the Riemann mapping theorem to prove a proposition which implies Theorem 1. But the main purpose of this proposition is to prove Theorem 1 in case of univalent functions. For bounded analytic functions we also have an alternative proof based on the theorem of F. Riesz on the boundary values of Blaschke products.

Let D be a simply connected "double comb" domain in the w-plane obtained from the square

$$\{w = u + \mathrm{i}v : 0 < u < 1,\ 0 < v < 1\}$$

by taking off the line segments $l_{2n} = \left\{u + \mathrm{i}v : u = \frac{1}{2n},\ 0 \le v \le \frac{3}{4}\right\}$ and $l_{2n+1} = \left\{u + \mathrm{i}v : u = \frac{1}{2n+1},\ \frac{1}{4} \le v \le 1\right\}$ for all values of n ($n = 1, 2, \ldots$). Denote by AB the closed set $\{\mathrm{i}v : 0 \le v \le 1\}$ (the left side of the original square). It contains no accessible boundary points of D. In other words, there is no Jordan arc in D ending at a point of AB (to approach to AB, a Jordan arc has to "oscillate").

Proposition 3. *Let φ map U conformally onto D. Then there exists a point ξ on T such that φ has no limit as z approaches ξ along any Jordan arc γ, $\gamma \setminus \{\xi\} \subset U$, ending at ξ.*[4]

Let Γ be a halfopen Jordan arc in D, oscillating and approaching to AB asymptotically. For instance, as such Γ, one can take the polygonal in D joining the sequence of the points $M_1(\frac{1}{2}, \frac{7}{8})$, $M_2(\frac{1}{3}, \frac{1}{8})$, $M_3(\frac{1}{4}, \frac{7}{8})$, $M_4(\frac{1}{5}, \frac{1}{8})$, ... We may assume that Γ is given by an equation $w = w(t)$, $0 \le t < 1$, where $w(t)$ is continuous on $[0, 1)$ and $w(0) \equiv M_1(\frac{1}{2}, \frac{7}{8})$ is the initial point of Γ. Let Γ_1 be a Jordan arc in D having the same initial point $w(0)$ as Γ and ending at an accessible boundary point w_1 of D ($\Gamma_1 \setminus \{w_1\} \subset D$), and such that $w(0)$ is the only common point of Γ and Γ_1.

The following proposition is obvious.

Proposition 4. *The set $\Gamma \cup \Gamma_1$ divides the domain D into two domains D_1 and D_2 such that the boundaries of both D_1 and D_2 contain either all segments l_{2n} or all segments l_{2n+1} except finitely many of such segments.*

Morera's theorem and the elementary (inner) uniqueness theorem immediately imply:

Proposition 5. *If f is continuous in a domain Ω and analytic in $\Omega \setminus L$ where L is a line segment, then f is analytic in Ω. If in addition $f(z) = c$ on L, then f is identically c on Ω.*

Proof of Proposition 3. Let $w = \varphi(z)$ be a conformal map of U onto D (as in Proposition 3). Denote by $z = \psi(w)$ the inverse of $w = \varphi(z)$. The image $\psi(\Gamma)$ of Γ is a halfopen Jordan arc in U given by the equation $w = \psi(w(t))$, $0 \le t < 1$. Note that $\psi(\Gamma)$ ends at a point $\xi \in T$, because otherwise $\psi(\Gamma)$ has to have two accumulating points a and b on T, and the function φ cannot have radial limits on one of the two complementary to a and b open arcs of T, which contradicts to Theorem A. Now we show that $\xi \in T$ has the property formulated in Proposition 3.

Let γ, $\gamma \setminus \{\xi\} \subset U$, be an arbitrary Jordan arc ending at ξ. If γ and $\psi(\Gamma)$ share points (other than ξ) at each neighborhood of ξ, then there is nothing to prove (because as Γ, the curve $\varphi(\gamma)$ too would be oscillating and approaching to AB in D). Thus, by deleting some initial portion of γ if necessary, we may assume that γ and $\psi(\Gamma)$ have no common point other than ξ. Let us join the initial points of γ and $\psi(\Gamma)$ by an arc $\delta \subset U$ such that δ has no other common point with γ or with $\psi(\Gamma)$. The curve $\psi(\Gamma) \cup \delta \cup \gamma$ divides U into two domains. One of them, let denote it by U_1, has only one point of T, namely ξ, on its boundary.

Assume by contrary that $\varphi(z)$ has a limit equal to q as z approaches ξ along γ. This means that q is an accessible boundary point of D (and the Jordan arc $\varphi(\gamma)$ ends at q). The Jordan arc $\varphi(\delta)$ joins in D the initial points of Γ and $\varphi(\gamma)$, and $\varphi(\delta)$ has no other common point with them. Let

[4] Note that the existence of ξ follows from the famous prime end theorem of Carathéodory (1913); simply take as ξ the point corresponding (under the mapping φ) to the prime end of D the impression of which is AB. Below we give an elementary proof of the same fact avoiding the concept of prime end.

us take $\varphi(\gamma)\cup\varphi(\delta)$ as Γ_1 and apply Proposition 4; we conclude that there exists either a segment l_{2n+1} or a segment l_{2n} lying on the boundary of the image $\varphi(U_1)$ of U_1. For briefness, denote this segment by l.

Because ξ is the only boundary point of U_1 belonging to T, the conformal mapping $\psi(w)$ of $\varphi(U_1)$ onto U_1 will be continuously extended to the set l once we put $\psi(w)=\xi$ on l. Now Proposition 5 implies that $\psi(w)$ is identically equal to ξ, which is impossible since ψ is a univalent function. This contradiction completes the proof of Proposition 3. □

We close this section by a discussion on Blaschke products. Let $\{a_n\}$ be a sequence of (non-zero) numbers in U such that $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$. It is well-known that this condition is necessary and sufficient for the infinite product $B(z)=\prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\frac{a_n-z}{1-\bar{a}_n z}$ to be uniformly convergent on compact subsets of U. Of course, $B(z)$ is bounded (by 1) and analytic in U, and the zeros of $B(z)$ are precisely the numbers of the sequence $\{a_n\}$. More generally, a Blaschke product is the following function $B(z)=z^m\prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\frac{a_n-z}{1-\bar{a}_n z}$, where $m\geq 0$ is an integer.

By Theorem A, of course, the radial limits of a Blaschke product exist a.e. on T. In 1923 F. Riesz [10] (cf. [3, Theorem 2.11]) proved the following result.

Theorem F (F. Riesz, 1923). *A Blaschke product B possesses radial limits of modulus* 1 *for almost all* $e^{i\theta}$ *on T*

An elementary proof of this theorem can be found in K. Hoffman's book[5] (see [5, bottom of p. 65 – top of p. 66]). We present this proof.

Proof of Theorem F. Denote by B_n the nth partial product of B. For natural $n>m$,

$$\|B_m-B_n\|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|B_m-B_n|^2\,dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}[|B_m|^2+|B_n|^2-2\Re B_n\bar{B}_m]\,dt.$$

Since $|B_n|=1$ on T,

$$\|B_m-B_n\|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[2-2\Re\frac{B_n}{B_m}\right]dt=2\left[1-\Re\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{B_n}{B_m}\,dt\right].$$

Because $n>m$, B_n/B_m is analytic on U. Thus $\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{B_n}{B_m}\,dt=\frac{B_n}{B_m}(0)=\prod_{j=m+1}^{n}|a_j|$, and

$$\|B_m-B_n\|^2=2\left(1-\prod_{j=m+1}^{n}|a_j|\right).$$

Since $\sum_{j=1}^{\infty}(1-|a_n|)<\infty$, the infinite product $\prod_{j=1}^{\infty}|a_j|$ converges, and, therefore, $\{B_n\}$ is Cauchy. Then $\{B_n\}$ converges to some g in H^2. This L^2 convergence on T implies the uniform convergence of $\{B_n\}$ to g on compact subsets of U. Thus $g(z)=B(z)$ on U, and so $g=B$. Since $\{B_n\}$ converges to B (on T) in L^2, a subsequence $\{B_{k_n}\}$ converges to B a.e. on T. Thus $|B|=1$ a.e. on T. The proof is over. □

3. Proofs

For zero free, bounded analytic functions the proof of Theorem 1 was given in above Subsection 1.1. Now we prove the theorem for the univalent functions; at the same time, this provides yet another proof for zero free, bounded analytic functions. Then we complete the proof of Theorem 1 by reducing the proof of the remaining case (of bounded analytic functions with zeros) to the case of zero free, bounded analytic functions.

[5] I am indebt to Don Marshall for calling my attention to this proof.

3.1. *Proof of Theorem 1*

Let f be bounded, univalent (or analytic and zero free) on U, and let $f(\mathrm{e}^{\mathrm{i}\theta}) = 0$ on E. (Since $f(\mathrm{e}^{\mathrm{i}\theta}) = 0$ on E, of course, f is zero free on U also in case if f is univalent.) We may assume f is bounded by 1. Then $f(z) = \mathrm{e}^{h(z)}$, where h is univalent (or analytic), $\Re h(z) < 0$ on U and $h(\mathrm{e}^{\mathrm{i}\theta}) = \infty$ on E. Let the univalent function φ and the point $\xi \in T$ be as in Proposition 3, and let $\psi(\zeta)$ be a fractional-linear mapping of the left half plane onto the unit disk under which ∞ corresponds to ξ. Then the function $g(z) = \varphi(\psi(h(z)))$ does not have radial limits on E. Next, g is bounded analytic, and if f is univalent, then with h also g is univalent. This completes the proof in case of univalent (or bounded analytic and zero free) functions.

Finally, let $f \neq 0$ be a bounded analytic function such that $f(\mathrm{e}^{\mathrm{i}\theta}) = 0$ on E, and let $\{a_n\}$ be the sequence of zeros (each zero repeated as often as it's multiplicity) of f in U. It is well-known that $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$. Let $B(z)$ be a Blaschke product which has zeros precisely at the points of the sequence $\{a_n\}$. Then $f(z) = B(z)f_1(z)$, where f_1 is bounded analytic and zero free on U. By Theorem F, $|B(\mathrm{e}^{\mathrm{i}\theta})| = 1$ a.e. on T. Thus, $f_1(\mathrm{e}^{\mathrm{i}\theta}) = 0$ on some $E_1 \subset E$ such that $E \setminus E_1$ is of Lebesgue measure zero. Therefore f_1 generates a bounded analytic g which does not have radial limits on E_1. The proof is over. $\square$

Remark 6. Since f_1 is zero free, $g(z) = \mathrm{e}^{-\mathrm{i}\log\log f_1(z)}$ as in Remark 2, and thus we have $g(z) = \mathrm{e}^{-\mathrm{i}\log\log\frac{f(z)}{B(z)}}$ as an explicit equation which connects f and g in case of bounded analytic function f.

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References

[1] A. Beurling, "Ensembles exceptionnels", *Acta Math.* **72** (1940), pp. 1–13.
[2] L. Carleson, "Sets of uniqueness for functions regular in the unit circle", *Acta Math.* **87** (1952), pp. 325–345.
[3] E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge University Press, 1966, pp. xi+211.
[4] P. Fatou, "Séries trigonométriques et séries de Taylor", *Acta Math.* **30** (1906), no. 1, pp. 335–400.
[5] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, 1962.
[6] N. N. Lusin, "Sur la représentation conforme", *Bull. Ivanovo-Vozn. Politech. Inst.* **2** (1919), pp. 77–80.
[7] N. N. Lusin and J. Priwaloff, "Sur l'unicité et la multiplicité des fonctions analytiques", *Ann. Sci. Éc. Norm. Supér.* **42** (1925), pp. 143–191.
[8] I. I. Privalov, *Intégrale de Cauchy*, Saratov, 1919. Online at https://archive.org/details/libgen_00292731/page/n3/mode/2up.

[9] I. I. Privalov, *Graničnye svoĭstva analitičeskih funkciĭ*, second edition, Gosudarstv. Izdat. Tehn.-Teor. Lit., 1950, p. 336.
[10] F. Riesz, “Über die Randwerte einer analytischen Funktionen”, *Math. Z.* **18** (1923), pp. 87–95.
[11] M. Riesz F.and Riesz, “Über die Randwerte einer analytischen Funktionen”, *Quatrième Congrès des Math. Scand. Stockholm* (1916), pp. 27–44.
[12] A. Zygmund, *Trigonometric series. Vol. I*, second edition, Cambridge University Press, 1959, pp. xii+383.