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Mathématique

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Volume 362 (2024), p. 863-881

Online since: 7 October 2024

https://doi.org/10.5802/crmath.610

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Comptes Rendus. Mathématique
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https://doi.org/10.5802/crmath.610

Research article / *Article de recherche*
Complex analysis and geometry / *Analyse et géométrie complexes*

Further than Descartes' rule of signs

Au-delà de la règle des signes de Descartes

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Abstract. The *sign pattern* defined by the real polynomial $Q := \Sigma_{j=0}^{d} a_j x^j$, $a_j \neq 0$, is the string $\sigma(Q) := (\mathrm{sgn}(a_d), \ldots, \mathrm{sgn}(a_0))$. The quantities pos and neg of positive and negative roots of Q satisfy Descartes' rule of signs. A couple $(\sigma_0, (\mathrm{pos}, \mathrm{neg}))$, where σ_0 is a sign pattern of length $d+1$, is *realizable* if there exists a polynomial Q with pos positive and neg negative simple roots, with $(d - \mathrm{pos} - \mathrm{neg})/2$ complex conjugate pairs and with $\sigma(Q) = \sigma_0$. We present a series of couples (sign pattern, pair $(\mathrm{pos}, \mathrm{neg})$) depending on two integer parameters and with $\mathrm{pos} \geq 1$, $\mathrm{neg} \geq 1$, which is not realizable. For $d = 9$, we give the exhaustive list of realizable couples with two sign changes in the sign pattern.

Résumé. La suite de signes des coefficients d'un polynôme réel $Q := \Sigma_{j=0}^{d} a_j x^j$, $a_j \neq 0$, est donnée par $\sigma(Q) := (\mathrm{sgn}(a_d), \ldots, \mathrm{sgn}(a_0))$. Les quantités pos et neg de racines positives et négatives de Q satisfont la règle des signes de Descartes. Un couple $(\sigma_0, (\mathrm{pos}, \mathrm{neg}))$, où σ_0 est une suite de signes de longueur $d+1$, est « réalisable » s'il existe un polynôme Q avec pos racines simples positives et neg racines simples négatives, avec $(d - \mathrm{pos} - \mathrm{neg})/2$ paires complexes conjuguées et avec $\sigma(Q) = \sigma_0$. Nous présentons une série de couples (suite de signes, paire $(\mathrm{pos}, \mathrm{neg})$) dépendant de deux paramètres entiers et avec $\mathrm{pos} \geq 1$, $\mathrm{neg} \geq 1$, qui ne sont pas réalisables. Pour $d = 9$, nous donnons la liste exhaustive des couples réalisables avec deux changements de signe dans la suite de signes.

Keywords. Real polynomial in one variable, hyperbolic polynomial, sign pattern, Descartes' rule of signs.

Mots-clés. Polynôme à une variable réelle, polynôme hyperbolique, suite de signes, règle de Descartes.

2020 Mathematics Subject Classification. 26C10, 30C15.

Manuscript received 15 February 2023, revised 14 January 2024, accepted 7 January 2024.

1. Introduction

This paper deals with a problem which is a natural extension of Descartes' rule of signs. We consider univariate real polynomials without vanishing coefficients. About such a degree d polynomial Q, Descartes' rule of signs states that the number pos of its positive roots is bounded by the number c of sign changes in the sequence of its coefficients, the difference $c - \mathrm{pos}$ being even, see [3, 6, 9, 10, 12, 17–20]. When applied to the polynomial $Q(-x)$, this rule yields the result that the number neg of its negative roots is bounded by the number p of sign preservations and the difference $p - \mathrm{neg}$ is also even. In the case of *hyperbolic* polynomials, i.e. real polynomials

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ISSN (electronic): 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

with all roots real, one has pos $= c$ and neg $= p$. Descartes' rule of signs proposes only necessary conditions. Our aim is to understand how far from sufficient they can be.

In what follows we use the quantity $\widetilde{\mu} := \min(\text{pos},\text{neg})$ and we denote by $\widetilde{\lambda}$ the number of complex conjugate pairs of roots of Q. Hence the quantities which we introduce satisfy the following relations in which $[.]$ denotes the integer part:

$$\begin{array}{ccc} \text{pos} \leq c\,, & \text{neg} \leq p\,, & \widetilde{\lambda} = (d - \text{pos} - \text{neg})/2\,, \\ c - \text{pos} \in 2\mathbb{Z}\,, & p - \text{neg} \in 2\mathbb{Z}\,, & \widetilde{\lambda} \leq [d/2]\,, \\ \text{sgn}(Q(0)) = (-1)^{\text{pos}} & \text{and} & c + p = d\,. \end{array} \tag{1}$$

Definition 1.

(1) *A* sign pattern *of length $d+1$ is a string of $d+1$ signs $+$ and/or $-$. We say that the polynomial $Q := \sum_{j=0}^{d} a_j x^j$ defines the sign pattern $\sigma(Q) := (\text{sgn}(a_d), \ldots, \text{sgn}(a_0))$. Most often we deal with monic polynomials in which case the first sign of the sign pattern is a $+$.*

(2) *Suppose that a sign pattern σ_0 is given, with c sign changes and p sign preservations. An* admissible pair *(i.e. a pair admissible for σ_0) is a pair* (pos, neg) *satisfying conditions* (1). *In this case we say that* $(\sigma_0, (\text{pos},\text{neg}))$ *is a* compatible couple *(or just* couple *for short).*

(3) *We say that this compatible couple is* realizable *if there exists a degree d real polynomial Q with $\sigma(Q) = \sigma_0$, with exactly* pos *positive and* neg *negative roots, all of them distinct.*

(4) *The pair* $(\text{pos},\text{neg}) = (c, p)$, *which is admissible for the sign pattern σ_0, is its* Descartes' pair.

We study the following problem:

Problem 2. *For a given degree d, which compatible couples are realizable?*

In spite of its simple formulation the problem is not trivial at all. Descartes' rule of signs gives only necessary conditions, and Problem 2 is a realization problem. For $d = 1$, 2 and 3, all compatible couples are realizable, but for $d = 4$, the couple $((+,-,-,-,+),(0,2))$ is not (see [11]). In fact, for $d = 4$, this is the only non-realizable couple up to the following equivalence.

Definition 3. *For a given degree d, the* $\mathbb{Z}_2 \times \mathbb{Z}_2$-action *on the set of compatible couples is defined by means of two commuting involutions. The involution*

$$i_m\ :\ (\sigma(Q),(\text{pos},\text{neg})) \longmapsto ((-1)^d \sigma(Q(-x)),(\text{neg},\text{pos})) \tag{2}$$

expresses the fact that the change of variable $x \mapsto -x$ changes the sign of every second coefficient and the signs of the real roots. The involution

$$i_r\ :\ (\sigma(Q),(\text{pos},\text{neg})) \longmapsto (\sigma(Q^R(x)/Q(0))(\text{pos},\text{neg}))\,,\ \ Q^R(x) := x^d Q(1/x) \tag{3}$$

is connected with the property the reverted *polynomial Q^R (i.e. Q read from right to left) to have the same numbers of positive and negative roots as Q (reversion changes the roots to their reciprocals). The normalizing factors $(-1)^d$ and $1/Q(0)$ are introduced to preserve the set of monic polynomials. The* orbit *of each compatible couple under this action consists either of 2 or of 4 couples which share the same values of $\widetilde{\lambda}$ and $\widetilde{\mu}$ and which are simultaneously realizable or not. This is why we use the same notation for couples and for their orbits. Sometimes we consider the orbits under the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action only of sign patterns, not of couples.*

Notation 4. We denote by $\Sigma_{m_1,m_2,\ldots,m_s}$, $m_k \in \mathbb{N}$, $m_1 + \cdots + m_s = d+1$, the sign pattern beginning with a sequence of m_1 signs $+$ followed by a sequence of m_2 signs $-$ followed by a sequence of m_3 signs $+$ etc. Example:

$$(+,+,-,-,-,+,-,+,+,+) = \Sigma_{2,3,1,1,3}\,.$$

Example 5. For $d = 3$, the orbit of the sign pattern $\sigma^\diamond := (+,+,-,+)$ consists of four sign patterns: $\sigma^\diamond$, $i_r(\sigma^\diamond) = (+,-,+,+)$, $i_m(\sigma^\diamond) = (+,-,-,-)$ and $i_r(i_m(\sigma^\diamond)) = (+,+,+,-)$. The orbit of the sign pattern $\Sigma_{1,d-1,1}$ ($d \geq 3$) consists of two sign patterns:

$$\Sigma_{1,d-1,1} = i_r(\Sigma_{1,d-1,1}) \text{ and } i_m(\Sigma_{1,d-1,1}) = i_m(i_r(\Sigma_{1,d-1,1})) = \Sigma_{2,1,\ldots,1,2}$$

($d-3$ units). For d even, both $\Sigma_{1,d-1,1}$ and $i_m(\Sigma_{1,d-1,1})$ are center-symmetric. For d odd, $\Sigma_{1,d-1,1}$ is center-symmetric and $i_m(\Sigma_{1,d-1,1})$ is center-antisymmetric.

Remark 6. For an orbit of length 2, one has either $i_r(\sigma(Q)) = \sigma(Q)$ or $i_r(i_m(\sigma(Q))) = \sigma(Q)$. One has always $i_m(\sigma(Q)) \neq \sigma(Q)$, because the second components of these two sign patterns are different.

Problem 2 is formulated for the first time in [2]. Up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, for $d = 5$ and $d = 6$, there exist exactly 1 and exactly 4 non-realizable orbits respectively, see [1]. For $d = 7$ (see [7]) and $d = 8$ (see [7] and [13]) these numbers equal 6 and 19. In all these cases one has $\widetilde{\mu} = 0$ and the exhaustive answer to Problem 2 is known. Details are given in Section 2.

For hyperbolic polynomials, the answer to Problem 2 is always positive, see [15, Proposition 1]. In other words, the orbit of any compatible couple in which the admissible pair is the Descartes' pair is realizable. A tropical analog of Descartes' rule of signs is studied in [8].

The first examples of non-realizable orbits with $\widetilde{\mu} = 1$ are found for $d = 11$ in [14] and for $d = 9$ in [4]. For $d = 9$, the orbit studied in [4] is the only non-realizable one with $\widetilde{\mu} = 1$. In this paper we present a series of non-realizable orbits (depending on two integer parameters) with $\widetilde{\mu} = 1$. The series includes the examples for $d = 11$ and $d = 9$ already found. Besides, for $d = 10$ and $d = 11$, we give the list of the orbits with $\widetilde{\mu} = 1$ which are either non-realizable or for which the answer to Problem 2 is not known. We also show that for $d \leq 14$, with the possible exception of one orbit with $d = 14$, there are no non-realizable orbits with $\widetilde{\mu} \geq 2$. Finally, we give the exhaustive answer to Problem 2 for $d = 9$, $c = 2$.

Definition 7. *For $d \geq 9$, we denote by $\mathcal{K}_{n,q}$ the orbit $(\Sigma_{1,n,q,1},(1,d-3))$, $n \geq 4$, $q \geq 4$, $n+q = d-1$, hence with Descartes' pair $(3,d-3)$. The orbits $\mathcal{K}_{n,q}$ and $\mathcal{K}_{q,n}$ being the same one can assume that $n \leq q$.*

Theorem 8. *For $d \geq 9$, the orbit $\mathcal{K}_{n,q}$ is not realizable.*

The theorem is proved in Section 3.

Theorem 9.

(1) *For $d \leq 14$, with the possible exception of the orbit $(\Sigma_{1,4,5,4,1},(2,10))$ about which the answer to Problem 2 is not known, there are no non-realizable orbits with $\widetilde{\mu} \geq 2$.*

(2) *For $d = 10$, for the orbits $\mathcal{K}_{4,5} = (\Sigma_{1,4,5,1},(1,7))$ and $(\Sigma_{1,4,4,2},(1,7))$, one has $(\widetilde{\mu},\widetilde{\lambda}) = (1,1)$. The first of them is not realizable (see Theorem 8), for the second one the answer to Problem 2 is not known. All other orbits with $d = 10$ and $\widetilde{\mu} = 1$ are realizable.*

(3) *For $d = 11$, the following two orbits (both with $\widetilde{\mu} = \widetilde{\lambda} = 1$) are not realizable:*

$$\mathcal{K}_{5,5} = (\Sigma_{1,5,5,1},(1,8)) \text{ and } \mathcal{K}_{4,6} = (\Sigma_{1,4,6,1},(1,8))\,.$$

For $d = 11$, with the exception of these two and of the following six orbits

$$(\Sigma_{1,4,5,2},(1,8))\,,\ (\widetilde{\mu},\widetilde{\lambda}) = (1,1)\,, \qquad (\Sigma_{1,5,4,2},(1,8))\,,\ (\widetilde{\mu},\widetilde{\lambda}) = (1,1)\,,$$

$$(\Sigma_{2,4,4,2},(1,8))\,,\ (\widetilde{\mu},\widetilde{\lambda}) = (1,1)\,, \qquad (\Sigma_{1,4,4,1,1,1},(1,6))\,,(\widetilde{\mu},\widetilde{\lambda}) = (1,2)\,,$$

$$(\Sigma_{1,4,1,1,4,1},(1,6))\,,\ (\widetilde{\mu},\widetilde{\lambda}) = (1,2)\,, \quad (\Sigma_{1,3,1,1,1,1,3,1},(1,4))\,,\ (\widetilde{\mu},\widetilde{\lambda}) = (1,3)\,,$$

(about which the answer to Problem 2 is not known), all other non-realizable orbits are with $\widetilde{\mu} = 0$.

The theorem is proved in Section 4. Our next result concerns orbits of sign patterns with two sign changes. Sign patterns with none or one sign change are realizable, see [5, Example 1 and Theorem 1], and the first examples of non-realizability are obtained for $c = 2$, see the beginning of Section 2.

Theorem 10. *For $d = 9$, the following orbits are not realizable:*

$$(\Sigma_{1,8,1},(0,k))\ ,\ k=3\ ,\ 5 \text{ and } 7\ ;$$
$$(\Sigma_{1,7,2},(0,\ell))\ ,\ (\Sigma_{1,6,3},(0,\ell)) \text{ and } (\Sigma_{2,6,2},(0,\ell))\ ,\ \ell=5 \text{ and } 7\ ;$$
$$(\Sigma_{1,5,4},(0,7))\ ,\ (\Sigma_{1,4,5},(0,7))\ ,\ (\Sigma_{2,4,4},(0,7))\ ,\ (\Sigma_{2,5,3},(0,7)) \text{ and } (\Sigma_{3,4,3},(0,7))\ .$$

All other orbits $(\Sigma_{m,n,q},(\mathrm{pos},\mathrm{neg}))$ with $m>0$, $n>0$, $q>0$, $m+n+q=10$, are realizable.

The theorem is proved in Section 5.

2. Preliminaries

We list here the non-realizable orbits for $4 \le d \le 8$. In all cases one has $\tilde{\mu} = 0$. For $d = 4$ and $d = 5$, the non-realizable orbits are:

$$(\Sigma_{1,3,1},(0,2)) \text{ and } (\Sigma_{1,4,1},(0,3))\ . \tag{4}$$

For $d = 6$, there are four such orbits:

$$(\Sigma_{1,5,1},(0,2))\ ,\ (\Sigma_{1,5,1},(0,4))\ ,\ (\Sigma_{1,1,1,3,1},(0,2)) \text{ and } (\Sigma_{2,4,1},(0,4))\ . \tag{5}$$

For $d = 7$, there are six non-realizable orbits:

$$\begin{matrix} (\Sigma_{2,5,1},(0,5))\ , & (\Sigma_{2,4,2},(0,5))\ , & (\Sigma_{3,4,1},(0,5))\ , \\ (\Sigma_{1,4,1,1,1},(0,3))\ , & (\Sigma_{1,6,1},(0,3))\ , & (\Sigma_{1,6,1},(0,5))\ . \end{matrix} \tag{6}$$

For $d = 8$, the non-realizable orbits are the following ones:

$$\begin{aligned} &\Sigma_{2,5,2}\ ,\ \Sigma_{1,6,2}\ ,\ \Sigma_{1,4,4}\ ,\ \Sigma_{1,5,3} \text{ and } \Sigma_{2,4,3} \text{ with } (0,6)\ , \\ &\Sigma_{1,1,1,3,1,1,1} \text{ and } \Sigma_{1,3,1,1,1,1,1} \text{ with } (0,2)\ , \\ &\Sigma_{1,5,1,1,1} \text{ and } \Sigma_{1,3,1,3,1} \text{ with } (0,2) \text{ and } (0,4)\ , \\ &\Sigma_{1,7,1} \text{ with } (0,2)\ ,\ (0,4) \text{ and } (0,6)\ , \\ &\Sigma_{1,4,1,2,1}\ ,\ \Sigma_{1,6,2}\ ,\ \Sigma_{1,4,2,1,1}\ ,\ \Sigma_{1,1,1,4,2} \text{ and } \Sigma_{1,4,1,1,2} \text{ with } (0,4)\ . \end{aligned} \tag{7}$$

The following *concatenation lemma* (proved in [7]) is a basic tool for proving the relizability of certain orbits.

Lemma 11. *Suppose that the monic polynomials P_1 and P_2 of degrees d_1 and d_2, with sign patterns represented in the form $(+,\sigma_1)$ and $(+,\sigma_2)$ respectively, realize the pairs $(\mathrm{pos}_1,\mathrm{neg}_1)$ and $(\mathrm{pos}_2,\mathrm{neg}_2)$. Here σ_j denotes what remains of the sign patterns when the initial sign $+$ is deleted. Then*

(1) *if the last position of σ_1 is $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ realizes the sign pattern $(+,\sigma_1,\sigma_2)$ and the admissible pair $(\mathrm{pos}_1+\mathrm{pos}_2,\mathrm{neg}_1+\mathrm{neg}_2)$;*
(2) *if the last position of σ_1 is $-$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ realizes the sign pattern $(+,\sigma_1,-\sigma_2)$ and the pair $(\mathrm{pos}_1+\mathrm{pos}_2,\mathrm{neg}_1+\mathrm{neg}_2)$. Here $-\sigma_2$ is obtained from σ_2 by changing each $+$ by $-$ and vice versa.*

Remark 12. We use the symbol $*$ to denote concatenation of couples or of sign patterns. Lemma 11 implies that when one concatenates the compatible realizable couples $(\Sigma_{m_1,\ldots,m_s},(a,b))$ and $(\Sigma_{n_1,\ldots,n_\ell},(c,d))$ one obtains the realizable couple

$$(\Sigma_{m_1,\ldots,m_{s-1},m_s+n_1-1,n_2,\ldots,n_\ell},(a+c,b+d)) = ((\Sigma_{m_1,\ldots,m_s},(a,b)) * (\Sigma_{n_1,\ldots,n_\ell},(c,d)) .$$

If one considers only sign patterns instead of couples, then one can write

$$\Sigma_{m_1,\ldots,m_{s-1},m_s+n_1-1,n_2,\ldots,n_\ell} = \Sigma_{m_1,\ldots,m_s} * \Sigma_{n_1,\ldots,n_\ell} .$$

When necessary we use more than two consecutive concatenations.

Consider a hyperbolic monic degree d polynomial without vanishing coefficients. Suppose that the moduli of its roots are all distinct. Consider the order of these moduli on the real positive half-axis. We note only at which positions the moduli of its negative roots are; this should be clear from the following example.

Example 13. The polynomial

$$(x-1)(x+2)(x-3)(x-4)(x+5)(x-6)(x+7)(x+8)(x-9)$$

has five positive and four negative roots. We note the relative positions of the moduli of its positive and negative roots by the letters P and N. The order of the moduli of the roots of the polynomial is

$$|1| < |-2| < |3| < |4| < |-5| < |6| < |-7| < |-8| < |9|$$

which we note as

$$P < N < P < P < N < P < N < N < P .$$

Given a sign pattern $(\alpha_d,\alpha_{d-1},\ldots,\alpha_0)$, $\alpha_j = \pm 1$, one can construct a hyperbolic degree d polynomial defining this sign pattern using Lemma 11. At the first step one constructs the linear polynomial $P_+ := x+1$ if $\alpha_d = \alpha_{d-1}$ or $P_- := x-1$ if $\alpha_d = -\alpha_{d-1}$. At each next step one concatenates the previously obtained polynomial (which plays the role of P_1 and which defines the sign pattern $(\alpha_d,\alpha_{d-1},\ldots,\alpha_j)$) and the polynomial P_+ or P_- as P_2 depending on whether $\alpha_j = \alpha_{j-1}$ or $\alpha_j = -\alpha_{j-1}$ respectively.

Hence the modulus of each next root which is added during this construction is much smaller than the moduli of the roots previously obtained. In the end the roots of the constructed polynomial define the *canonical order* corresponding to the given sign pattern: one reads the sign pattern from the right, to each consecutive equal (resp. different) signs of coefficients puts in correspondence the letter N (resp. P) and then inserts between any two consecutive letters the sign $<$. E. g. the sign pattern $(+,-,-,+,-,+,+,+,-)$ defines the canonical order

$$P < N < N < P < P < P < N < P .$$

Thus for every sign pattern, there exists a hyperbolic polynomial the moduli of whose roots define the corresponding canonical order, see [15, Proposition 1].

3. Proof of Theorem 8

Proof.

(A). Suppose that a couple $\mathcal{K}_{n,q}$ is realizable by some polynomial Q. Using if necessary a linear transformation $x \mapsto hx$, $h > 0$, one can assume that one of the roots of Q is at 1. (We remind that Q has three sign changes in the sequence of its coefficients, so by Descartes' rule of signs Q has either 3 or 1 positive roots.) We denote by $-x_i$ the negative roots of Q, $0 < x_1 \le x_2 \le \cdots \le x_{d-3}$, and we set

$$\sum_{j=1}^{d} a_j x^j =: Q := (x^{d-3} + e_1 x^{d-4} + \cdots + e_{d-4}x + e_{d-3})(x^2 - ux + v)(x-1) ,$$

where e_k are the elementary symmetric polynomials of the quantities x_i. Hence $e_k > 0$ and $a_d = 1$. We explicit some of the first and some of the last coefficients a_j:

$$\begin{array}{llrcrcr} a_{d-1} & = & (e_1-1) & - & u\,, & & \\ a_{d-2} & = & (e_2-e_1) & - & (e_1-1)u & + & v\,, \\ a_{d-3} & = & (e_3-e_2) & - & (e_2-e_1)u & + & (e_1-1)v\,, \\ a_{d-4} & = & (e_4-e_3) & - & (e_3-e_2)u & + & (e_2-e_1)v\,, \\ a_3 & = & (e_{d-3}-e_{d-4}) & - & (e_{d-4}-e_{d-5})u & + & (e_{d-5}-e_{d-6})v\,, \\ a_2 & = & -e_{d-3} & - & (e_{d-3}-e_{d-4})u & + & (e_{d-4}-e_{d-5})v\,, \\ a_1 & = & & & e_{d-3}u & + & (e_{d-3}-e_{d-4})v\,, \\ a_0 & = & & & & & -e_{d-3}v\,. \end{array} \tag{8}$$

(B). We introduce some notation:

Notation 14. In the plane of the variables (u, v) we consider the parabola $\mathscr{P} : v = u^2/4$ and the straight lines (a_j) defined by the respective equations of the form $a_j = \cdots$ from the list (8). We set

$$\begin{array}{ll} c_- := 23 - 4\sqrt{30} = 1.09\ldots\,, & c_+ := 23 + 4\sqrt{30} = 44.90\ldots\,, I := (c_-, c_+)\,, \\ E_1 := \sum_{\nu=1}^{d-3} 1/x_\nu\,, & E_2 := \sum_{1\le i<j\le d-3} 1/(x_i x_j)\,. \end{array} \quad \text{(see (11)}$$

Hence $e_{d-4} = e_{d-3} \cdot E_1$ and $e_{d-5} = e_{d-3} \cdot E_2$.

Remark 15. Points above (resp. below) the parabola $\mathscr{P}$ correspond to polynomials $x^2 - ux + v$ having two complex conjugate (resp. two real distinct) roots. Any polynomial from the parabola has a double real root. Polynomials between the parabola and the u-axis have two positive roots if $u > 0$ and two negative roots if $u < 0$. Polynomials below the u-axis have two roots of opposite signs. Our aim is to show that the domain defined by the inequalities of the form $a_j > 0$ or $a_j < 0$ resulting from the sign pattern $\Sigma_{1,n,q,1}$ does not intersect the domain $\{v > u^2/4\}$. This is why in what follows we assume that $v > 0$.

We consider first the case $e_1 > c_-$. As $e_1 > 1$ and $a_{d-1} < 0$, one has $u > 0$ (see (8)).

For $e_1 \in I$, from the definition of $e_1 = x_1 + \cdots + x_{d-3}$ and E_1 follows that

$$\begin{cases} e_1 \cdot E_1 \ge (d-3)^2 \ge 49 \text{ hence } E_1 \ge 49/c_+ > 1 \text{ and} \\ e_{d-4} = e_{d-3} \cdot E_1 > e_{d-3}\,. \end{cases} \tag{9}$$

Lemma 16.

(1) *For $e_1 \in I$, the intersection point S of the straight lines (a_1) and (a_{d-1}) is below the parabola $\mathscr{P}$.*

(2) *Suppose that $e_1 \ge c_+$ and that $x_1 \le 1$. Then the point S is below the parabola $\mathscr{P}$.*

Proof of Lemma 16.

Part (1). One has (see (8))

$$(a_1) \cap (a_{d-1}) \;=\; S := (\, e_1 - 1\,,\, e_{d-3}(e_1-1)/(e_{d-4}-e_{d-3})\,)$$

and $e_{d-4} > e_{d-3}$, see (9). The point S is below the parabola $\mathscr{P}$. This follows from

$$e_{d-3}(e_1-1)/(e_{d-4}-e_{d-3}) < (e_1-1)^2/4$$

which is equivalent to

$$e_{d-3}(e_1+3) < e_{d-4}(e_1-1) \tag{10}$$

or to $e_1+3<(e_1-1)E_1$. However (see (9))

$$(e_1-1)E_1 \geq (e_1-1)(49/e_1) = 49-49/e_1 > e_1+3\ ;$$

the second of these inequalities is equivalent to

$$e_1^2-46e_1+49=(e_1-c_-)(e_1-c_+)<0\ ,\ \text{i.e. to } e_1\in I\ . \tag{11}$$

Part (2). For fixed sum $x_2+\cdots+x_{d-3}$, the sum $\sum_{\nu=2}^{d-3}1/x_\nu$ is minimal if $x_2=\cdots=x_{d-3}$. Hence for $x_1\leq 1$, one has

$$E_1 \geq 1+(d-4)/(e_1/(d-4)) = 1+(d-4)^2/e_1 \geq 1+36/e_1\ ,$$

so again $e_{d-4}>e_{d-3}$. Inequality (10) can be given the equivalent form

$$1+4/(e_1-1) < e_{d-4}/e_{d-3} = E_1\ .$$

However the inequalities $E_1\geq 1+36/e_1 > 1+4/(e_1-1)$ hold true for $e_1\geq c_+$ from which part (2) of the lemma follows. □

There exist no couples $\mathcal{K}_{n,q}$ satisfying the assumptions of Lemma 16 since the lemma implies that the domain defined by the inequalities

$$v>u^2/4,\quad a_1=e_{d-3}u+(e_{d-3}-e_{d-4})v>0 \quad\text{and}\quad a_{d-1}=e_1-1-u<0\ ,\ \text{i.e. } u>e_1-1\ ,$$

is void. Indeed, the straight line (a_1) has a positive slope, see (9)); it intersects the parabola $\mathcal{P}$ at the origin and at a point with $u>0$. Hence the whole sector $\{a_1>0,\ u>e_1-1\}$ is below the parabola $\mathcal{P}$.

(C). Suppose that $x_\nu\geq 1$, $1\leq\nu\leq d-3$, so $e_1>c_-$. We consider the intersection point

$$T\ :=\ (a_{d-4})\cap(a_{d-1})\ =(\ e_1-1\ ,\ ((e_3-e_2)(e_1-1)-(e_4-e_3))/(e_2-e_1)\ )\ .$$

Observe first that for $d\geq 10$ and $x_\nu\geq 1$, one has $e_2-e_1>0$ and $e_3-e_2>0$. We show that the point T is below the parabola $\mathcal{P}$. The straight line (a_{d-4}) has a positive slope. Hence the domain defined by the three inequalities

$$v>u^2/4,\quad a_{d-1}=e_1-1-u<0 \quad\text{and}\quad a_{d-4}<0,\ \text{i.e. } v<((e_3-e_2)u-(e_4-e_3))/(e_2-e_1),$$

is void, so there exist no couples $\mathcal{K}_{n,q}$ with $e_1\geq c_+$ and $x_\nu\geq 1$.

The point T is under the parabola $\mathcal{P}$ exactly when

$$((e_3-e_2)(e_1-1)-(e_4-e_3))/(e_2-e_1) < (e_1-1)^2/4\ ,$$

i.e.

$$\Psi := 4(e_3-e_2)(e_1-1)-4(e_4-e_3)-(e_1-1)^2(e_2-e_1)<0\ .$$

We consider the quantity Ψ as a function of one of the variables x_ν (say, x_1) when the other variables x_ν are fixed. We denote here $\mathrm{d}e_j/\mathrm{d}x_1$ by e_j'. Clearly $e_j'=f_{j-1}$, where f_{j-1} is the $(j-1)$st elementary symmetric polynomial of the quantities $x_2,\ldots,x_{d-3}$; we set $e_0=f_0=1$. Thus $e_j=f_j+x_1f_{j-1}$,

$$\Psi = e_1^3-e_1^2e_2-2e_1^2-2e_1e_2+4e_1e_3+e_1+3e_2-4e_4 \quad\text{and}$$
$$\Psi' = 3e_1^2-2e_1e_2-e_1^2f_1-4e_1-2f_1e_1-2e_2+4f_2e_1+4e_3+1+3f_1-4f_3\ .$$

Substituting $f_j+x_1f_{j-1}$ for e_j in Ψ' gives

$$\begin{aligned}\Psi' &= -(f_1+3x_1-1)(f_1^2+x_1f_1-2f_2-x_1+1)\\ &= -(f_1+3x_1-1)(x_1(f_1-1)+f_1^2-2f_2+1)<0\ ,\end{aligned}$$

because $f_1 > 1$ and $f_2 < f_1^2/2$. Thus $\Psi' < 0$. Hence if one considers Ψ as a function in all the variables x_j, one finds that $\partial\Psi/\partial x_j < 0$. Hence Ψ takes its maximal value for $x_1 = \cdots = x_{d-3} = 1$. In this case $e_j = \binom{d-3}{j}$ and

$$\Psi = -(d-2)(d-3)(d-4)/2 < 0 \, .$$

Thus for $x_\nu \geq 1$, one has $\Psi < 0$, $e_1 > c_-$ and the point T is below the parabola $\mathscr{P}$.

(D). Up to now we showed that there are no couples $\mathscr{K}_{n,q}$ for $e_1 > c_-$. For $e_1 \in I$, this was deduced from part (2) of Lemma 16; for $e_1 \geq c_+$, this follows from part (2) of Lemma 16 (if $x_1 \leq 1$) and from (C) (if $x_1 > 1$). Suppose now that $0 < e_1 \leq c_-$. The involution i_r (see (3)) transforms the polynomial Q into a polynomial with sign pattern $\Sigma_{1,q,n,1}$ and with $e_1 > c_-$; the factor $x-1$ is preserved and each of the other two factors of Q is replaced by a factor of the same form. For $e_1 \leq c_-$, at least $d-4$ of the quantities x_ν are < 1, so after applying the involution i_r they become $1/x_\nu > 1$ and e_1 becomes larger than $d-4 \geq 6$. Thus the proof of Theorem 8 for $0 < e_1 \leq c_-$ results directly from its proof for $e_1 > c_-$. □

4. Proof of Theorem 9

(A). The proof of Theorem 9 is organised as follows. The proof of part (1) is given for each degree from 10 to 14 in (B) – (F) respectively. Parts (B) and (C) of the proof are subdivided into (B1), (C1) (in which we prove part (1) of the theorem for $d = 10$ and $d = 11$) and (B2), (C2) containing the proofs of parts (2) and (3) of the theorem. Using the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action (see Definition 3) one can assume that pos ≤ neg. We remind that:

(i) For $d \leq 8$, there are no non-realizable orbits with $\widetilde{\mu} \geq 1$ (see [1, 7] and [13]).

(ii) For $d = 9$, the only non-realizable orbit with $\widetilde{\mu} \geq 1$ is (see [4])

$$(\Sigma_{1,4,4,1}, (1,6)) \, , \text{ with } \widetilde{\mu} = 1 \text{ and } \widetilde{\lambda} = 1 \, . \tag{12}$$

(iii) If the admissible pair of a given orbit is $(0,0)$, $(0,1)$, $(1,0)$ or $(1,1)$, then the orbit is realizable. Indeed, if the admissible pair equals $(0,0)$ or $(0,1)$ (resp. $(1,0)$), then one chooses a polynomial with the given sign pattern σ and adds to it a sufficiently large positive (resp. negative) constant. If the admissible pair is $(1,1)$, then one represents the sign pattern in the form $\sigma = (\sigma_\dagger, \alpha, \beta)$, where α and β are its last two signs of σ. If $\alpha = \beta$ (resp. $\alpha = -\beta$), then one uses the concatenation $(\sigma_\dagger, \alpha), (1,0)) * (\Sigma_2, (0,1))$ (resp. $(\sigma_\dagger, \alpha), (0,1)) * (\Sigma_{1,1}, (1,0)))$.

(B). Suppose that $d = 10$.

(B1). Consider any compatible couple $\mathscr{K}_\flat := (\sigma_\flat, (\text{pos}, \text{neg}))$ with $2 \leq \text{pos} \leq \text{neg}$. Represent the sign pattern $\sigma_\flat$ as above in the form $(\sigma_\dagger, \alpha, \beta)$.

If $\alpha \neq \beta$ and the couple $\mathscr{K}_\triangle := ((\sigma_\dagger, \alpha), (\text{pos}-1, \text{neg}))$ is realizable, then the couple $\mathscr{K}_\flat$ is also realizable as $\mathscr{K}_\triangle * (\Sigma_{1,1}, (1,0))$.

If $\alpha = \beta$ and the couple $\mathscr{K}_\diamond := ((\sigma_\dagger, \alpha), (\text{pos}, \text{neg}-1))$ is realizable, then one can realize $\mathscr{K}_\flat$ as $\mathscr{K}_\diamond * (\Sigma_2, (0,1))$. However the couple $\mathscr{K}_\diamond$ is realizable. Indeed, it is with $d = 9$ and either with $\widetilde{\mu} = 1$ when pos = neg = 2 (and there exist no such non-realizable couples) or with $\widetilde{\mu} \geq 2$ when neg ≥ 3, hence again realizable. So the only situation in which one does not know whether the couple $\mathscr{K}_\flat$ is realizable or not is when

$$\alpha \neq \beta \text{ and } \mathscr{K}_\triangle = \mathscr{K}_{4,4} = (\Sigma_{1,4,4,1}, (1,6)) \, .$$

This means that $\mathscr{K}_\flat = (\Sigma_{1,4,4,1,1}, (2,6))$ which is realizable as $(\Sigma_{1,4,4}, (2,6)) * (\Sigma_{1,1,1}, (0,0))$. Hence all couples with $d = 10$ and $\widetilde{\mu} \geq 2$ are realizable.

(B2). We need some more notation:

Notation 17. We denote by $\mathcal{K}_\bullet$ a couple with $\tilde{\mu} = 1$ and by $\sigma_\bullet := (\alpha_d, \ldots, \alpha_0)$ its sign pattern, where $\alpha_j = +$ or $-$, $\alpha_d = +$. We set $\sigma^s := \alpha_s(\alpha_s, \ldots, \alpha_0)$ and $\sigma_s := (\alpha_d, \ldots, \alpha_s)$. We discuss the possibility to realize $\mathcal{K}_\bullet$ as $\mathcal{K}_s * \mathcal{K}^s$, where the couples $\mathcal{K}_s$, $\mathcal{K}^s$ correspond to degree $d-s$ and s polynomials with sign patterns σ_s and σ^s.

Remark 18. If $\alpha_s = +$, then the admissible pair of $\mathcal{K}_s$ (resp. of $\mathcal{K}^s$) is of the form $(0,.)$ (resp. $(1,.)$). If $\alpha_s = -$, then these admissible pairs are of the form $(1,.)$ and $(0,.)$ respectively. We remind that for $d \leq 8$, the admissible pairs of the form $(1,.)$ are realizable, see the beginning of Section 2 and part (A) of this proof.

Suppose first that $s = 5$ and $\alpha_5 = +$. Then the couple $\mathcal{K}^5$ is realizable. If $\mathcal{K}_5$ is realizable, then such is $\mathcal{K}_\bullet$ as well. The only possibility $\mathcal{K}_5$ not to be realizable is when $\mathcal{K}_5 = (\Sigma_{1,4,1}, (0,3))$, see (4). So we assume that $\sigma_\bullet = (+,-,-,-,-,+,?,?,?,?,-)$. If $\alpha_4 = -$, then the couple $\mathcal{K}_4$ is realizable and $\mathcal{K}^4$ is not realizable only when $\mathcal{K}^4 = (\Sigma_{1,3,1}, (0,2))$. This means that $\sigma_\bullet = \Sigma_{1,4,1,1,3,1}$, with Descartes' pair $(5,5)$. This sign pattern is realizable with the admissible pairs $(1,a)$, $a = 3$ or 5:

$$(\Sigma_{1,4,1,1,3,1}, (1,a)) = ((\Sigma_{1,3}, (1, a-3)) \ * \ (\Sigma_{2,1,1,3,1}, (0,3)) \ . \tag{13}$$

If $\alpha_5 = \alpha_4 = +$, then $\mathcal{K}^4$ is realizable whereas the couple $\mathcal{K}_4$ is not realizable only when it corresponds to one of the four cases listed in (5). In each of them one has to take into account the involution i_r (see (3)), but not the involution i_m, because the latter makes the first component of the admissible pair larger than 1. Hence $\sigma_4 = \Sigma_{1,4,2}$. Then we consider the couples $\mathcal{K}_3$ and $\mathcal{K}^3$. The latter is always realizable, so we assume that this is not the case of $\mathcal{K}_3$. Hence σ_3 is obtained from σ_4 by adding a sign $+$ to the right, i.e. $\sigma_3 = \Sigma_{1,4,3}$.

If $\sigma_2 = \Sigma_{1,4,3,1}$, then both $\mathcal{K}_2$ and $\mathcal{K}^2$ are realizable, so we need to consider only the possibility $\sigma_2 = \Sigma_{1,4,4}$. Thus in the end $\sigma_\bullet = \Sigma_{1,4,5,1}$ or $\Sigma_{1,4,4,2}$. These two sign patterns are realizable with the admissible pairs $(1,a)$, $a = 3$ and 5:

$$(\Sigma_{1,4,5,1}, (1,a)) = (\Sigma_{1,4,1}, (0,1)) \ * \ (\Sigma_{5,1}, (1, a-1)) \ ,$$
$$(\Sigma_{1,4,4,2}, (1,a)) = (\Sigma_{1,4,1}, (0,1)) \ * \ (\Sigma_{4,2}, (1, a-1)) \ .$$

The first of them is not realizable with the admissible pair $(1,7)$ (see Theorem 8), for the second one the answer is not known.

Suppose now that $\alpha_5 = -$. Then $\mathcal{K}_5$ is realizable while $\mathcal{K}^5$ might not be only if $\sigma^5 = \Sigma_{1,4,1}$. In this case we consider $\mathcal{K}_6$ and $\mathcal{K}^6$. For $\alpha_6 = +$, the couple $\mathcal{K}^6$ is realizable whereas $\mathcal{K}_6$ could be non-realizable, if $\sigma_6 = \Sigma_{1,3,1}$ hence $\sigma_\bullet = \Sigma_{1,3,1,1,4,1}$. This is the case (13) to which one has applied the involution i_r, so it is realizable. For $\alpha_5 = \alpha_6 = -$, one is, up to the involution i_r, in the case $\alpha_{10} = \alpha_5 = \alpha_4 = +$, $\alpha_0 = -$, which was already considered.

(C). Suppose that $d = 11$.

(C1). We use the same notation as the one used in part (B1) of this proof; in particular, we denote by $\mathcal{K}_\flat := (\sigma_\flat, (\text{pos}, \text{neg}))$ a compatible couple with $\tilde{\mu} \geq 2$, where $\sigma_\flat = (\sigma_\dagger, \alpha, \beta)$. As in part (B1) of this proof we show that the only cases in which the couple $\mathcal{K}_\flat$ is possibly non-realizable are when $\alpha \neq \beta$ and

$$\mathcal{K}_\triangle = ((\sigma_\dagger, \alpha), (\text{pos}-1, \text{neg})) = (\Sigma_{1,4,5,1}, (1,7)) = \mathcal{K}_{4,5} \ \text{or} \ (\Sigma_{1,4,4,2}, (1,7)) \ ,$$

that is, $\mathcal{K}_\triangle$ corresponds to one of the two cases mentioned in part (2) of the present theorem. In these cases $\mathcal{K}_\flat$ is realizable, because it equals respectively

$$(\Sigma_{1,4,5,1,1}, (2,7)) = (\Sigma_{1,4,1}, (2,3)) * (\Sigma_{5,1,1}, (0,4)) \quad \text{and}$$
$$(\Sigma_{1,4,4,2,1}, (2,7)) = (\Sigma_{1,4,1}, (2,3)) * (\Sigma_{4,2,1}, (0,4)) \ .$$

(C2). We use the notation and method of proof as developed in part (B2) of this proof. We are looking for non-realizable couples $\mathcal{K}_{\bullet}$. Suppose first that

1. $\alpha_5 = +$. Then the couple $\mathcal{K}^5$ is realizable and the couple $\mathcal{K}_5$ might not be realizable only if σ_5 corresponds to one of the cases (5) up to the involution i_r. This means that σ_5 is among the sign patterns

$$\Sigma_{1,5,1}\ ,\ \Sigma_{1,1,1,3,1}\ ,\ \Sigma_{1,3,1,1,1}\ ,\ \Sigma_{2,4,1} \text{ and } \Sigma_{1,4,2}\ .$$

1.1. If $\alpha_4 = +$, then $\mathcal{K}^4$ is realizable and σ_4 is one of the sign patterns

$$\Sigma_{1,5,2}\ ,\ \Sigma_{1,1,1,3,2}\ ,\ \Sigma_{1,3,1,1,2}\ ,\ \Sigma_{2,4,2} \text{ and } \Sigma_{1,4,3}\ .$$

1.1.1. The second and third sign patterns of this list do not correspond to non-realizable cases, see (6).

1.1.2. In the cases $\sigma_4 = \Sigma_{1,5,2}$ and $\sigma_4 = \Sigma_{2,4,2}$ the first component of the sign pattern σ^8 must be 3 and the second must be ≥ 2; the sign pattern σ_8 corresponds to realizable couples. Then one concludes from (7) that $\sigma_{\bullet}$ is one of the sign patterns

$$\Sigma_{1,5,5,1}\ ,\ \Sigma_{1,5,4,2}\ ,\ \Sigma_{2,4,5,1} \text{ and } \Sigma_{2,4,4,2}\ .$$

The first of them is not realizable with the admissible pair $(1,8)$, see Theorem 8; for the second, third and fourth one the answer to this question is not known. (The second and third are in one and the same orbit.) We show the realizability of the four sign patterns with the other admissible pairs $(1,a)$, $a = 6$, 4 or 2:

$$\begin{aligned}
(\Sigma_{1,5,5,1},(1,a)) &= (\Sigma_{1,5,3},(0,a-2))\ *\ (\Sigma_{3,1},(1,2))\\
(\Sigma_{1,5,4,2},(1,a)) &= (\Sigma_{1,5,3},(0,a-2))\ *\ (\Sigma_{2,2},(1,2))\\
(\Sigma_{2,4,4,2},(1,a)) &= (\Sigma_{2,4,3},(0,a-2))\ *\ (\Sigma_{2,2},(1,2))\ .
\end{aligned}$$

1.1.3. In the case $\sigma_4 = \Sigma_{1,4,3}$ the first component of σ^8 equals 2 and the second is ≥ 3. Using the list (7) one obtains the following possibilities for $\sigma_{\bullet}$:

$$\Sigma_{1,4,5,2}\ ,\ \Sigma_{1,4,6,1}\ ,\ \Sigma_{1,4,4,3} \text{ and } \Sigma_{1,4,4,1,1,1}\ .$$

The second of them is not realizable with the admissible pair $(1,8)$ (see Theorem 8). The third sign pattern is realizable with the admissible pair $(1,8)$. Indeed, by [5, Theorem 3, part (iii)], the couple $(\Sigma_{4,4,3},(0,8))$ is realizable, so one can set

$$(\Sigma_{1,4,4,3},(1,8)) = ((+,-),(1,0)) * (\Sigma_{4,4,3},(0,8))\ .$$

The realizability of other possible cases is given below; for the cases which are not covered by this list the answer is not known.

$$\begin{aligned}
(\Sigma_{1,4,5,2},(1,a)) &= (\Sigma_{1,4,4},(0,a-2))\ *\ (\Sigma_{2,2},(1,2))\ , && a = 6,\ 4 \text{ or } 2\ ;\\
(\Sigma_{1,4,6,1},(1,a)) &= (\Sigma_{1,4,4},(0,a-2))\ *\ (\Sigma_{3,1},(1,2))\ , && a = 6,\ 4 \text{ or } 2\ ;\\
(\Sigma_{1,4,4,3},(1,a)) &= (\Sigma_{1,4,4},(0,a-2))\ *\ (\Sigma_{1,3},(1,2))\ , && a = 6,\ 4 \text{ or } 2\ ;\\
(\Sigma_{1,4,4,1,1,1},(1,a)) &= (\Sigma_{1,4,4},(0,a-2))\ *\ (\Sigma_{1,1,1,1},(1,0))\ , && a = 4 \text{ or } 2\ .
\end{aligned}$$

1.2. If now $\alpha_4 = -$, the couple $\mathcal{K}_4$ is realizable, so one has to treat only the situation $\sigma^4 = \Sigma_{1,3,1}$ in which $\mathcal{K}^4$ is not realizable with the admissible pair $(0,2)$. This, combined with the list (5), gives the following sign patterns $\sigma_{\bullet}$:

$$\Sigma_{1,5,1,1,3,1}\ ,\ \Sigma_{1,1,1,3,1,1,3,1}\ ,\ \Sigma_{1,3,1,1,1,1,3,1}\ ,\ \Sigma_{2,4,1,1,3,1} \text{ and } \Sigma_{1,4,2,1,3,1}\ .$$

1.2.1. The first, fourth and fifth sign patterns are realizable with the admissible pairs $(1,6)$, $(1,4)$ and $(1,2)$:

$$(\Sigma_{1,5,1,1,3,1},(1,a+b)) = (\Sigma_{1,4},(1,a)) * (\Sigma_{2,1,1,3,1},(0,b))\ ,\ a,b = 1 \text{ or } 3\ ,$$
$$(\Sigma_{2,4,1,1,3,1},(1,a+b)) = (\Sigma_{2,3},(1,a)) * \Sigma_{2,1,1,3,1},(0,b))\ ,\ a,b = 1 \text{ or } 3\ ,$$
$$(\Sigma_{1,4,2,1,3,1},(1,a+b)) = (\Sigma_{1,4},(1,a)) * (\Sigma_{1,2,1,3,1},(0,b))\ ,\ a,b = 1 \text{ or } 3\ .$$

1.2.2. The second (resp. the third) sign patterns is realizable with the admissible pairs $(1,4)$ and $(1,2)$ (resp. $(1,2)$); for the third sign pattern and for the admissible pair $(1,4)$, the answer is not known:

$$(\Sigma_{1,1,1,3,1,1,3,1},(1,4)) = (\Sigma_{1,1,1},(0,0))\ *\ (\Sigma_{1,3,1,1,3,1},(1,4))\ ,$$
$$(\Sigma_{1,1,1,3,1,1,3,1},(1,2)) = (\Sigma_{1,1,1,3,1,1,1,1},(0,0))\ *\ (\Sigma_{3,1},(1,2))\ ,$$
$$(\Sigma_{1,3,1,1,1,1,3,1},(1,2)) = (\Sigma_{1,3,1,1,1,1,1,1},(0,0))\ *\ (\Sigma_{3,1},(1,2))\ .$$

2. $\alpha_5 = -$. The couple $\mathcal{K}_5$ is realizable while $\mathcal{K}^5$ is not only if $\sigma^5 = \Sigma_{1,4,1}$.

2.1. If $\sigma^5 = \Sigma_{1,4,1}$ and $\alpha_6 = +$, then $\mathcal{K}^6$ is realizable whereas $\mathcal{K}_6$ is not realizable only if $\sigma_6 = \Sigma_{1,4,1}$, so $\sigma_\bullet = \Sigma_{1,4,1,1,4,1}$. This sign pattern is realizable with the admissible pairs $(1,4)$ and $(1,2)$ (for $(1,6)$, the answer remains unknown):

$$(\Sigma_{1,4,1,1,4,1},(1,4) \text{ or } (1,2)) = (\Sigma_{1,3},(1,2))\ *\ (\Sigma_{2,1,1,4,1},(0,2) \text{ or } (0,0))\ .$$

2.2. If $\alpha_5 = -$, $\sigma^5 = \Sigma_{1,4,1}$ and $\alpha_6 = -$, i.e. $\sigma^6 = \Sigma_{2,4,1}$, then applying the involution i_r (see (3)) one transforms this case into $\sigma_5 = \Sigma_{1,4,2}$, $\alpha_0 = -$. This case was treated in 1.

(D). Suppose that $d = 12$. One can try to represent a given couple $\mathcal{K}_\bullet$ with $\widetilde{\mu} \geq 2$ as a concatenation of couples $\mathcal{K}'$ and $\mathcal{K}''$ of degree 10 and 2 respectively. If $\mathcal{K}'$ is realizable, then such is $\mathcal{K}_\bullet$ as well, because all compatible couples of degree 2 are realizable. So we assume that $\mathcal{K}'$ is not realizable. Then one can assume that $\mathcal{K}'$ is one of the two couples of part (2) of the theorem. Hence the sign pattern of $\mathcal{K}'$ is among the following ones:

$$\Sigma_{1,4,5,3}\ ,\ \Sigma_{1,4,5,2,1}\ ,\ \Sigma_{1,4,5,1,2}\ ,\ \Sigma_{1,4,5,1,1,1}\ ,$$
$$\Sigma_{1,4,4,4}\ ,\ \Sigma_{1,4,4,3,1}\ ,\ \Sigma_{1,4,4,2,2}\ ,\ \Sigma_{1,4,4,2,1,1}\ .$$

Represent these sequences of 4, 5 or 6 numbers in the forms $(1,A)$ and $(1,4,B)$, where A is the sequence of the last 3, 4 or 5 of them, and B of the last 2, 3 or 4 respectively. If $3 \leq \text{pos} \leq \text{neg}$, then

$$\mathcal{K}_\bullet = (\Sigma_{1,1},(1,0))\ *\ \mathcal{K}^\sharp\ ,\ \text{ where } \mathcal{K}^\sharp = (\Sigma_{1,A},(\text{pos}-1,\text{neg}))\ . \tag{14}$$

The couple $\mathcal{K}^\sharp$ is with $d = 11$ and $\widetilde{\mu} \geq 2$ hence realizable. Hence $\mathcal{K}_\bullet$ is also realizable.

Suppose that $2 = \text{pos} \leq \text{neg}$. Then one can write

$$\mathcal{K}_\bullet = (\Sigma_{1,4},(1,3) \text{ or } (1,1))\ *\ (\Sigma_{1,B},(a,b))\ , \tag{15}$$

where the admissible pair of $\mathcal{K}_\bullet$ is $(a+1,b+3)$ if $\text{neg} \geq 4$ or $(a+1,b+1)$ if $\text{neg} = 2$ or 3. Each couple $(\Sigma_{1,4},(1,3))$, $(\Sigma_{1,4},(1,1))$ and $(\Sigma_{1,B},(a,b))$ is realizable (for the latter one has $d = 8$ and $\widetilde{\mu} \geq 1$), so $\mathcal{K}_\bullet$ is realizable.

(E). Suppose that $d = 13$ or $d = 14$. (For $d = 14$, we assume that the theorem is proved for $d = 13$.) Similarly to part (D), one can try to represent a given couple $\mathcal{K}_\bullet$ with $\widetilde{\mu} \geq 2$ as a concatenation of couples $\mathcal{K}'$ and $\mathcal{K}''$ of degree 10 and 3 respectively (or 10 and 4 if $d = 14$). As in part (D), we assume that $\mathcal{K}'$ is one of the couples of part (2) of the theorem. We denote the sign pattern of $\mathcal{K}_\bullet$ by $(1,A)$ and $(1,4,B)$ with similar definition of A and B.

For $3 \leq \text{pos} \leq \text{neg}$, we use formula (14). The couple $\mathcal{K}^\sharp$ is with $d = 12$ (resp. $d = 13$) and $\widetilde{\mu} \geq 2$ hence realizable. For $2 = \text{pos} \leq \text{neg}$, we can use formula (15) with the reasoning after it except in the case when $(\Sigma_{1,B},(a,b))$ is of the orbit of the couple (12) (resp. of one of the orbits of the couples of part (2) of the theorem). The first 8 signs of the sign pattern of $\mathcal{K}_\bullet$ are $(+,-,-,-,-,+,+,+)$. Further we treat separately the cases $d = 13$ and $d = 14$.

Suppose that $d = 13$. These first 8 signs imply that from the orbit of the sign pattern $\Sigma_{1,4,4,1} = i_r(\Sigma_{1,4,4,1})$ one has to choose $\Sigma_{1,4,4,1}$ (and not $\Sigma_{2,1,1,2,1,1,2} = i_m(\Sigma_{1,4,4,1}) = i_r(\Sigma_{2,1,1,2,1,1,2})$) to be equal to $\Sigma_{1,B}$. Thus the sign pattern of $\mathcal{K}_\bullet$ is $\Sigma_{1,4,4,4,1}$. The admissible pair equals $(2,\nu)$, $\nu = 9$, 7, 5 or 3. We set

$$(\Sigma_{1,4,4,4,1},(2,\nu)) = (\Sigma_{1,2},(1,1)) \ * \ \mathcal{K}^* \ * \ (\Sigma_{1,1},(1,0)) \ , \ \mathcal{K}^* := (\Sigma_{3,4,4},(0,\nu-1))$$

The first and the third couple in this concatenation are realizable. The couple $\mathcal{K}^*$ is realizable for $\nu = 9$, see [5, Theorem 4]. Hence there exists a real polynomial $Q_\natural$ with sign pattern $\Sigma_{3,4,4}$ and having 8 negative distinct roots and a complex conjugate pair. One can perturb the coefficients of $Q_\natural$ without changing its sign pattern so that all its critical values become distinct. Hence for suitable positive values of t one obtains polynomials $Q_\natural + t$ having the sign pattern $\Sigma_{3,4,4}$ and with exactly 6, 4 or 2 distinct negative roots and, respectively, 2, 3 or 4 conjugate pairs. Thus the couple $\mathcal{K}^*$ (and $(\Sigma_{1,4,4,4,1},(2,\nu))$ as well) is realizable for $\nu = 9$, 7, 5 or 3.

(F). Suppose that $d = 14$. The first 8 signs of the sign pattern of $\mathcal{K}_\bullet$ (see (E)) mean that the sign pattern $\Sigma_{1,B}$ is in one of the orbits of the sign patterns $\Sigma_{1,4,5,1}$ or $\Sigma_{1,4,4,2}$, see part (2) of the theorem; the first component of B must be ≥ 3. Hence $\Sigma_{1,B}$ is among the following sign patterns:

$$\Sigma_{1,4,5,1} \ , \ \Sigma_{1,5,4,1} \text{ or } \Sigma_{1,4,4,2} \ .$$

For $\Sigma_{1,B} = \Sigma_{1,4,4,2}$, one has

$$\mathcal{K}_\bullet = (\Sigma_{1,4,4,4,2},(2,\nu+1)) = (\Sigma_{1,4,4,4,1},(2,\nu)) \ * \ (\Sigma_2,(0,1)) \ ,$$

where $\nu = 9$, 7, 5 or 3 and the first concatenation factor is realizable (see (E)). Hence $\mathcal{K}_\bullet$ is also realizable. For $\Sigma_{1,B} = \Sigma_{1,4,5,1}$, one has

$$\mathcal{K}_\bullet = (\Sigma_{1,4,4,5,1},(2,\nu+1)) = (\Sigma_{1,2},(1,1)) \ * \ \mathcal{K}^* \ * \ (\Sigma_{2,1},(1,1))$$

with $\mathcal{K}^*$ as in (E), so this case is also realizable. For $\Sigma_{1,B} = \Sigma_{1,4,5,1}$, the sign pattern of $\mathcal{K}_\bullet$ equals $\Sigma_{1,4,5,4,1}$. It is realizable with the admissible pairs $(2,\rho)$, $\rho = 2$, 4, 6 or 8:

$$(\Sigma_{1,4,5,4,1},(2,\rho)) = (\Sigma_{1,4,1},(0,1)) \ * \ (\Sigma_{5,4,1},(2,\rho-1)) \ .$$

For $\rho = 10$, the answer to Problem 2 remains unknown.

5. Proof of Theorem 10

(A). We consider first the (non)-realizable cases with the admissible pairs $(0,1)$ and $(0,3)$. Every couple $(\Sigma_{m,n,q},(0,1))$ with $m+n+q = 2\ell$, $\ell \geq 2$, $m > 0$, $n > 0$, $q > 0$, is realizable – it suffices to choose a polynomial with sign pattern $\Sigma_{m,n,q}$ and to add to it a large positive constant. Further we suppose that $q \geq m$, otherwise one considers the couple from the same orbit $(\Sigma_{q,n,m},(0,3))$ using the involution i_r.

On the other hand any couple of the form $(\Sigma_{d+1},(0,k))$ is realizable ([5, Example 1]). This implies that for $d = 9$, any couple with $q \geq 3$ and admissible pair $(0,3)$ is realizable as

$$(\Sigma_{m,n,q},(0,3)) = (\Sigma_{m,n,q-2},(0,1)) * (\Sigma_3,(0,2)) \ .$$

The couple $(\Sigma_{2,6,2},(0,3))$ is realizable as

$$(\Sigma_{2,6,2},(0,3)) = (\Sigma_{2,6,1},(0,2)) * (\Sigma_2,(0,1)) \ ,$$

where the first factor is from the same orbit as

$$(\Sigma_{1,6,2},(0,2)) = (\Sigma_{1,6,1},(0,1)) * (\Sigma_2,(0,1))$$

hence this orbit is realizable. We prove realizability of the orbit $(\Sigma_{1,7,2},(0,3))$ by direct construction. We set

$$G_0 := x(x^2-1)^2(x^4+2x^2+1) = x^9-2x^5+x\ ,$$

and then $G_1 := G_0 - 0.0001(x+1)^8 + 0.2$ which has three negative roots $-1.09\ldots$, $-0.84\ldots$ and $-0.20\ldots$ and three complex conjugate pairs of roots. The sign pattern defined by G_1 is $\Sigma_{1,7,2}$, because

$$G_1 = x^9 - 0.0001x^8 - 0.0008x^7 - 0.0028x^6 - 2.0056x^5 - 0.0070x^4 \\ - 0.0056x^3 - 0.0028x^2 + 0.9992x + 0.1999\ .$$

Finally, the orbit $(\Sigma_{1,8,1},(0,3))$ is not realizable, see [16, part (i) of Theorem 4] (one has to apply the involution i_m to the series of non-realizable examples described there).

(B). Suppose that the admissible pair is $(0,5)$. If $q \geq 5$, then one realizes the couple by the concatenation

$$(\Sigma_{m,n,q-4},(0,1)) * (\Sigma_5,(0,4))\ .$$

Suppose that either $q=4$ or $q=3$ and $m=2$ or 3. Then by [7, Theorem 9] the corresponding couple $(\Sigma_{m,n,q-2},(0,3))$ is realizable and one sets

$$(\Sigma_{m,n,q},(0,5)) = (\Sigma_{m,n,q-2},(0,3)) * (\Sigma_3,(0,2))\ .$$

The couple $(\Sigma_{1,8,1},(0,5))$ is not realizable (see [16, part (i) of Theorem 4]). The following proposition settles the remaining cases:

Proposition 19. *The couples* $(\Sigma_{1,7,2},(0,5))$, $(\Sigma_{1,6,3},(0,5))$ *and* $(\Sigma_{2,6,2},(0,5))$ *are not realizable.*

The proposition is proved in Section 6.

(C). Suppose that $d=9$ and the admissible pair is $(0,7)$. The couple $(\Sigma_{1,8,1},(0,7))$ is not realizable, see [16, part (i) of Theorem 4]. It follows from [5, Theorem 3] that if $n=1$, 2 or 3, then such a sign pattern is realizable with the admissible pair $(0,7)$. The same theorem implies that for $m=1$, $n\geq 4$, the couple $(\Sigma_{m,n,q},(0,7))$ is not realizable. Proposition 1 in [5] says that the couples $(\Sigma_{3,4,3},(0,7))$ and $(\Sigma_{2,4,4},(0,7))$ are not realizable. The remaining couples $(\Sigma_{2,5,3},(0,7))$ and $(\Sigma_{2,6,2},(0,7))$ are not realizable by [7, Proposition 6].

6. Proof of Proposition19

Definition 20.

(1) *A* generalized sign pattern *is a string of signs* $+$, $-$ *and/or* 0 *beginning with a* $+$. *A real polynomial* P *with positive leading coefficient is said to define a given generalized sign pattern* σ *if the components of* σ *are equal to the signs of the corresponding coefficients of* P. *Given a sign pattern* σ_1 *and a generalized sign pattern* σ_2 *of the same length,* σ_2 *is called* adjacent *to* σ_1 *if it is obtained from* σ_1 *by replacing some of its components (excluding the initial* $+$*) by zeros. The* closure *of a given sign pattern is the set containing the sign pattern and all generalized sign patterns adjacent to it.*

(2) *We call* simultaneous shift *a map* $\tau_\pm : x \mapsto x \pm \varepsilon$, *where* $\varepsilon > 0$. *We usually consider* ε *to be sufficiently small.*

Lemma 21. *Given a real monic polynomial* $W := \sum_{j=0}^{d} a_j x^j$ *with at least one vanishing coefficient, there exists a simultaneous shift* $\tau_\pm$ *after which all coefficients of* W *are non-zero and the initially non-zero coefficients keep their signs. Moreover, if* $a_k = 0 \neq a_{k+1}$, *then one can choose the sign* $+$ *or* $-$ *in the definition of* $\tau_\pm$ *so that after the shift the sign of the new coefficient* a_k *be the desired one* ($+$ *or* $-$).

Proof. After the shift all coefficients of W become non-constant polynomials in ε, so with the exception of finitely-many values of ε they are all non-zero. After the shift the coefficient a_k becomes $a_k \pm (k+1)a_{k+1}\varepsilon + o(\varepsilon)$ from which the last statement of the lemma follows. □

Lemma 22. *For $d = 7$, there exists no monic polynomial $P := \sum_{j=0}^{7} a_j x^j$ satisfying simultaneously the following conditions:*

(1) *one has $P(0) > 0$ and P defines either one of the sign patterns $\Sigma_{1,6,1}$ or $\Sigma_{1,5,2}$ or a generalized sign pattern adjacent to one of them;*
(2) *P has 5 negative roots counted with multiplicity;*
(3) *P has either a complex conjugate pair of roots or a double positive root.*

Proof. Suppose first that P has a complex conjugate pair of roots. Denote by $-\eta_j$ the negative roots of P. Set $P_1 := x(x+\eta_1)\cdots(x+\eta_5)$. Hence all coefficients of P_1 are positive. For $\varepsilon > 0$ small enough, the polynomial $P_2 := P - \varepsilon P_1$ defines one of the sign patterns $\Sigma_{1,6,1}$ or $\Sigma_{1,5,2}$ and has a complex conjugate pair. One can perturb the negative roots of P_2 to make them all distinct while keeping the signs of its coefficients the same and the presence of a complex conjugate pair. However such a polynomial P_2 does not exist, see [7, Theorem 9].

Suppose that P has a double positive root. Hence P is hyperbolic. If P has no vanishing coefficients, then one can perturb the negative roots of P to make them all distinct without changing the signs of the coefficients. After this one considers the polynomial $P_3 := P + \delta x^4$ with 5 distinct negative roots and defining the same sign pattern as P when $\delta > 0$ is small enough. The polynomial P_3 has no double positive root, but a complex conjugate pair close to the double root of P. Again by [7, Theorem 9] such a polynomial P_3 does not exist.

Suppose that P has a double positive root and at least one vanishing coefficient. We remind that P cannot have two consecutive vanishing coefficients ([14, Lemma 7]). For $\varepsilon > 0$ small enough, all coefficients of P can be made non-zero as a result of a shift $\tau_\pm$. We consider the following cases:

Case 1. One has $a_6 < 0$ and $a_1 < 0$ (resp. $a_2 < 0$). A shift τ_+ or τ_- makes all coefficients between a_6 and a_1 (resp. between a_6 and a_2) non-zero. Hence they are all negative, otherwise by Descartes' rule of signs P cannot have 5 negative roots. One perturbs the negative roots of P to make them distinct while keeping the sign pattern. Then for $\delta > 0$ small enough, the polynomial $P + \delta x^4$ has still 5 distinct negative roots and the same sign pattern, but the double root gives birth to a complex conjugate pair close to it. Such a polynomial does not exist, see [7, Theorem 9].

Case 2. One has $a_6 = 0$ and $a_1 < 0$ (resp. $a_2 < 0$). A shift $\tau_\pm$ with suitably chosen sign + or − makes all coefficients between a_7 and a_1 (resp. between a_7 and a_2) non-zero, and in particular a_6 becomes negative. Again by Descartes' rule of signs the rest of the coefficients between a_7 and a_1 (resp. between a_7 and a_2) are negative and as in case 1 one concludes that such a polynomial does not exist.

Case 3. One has $a_6 < 0$ and $a_1 = 0$ (resp. $a_2 = 0$). A shift $\tau_\pm$ with suitably chosen sign + or − makes all coefficients between a_6 and a_0 (resp. between a_6 and a_1) non-zero, and in particular a_1 (resp. a_2) becomes negative. The rest of the reasoning is as in case 2.

Case 4. One has $a_6 = a_1 = 0$ (resp. $a_6 = a_2 = 0$). Then $a_2 \neq 0$ (resp. $a_3 \neq 0$), see [14, Lemma 7], hence $a_2 < 0$ (resp. $a_3 < 0$). A shift $\tau_\pm$ with suitably chosen sign + or − makes all coefficients between a_7 and a_0 (resp. between a_7 and a_1) non-zero, and in particular a_6 becomes negative. Hence the sign pattern of P is now $\Sigma_{1,6,1}$, $\Sigma_{1,5,2}$ or $\Sigma_{1,4,3}$. One perturbs the negative roots of P to make them distinct while preserving the sign pattern, and then adds to P the monomial δx^4 with $\delta > 0$ small enough. The double root gives birth to a complex conjugate pair and P realizes one of the couples $(\Sigma_{1,k,7-k}, (0,5))$, $k = 4$, 5 or 6, which by [7, Theorem 9] is impossible. □

Lemma 23. *There exists no real monic degree* 9 *polynomial* U *satisfying the following conditions:*

(1) *its sign pattern is* $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ *or* $\Sigma_{2,6,2}$ *or it is a generalized sign pattern adjacent to one of these sign patterns;*
(2) *it has* 7 *negative roots counted with multiplicity;*
(3) *it has either a double positive root or a complex conjugate pair.*

Proof. First of all we observe that the last three components of the (generalized) sign pattern cannot be $(+,0,+)$, because in this case the polynomial $U(-x)$ has less than 7 sign changes and by Descartes' rule of signs the polynomial cannot have 7 negative roots. Hence the coefficients of $x^8, \ldots, x^3$ (or $x^7, \ldots, x^3$ in the case of $\Sigma_{2,6,2}$) are non-positive. The one of x is non-negative and can be 0 only if the one of x^2 is non-positive.

If the polynomial U has a double positive root, then it is hyperbolic. Recall that by [14, Lemma 7] the polynomial U has no two consecutive vanishing coefficients. One can perform a simultaneous shift to make all coefficients non-zero and so that the coefficient of x^8 (or of x^7 in the case of $\Sigma_{2,6,2}$) is negative. By Descartes' rule of signs the sign pattern of the polynomial is now $\Sigma_{1,5,4}$, $\Sigma_{1,6,3}$, $\Sigma_{1,7,2}$, $\Sigma_{2,6,2}$ or $\Sigma_{2,7,1}$. One can perturb its negative roots to make them distinct and then add to the polynomial a monomial δx^4, where $\delta > 0$ is small enough. Thus the sign pattern is preserved and the double positive root gives birth to a complex conjugate pair close to it. However such a polynomial does not exist, see [7, Proposition 6].

If U has a complex conjugate pair, then we change it to $U_1 := U - \varepsilon \prod_{j=1}^{7}(x+\eta_j)$, $-\eta_j$ being the negative roots of U. For $\varepsilon > 0$ small enough, all coefficients of U_1 with the possible exception only of the one of x^8 are non-zero. After a simultaneous shift the coefficient of x^8 becomes negative and the sign pattern now equals $\Sigma_{1,6,3}$, $\Sigma_{1,7,2}$, $\Sigma_{1,8,1}$ or $\Sigma_{2,6,2}$. Then one perturbs the negative roots to make them distinct, so the polynomial realizes one of the couples $(\Sigma_{1,6,3},(0,7))$, $(\Sigma_{1,7,2},(0,7))$, $(\Sigma_{1,8,1},(0,7))$ or $(\Sigma_{2,6,2},(0,7))$ which is impossible, see [7, Proposition 6]. □

Definition 24. *We call* multiplicity vector *the vector whose components are equal to the multiplicities of the negative roots of a real polynomial listed in the increasing order. The multiplicity vector* $\vec{v}_1$ *is* adjacent *to the multiplicity vector* $\vec{v}_2$ *if* $\vec{v}_1$ *is obtained by applying one or several times the operation of replacing two consecutive components by their sum. Example:* (5) *is adjacent to* $(3,2)$ *which in turn is adjacent to* $(3,1,1)$.

Lemma 25. *Suppose that there exists a degree* 9 *real monic polynomial* V *realizing the couple* $(\Sigma_{1,7,2},(0,5))$ *(resp.* $(\Sigma_{1,6,3},(0,5))$ *or* $(\Sigma_{2,6,2},(0,5))$*). Then there exists a real monic degree* 9 *polynomial satisfying the following conditions:*

(1) *it defines the sign pattern* $\Sigma_{1,7,2}$ *(resp.* $\Sigma_{1,6,3}$ *or* $\Sigma_{2,6,2}$*);*
(2) *it has a double positive root;*
(3) *the multiplicity vector of its negative roots is among the following ones:*

$$(1,2,2)\,,\ (2,1,2)\,,\ (3,1,1)\,,\ (1,1,3)\,,\ (3,2)\,,\ (2,3)\,.$$

Proof. The polynomial V has either 4 or 6 critical points for $x < 0$ (counted with multiplicity). We denote them by $-\xi_4 < -\xi_3 < -\xi_2 < -\xi_1$; if they are 6, then between two consecutive negative roots of V there are three critical points of which we choose the rightmost one. We denote the corresponding critical values by η_i, where $\eta_4 > 0$, $\eta_3 < 0$, $\eta_2 > 0$ and $\eta_1 < 0$.

Suppose that $\eta_3 = \eta_1$. Then we add to V a positive constant (this does not change the sign pattern) to obtain a polynomial with multiplicity vector $(1,2,2)$.

Suppose that $\eta_3 < \eta_1$. We consider the family of polynomials $V_t := V + tx$, $t \geq 0$ in which the sign of the coefficient of x is $+$. As t increases, the critical value η_4 decreases faster than η_3 and η_2 decreases faster than η_1. Denote by $t_0 > 0$ the smallest value of t for which one of the following things happens:

(1) One has $\eta_4 = \eta_1$. In this case we add to V_{t_0} a positive constant to obtain a polynomial with multiplicity vector $(2,1,2)$.

(2) One has $\eta_2 = \eta_1$. Then $-\xi_1$ is a degenerate critical point of V_{t_0}. We add to V_{t_0} a positive constant and get a polynomial with multiplicity vector $(1,1,3)$.

Suppose that $\eta_3 > \eta_1$. Then for $t = t_0$, one of the following things can take place in the family V_t:

(3) One has $\eta_3 = \eta_1$. We add to V_{t_0} a positive constant and obtain a polynomial with multiplicity vector $(1,2,2)$.

(4) One has $\eta_4 = \eta_3$. Then $-\xi_3$ is a degenerate critical point of V_{t_0}. We add to V_{t_0} a positive constant and get a polynomial with multiplicity vector $(3,1,1)$.

If (1) and (2) (resp. (3) and (4)) take place simultaneously, then the multiplicity vector of V_{t_0} is $(2,3)$ (resp. $(3,2)$). It is not possible for $t = t_0$ to obtain an equality between η_1, η_2, η_3 or η_4 and one of the two possible other critical values of V_{t_0} (if V_{t_0} has 6 and not 4 negative critical points), because then one can add a positive constant to V_{t_0} and get a polynomial with 7 negative roots, one conjugate pair and sign pattern $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ or $\Sigma_{2,6,2}$ which by Lemma 23 is impossible. □

Proposition 26. *There exists no real monic degree* 9 *polynomial having* 5 *negative roots with multiplicity vector having* 1, 2 *or* 3 *components, from the closure of the sign pattern* $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ *or* $\Sigma_{2,6,2}$, *and having one double positive root.*

Proof.

(A). We explain the method of proof on the example of the multiplicity vector (5) (one five-fold negative root). We want to prove the non-existence of a polynomial

$$H_5 := (x+1)^5((x+u)^2+v)(x-c)^2\ ,\ v>0\ ,\ c>0\ ,\ u\in\mathbb{R}\ .$$

One rescales the x-axis to make the negative root equal to -1. The index 5 corresponds to the multiplicity vector (5). Hence the non-existence of such a polynomial $H_5 := \sum_{j=0}^{9} a_j x^j$ is tantamount to the emptyness of the domain $E_5 \cap D_5$, where $D_5 \subset \mathbb{R}^3$ is defined by the inequalities

$$D_5\ :\ \{\ (u,v,c)\ |\ v>0\ ,\ c>0\ \}$$

and E_5 is the closed domain in $\mathbb{R}^3$ defined by the condition the signs of the coefficients of H_5 to correspond to the closure of one of the sign patterns $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ or $\Sigma_{2,6,2}$. We remind that each coefficient of H_5 is a polynomial in the variables (u,v,c), with $a_0 = c^2(v+u^2)$, …, $a_8 = 5+2u-2c$. If the interior of the domain E_5 is non-empty, then E_5 has a priori the structure of a stratified manifold. Its stratum of maximal dimension corresponds to polynomials defining the sign pattern $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ or $\Sigma_{2,6,2}$. The closure $\overline{D}_5$ of the set D_5 is defined by the inequalities $v \geq 0$, $c \geq 0$ and its border ∂D_5 by $v \geq 0$, $c \geq 0$, $cv = 0$.

No polynomial $H_5 \in \partial D_5$ is from the closure of the sign pattern $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ or $\Sigma_{2,6,2}$. Indeed, for $c = 0 < v$ or $c > 0 = u = v$, one has $H_5 = x^2 H_5^1$, where $H_5^1(0) > 0$. Then the (generalized) sign pattern of H_5 cannot be adjacent to $\Sigma_{1,7,2}$ or $\Sigma_{2,6,2}$, but only to $\Sigma_{1,6,3}$ and the one of H_5^1 is adjacent to $\Sigma_{1,6,1}$. By Lemma 22 such a polynomial H_5^1 does not exist. For $v = 0 < c$, $u > 0$, non-existence of H_5 follows from Lemma 23. For $v = 0 < c$, $u < 0$, the polynomial H_5 has four positive roots (counted with multiplicity) which by Descartes' rule of signs requires at least four sign changes in the (generalized) sign pattern – a contradiction. Thus $c = 0$ or $v = 0$ is impossible.

(B). Next we consider the two subdomains $D_5^+ := \overline{D}_5 \cap \{u \geq 0\}$ and $D_5^- := \overline{D}_5 \cap \{u \leq 0\}$. They are convex. To show that no polynomial $H_5 \in D_5^+$ (resp. $H_5 \in D_5^-$) has the necessary (generalized) sign pattern we consider the planes $T_r^+ : u+v+c = r$ (resp. $T_r^- : -u+v+c = r$), $r \in \mathbb{R}$. It is clear that for $r < 0$, one has $T_r^\pm \cap D_5^\pm = \emptyset$ and $T_0^\pm \cap D_5^\pm = \{(0,0,0)\}$, with $(0,0,0) \in \partial D_5$.

We suppose that there exists a polynomial $H_5 \in D_5^\pm$ having the necessary (generalized) sign pattern. Then it is not in $\partial D_5^\pm$ and belongs to some plane $T_r^\pm$ for some $r = r_0 > 0$. At the point

$(0,0,0)$ at least one coefficient a_j of H_5 has the wrong sign. The least possible value of r_0 is the least one for which $H_5 \in E_5$, i.e. where the signs of all coefficients correspond to the closure of the sign pattern $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ or $\Sigma_{2,6,2}$. So for $r = r_0$, at least one of the coefficients a_j of H_5 vanishes, because the corresponding polynomial(s) H_5 belong to the border, but not to the interior of the set E_5.

We use the method of Lagrange's multipliers as follows. We are looking for the minimal value of the function $T_0^{\pm}$ on the hypersurface $\{a_j = 0\}$. We construct the function

$$\tilde{T}_j^{\pm} := \pm u + v + c + \lambda a_j \ ,$$

where λ is a Lagrange multiplier. For $j = 0, \dots, 8$, we consider the system of equations

$$\partial \tilde{T}_j^{\pm}/\partial \lambda = a_j = \partial \tilde{T}_j^{\pm}/\partial u = \partial \tilde{T}_j^{\pm}/\partial v = \partial \tilde{T}_j^{\pm}/\partial c = 0 \ .$$

In each case we are looking for a solution with $\lambda \in \mathbb{R}$ and $(u, v, c) \in D_5^{\pm}$. It turns out that in each case either there is no solution or the sign of u, v or c of the solution is not the right one or the signs of the coefficients of the obtained polynomial H_5 are in contradiction with the closure of the sign pattern $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ or $\Sigma_{2,6,2}$. This can be established using computer algebra.

(C). The multiplicity vector (5) is adjacent to four multiplicity vectors with two components: $(4,1)$, $(3,2)$, $(2,3)$ and $(1,4)$. We prove the non-existence of polynomials

$$H_{k,5-k} := (x+\mu)^k (x+1)^{5-k}((x+u)^2 + v)(x-c)^2 \ , \ \ k = 1,\ 2,\ 3,\ 4 \ ,$$

with $\mu > 1$ and with (u, v, c) as above. Hence

$$\mathbb{R}^4 \supset D_{k,5-k} = \{(u, v, c, \mu) \mid v > 0 \ , \ c > 0 \ , \ \mu > 1 \} \ .$$

As in (A) one shows that no polynomial $H_{k,5-k} \in \partial D_{k,5-k}$ with $c = 0$ or $v = 0$ has signs of the coefficients from the closure of $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ or $\Sigma_{2,6,2}$. For $\mu = 1$, one is looking in fact for a polynomial H_5 about which it was shown in (A)–(B) that it does not exist. Hence $E_{k,5-k} \cap \partial D_{k,5-k} = \emptyset$.

(D). To prove that $E_{j,5-j} \cap D_{j,5-j} = \emptyset$ we use again the method of Lagrange multipliers. We set $D_{k,5-k}^+ := D_{k,5-k} \cap \{u > 0\}$, $D_{k,5-k}^- := D_{k,5-k} \cap \{u < 0\}$, $S_r^{\pm} : \pm u + v + c + (\mu - 1) = r$ $(r \in \mathbb{R})$ and $\tilde{S}_j^{\pm} := \pm u + v + c + (\mu - 1) + \lambda a_j$. In this case

$$a_8 = j\mu + (5 - j) + 2u - 2c \ , \ \dots \ , \ a_0 = \mu^j c^2 (v + u^2) \ .$$

For $j = 0, \dots, 8$, and for k as above, we consider the system of equations

$$\partial \tilde{S}_j^{\pm}/\partial \lambda = a_j = \partial \tilde{S}_j^{\pm}/\partial u = \partial \tilde{S}_j^{\pm}/\partial v = \partial \tilde{S}_j^{\pm}/\partial c = \partial \tilde{S}_j^{\pm}/\partial \mu = 0 \ .$$

We are looking for a solution with $\lambda \in \mathbb{R}$ and $(u, v, c, \mu) \in D_{k,5-k}^{\pm}$. In each case either there is no real solution or the sign of u, v, c or $\mu - 1$ of the solution is not the right one.

(E). The possible multiplicity vectors with three components are $(2,2,1)$, $(2,1,2)$, $(1,2,2)$, $(3,1,1)$, $(1,3,1)$ and $(1,1,3)$. The polynomials $H_{j,k,5-j-k}$ defined after the multiplicity vectors $(j, k, 5-j-k)$ are

$$H_{j,k,5-j-k} := (x+\mu_2)^j (x+\mu_1)^k (x+1)^{5-j-k}((x+u)^2 + v)(x-c)^2 \ ,$$

with (u, v, c) as above and $1 < \mu_1 < \mu_2$. We set

$$\mathbb{R}^5 \supset D_{j,k,5-j-k} := \{(u, v, c, \mu_1, \mu_2) \mid v > 0 \ , \ c > 0 \ , \ 1 < \mu_1 \ , 1 < \mu_2 \} \ ,$$

i.e. we consider a domain larger than the strict analog of the domains D_5 and $D_{j,5-j}$. (The strict analog would be defined by $1 < \mu_1 < \mu_2$ instead of $1 < \mu_1$, $1 < \mu_2$.) This is done with the aim to simplify the computations. We set also $D_{j,k,5-j-k}^- := D_{j,k,5-j-k} \cap \{u < 0\}$, $K_r^{\pm} : \pm u + v + c + (\mu_1 - 1) = r$

($r \in \mathbb{R}$) and $\widetilde{K}_i^{\pm} := \pm u + v + c + (\mu - 1) + (\mu_2 - 1) + \lambda a_i$. For $i = 0, \ldots, 8$, and for (j,k) as above, we consider the system of equations

$$\partial \widetilde{K}_i^{\pm} / \partial \lambda = a_i = \partial \widetilde{K}_i^{\pm} / \partial u = \partial \widetilde{K}_i^{\pm} / \partial v = \partial \widetilde{K}_i^{\pm} / \partial c = \partial \widetilde{K}_i^{\pm} / \partial \mu_1 = \partial \widetilde{K}_i^{\pm} / \partial \mu_2 = 0 \, .$$

We are looking for a solution with $\lambda \in \mathbb{R}$ and $(u, v, c, \mu_1, \mu_2) \in D^{\pm}_{j,k,5-j-k}$. In each case either there is no real solution or the sign of u, v, c, $\mu_1 - 1$ or $\mu_2 - 1$ of the solution is not the right one or, when a solution exists, the obtained polynomial $H_{j,k,5-j-k}$ does not have a (generalized) sign pattern from the closure of $\Sigma_{1,7,2}$, $\Sigma_{1,6,3}$ or $\Sigma_{2,6,2}$. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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