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Purity and almost strict purity of Anderson t-modules

Pureté et presque stricte pureté des t-modules d'Anderson

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Abstract. We study the relations between the notion of purity of a t-module introduced by Anderson and that of almost strict purity for a t-module introduced by Namoijam and Papanikolas (concept already mentioned by G. Anderson and D. Goss).

Résumé. On étudie les relations entre la notion de pureté d'un t-module introduite par Anderson et celle de presque pureté pour un t-module introduite par Namoijam et Papanikolas (concept déjà mentionné par G. Anderson et D. Goss).

Keywords. Purity, almost strict purity, Anderson t-modules, t-motive, Newton polygons.

Mots-clés. Pureté, presque stricte pureté, t-modules d'Anderson, t-motifs, polygone de Newton.

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1. Introduction

In [1], G.W. Anderson defined t-modules and the notion of purity. Related to t-modules, C. Namoijam and M. A. Papanikolas defined the notion of almost strict purity in [4, Remark 4.5] (concept already mentioned by G.Anderson in [1, Section 2.2] and D.Goss in [2, Remark 5.5]), then they proved that an almost strictly pure t-module is pure. We are interested here in the reciprocal, and with the help of the work of A. Maurischat in [3] we show that these two notions are not equivalent by presenting a counter-example (see Theorem 3).

2. Purity

Let K a perfect field containing $\mathbb{F}_q$. We let $\tau : K \to K$ denote the q-th power Frobenius map and $K\{\tau\}$ be the ring of twisted polynomials in τ over K, subject to the relation, $\tau a = a^q \tau$ for any $a \in K$.

We further consider the skew Laurent series ring over K in $\sigma := \tau^{-1}$

$$K(\{\sigma\}) := \left\{ \sum_{i=k}^{\infty} a_i \sigma^i \;\middle|\; k \in \mathbb{Z}, a_i \in K \right\}.$$

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Consider $\ell : \mathbb{F}_q[t] \to K$ a homomorphism of $\mathbb{F}_q$-algebras and denote $\sigma = \tau^{-1}$.

A t-module (E, φ) over K of dimension d is by definition an $\mathbb{F}_q$-vector space scheme E over K isomorphic to $\mathbb{G}_a^d$ together with a homomorphism of $\mathbb{F}_q$-algebras $\varphi : \mathbb{F}_q[t] \to \mathrm{End}_{\mathrm{grp},\mathbb{F}_q}(E)$ into the ring of $\mathbb{F}_q$-vector space scheme endomorphisms of E, such that for all $a \in \mathbb{F}_q[t]$, the endomorphism $d\varphi_a$ on $\mathrm{Lie}(E)$ induced by φ_a fulfills the condition that $d\varphi_a - \ell(a)$ is nilpotent.

We will fix in the following (E, ϕ) a t-module on K of dimension d as well as a coordinate system κ, i.e. an isomorphism of schemes in $\mathbb{F}_q$-vector spaces $\kappa : E \simeq \mathbb{G}_a^d$ defined on K. With respect to this coordinate system, we can represent ϕ_t by a matrix $D \in M_d(K\{\tau\})$.

Let $\mathrm{pr}_i : \mathbb{G}_a^d \to \mathbb{G}_a$ $(1 \le i \le d)$ be the projection to the i-th component of $\mathbb{G}_a^d$, and let $\kappa_i = \kappa \circ \mathrm{pr}_i$. Let $\widehat{\kappa}_j : \mathbb{G}_a \to E$ be defined by $\widehat{\kappa}_j = \kappa^{-1} \circ \mathrm{inj}_j$ where $\mathrm{inj}_j : \mathbb{G}_a \to \mathbb{G}_a^d$ is the natural injection into the $j-$th component.

We say that E is almost strictly pure if there is some integer $s \ge 1$ such that

$$D^s = A_0 + A_1\tau + \cdots + A_r\tau^r$$

with $A_r \in GL_d(K)$.

The t-motive $M(E)$ of E is the free $K\{\tau\}$-module of rank d with base $\{\kappa_1, \ldots, \kappa_d\}$ with a t-action on this base defined by

$$t.\begin{pmatrix}\kappa_1\\ \vdots\\ \kappa_d\end{pmatrix} = D\begin{pmatrix}\kappa_1\\ \vdots\\ \kappa_d\end{pmatrix}.$$

We define in a similar way the dual t-motive $\mathscr{M}$ as the free $K\{\tau\}$-module of rank d of basis $\{\widehat{\kappa}_1, \ldots, \widehat{\kappa}_d\}$ whose t-action (on the right) on this basis is defined by

$$\begin{pmatrix}\widehat{\kappa}_1 & \cdots & \widehat{\kappa}_d\end{pmatrix} \cdots t = \begin{pmatrix}\widehat{\kappa}_1 & \cdots & \widehat{\kappa}_d\end{pmatrix} D.$$

We say that E is abelian if $M(E)$ is a finitely generated $K[t]$-module. In this case, we define $w(M)$ the weight of M by

$$w(M) = \frac{d}{\mathrm{rk}(E)}$$

where $\mathrm{rk}(E)$ is the rank of M as a $K[t]$-module (that is finite because E is abelian).

We moreover consider:

- The ring of formal power series in $\frac{1}{t}$ with coefficients in K denoted by $K[[\frac{1}{t}]]$.
- The field of Laurent series in $\frac{1}{t}$ with coefficients in K denoted by $K((\frac{1}{t}))$ (that is the field of fractions of $K[[\frac{1}{t}]]$).

The t-motive $M(E)$ and the t-module E are called pure if there exists a $K[[\frac{1}{t}]]$-lattice Λ in $K((\frac{1}{t})) \otimes_{K[t]} M$ as well as positive integers $u, v \in \mathbb{N}$ such that

$$t^u\Lambda = \tau^v\Lambda.$$

We will use the following result, proved by A. Maurischat in [3, Theorem 6.6, Theorem 7.2], characterizing the fact of being abelian and being pure using Newton polygons.

Theorem 1 (Maurischat). *The t-module E is abelian if and only if the Newton polygon N_{λ_d} of the last invariant factor λ_d of the matrix D has only positive slopes. In this case, E is pure if and only if N_{λ_d} has exactly one edge. Then we have that the weight of M equals the reciprocal of the slope of the edge.*

Here we recall that the invariant factors of a matrix $D \in M_d(K\{\tau\})$ are obtained by diagonalizing the matrix $tI_d - D \in M_d(K(\{\sigma\})[t])$ by performing elementary operations on the rows and columns in $K(\{\sigma\})[t]$ that are the following (denote by L_i (resp C_i) the i-th row (resp the i-th column)):

- add to the i-th row L_i the j-th row L_j multiplied on the left by $a \in K(\{\sigma\})[t]$ denote by $L_i \to L_i + a.L_j$ (resp add to i-th column C_i the j-th column C_j multiplied on the right by $a \in K(\{\sigma\})[t]$ denote by $C_i \to C_i + C_j \cdot a$),
- multiply on the left i-th the row L_i by an element u of $K(\{\sigma\})^*$ denoted by $L_i \to uL_i$ (resp multiply on the right the i-th column C_i by u denoted by $C_i \to C_i u$),
- exchanging two lines L_i and L_j (resp two columns C_i and C_j) denote by $L_i \leftrightarrow L_i$ (resp $C_i \leftrightarrow C_j$).

Contrary to the commutative case the invariant factors are only unique up to similarity.

Example. In [3], Maurischat defined the t-module given by the matrix

$$M := \begin{pmatrix} \theta & 0 \\ 1 & \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \tau + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \tau^2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \tau^3 \in M_2(K\{\tau\}).$$

By diagonalizing the matrix $tI_2 - M$ we get the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where

$$\lambda_2 = \left(-\sigma^{-3} + (\theta + \theta^{q^2})\sigma^{-2} + \theta^{q^3+1}\right) - \left(2\sigma^{-2} + \theta^{q^{-3}} + \theta\right) \cdot t + t^2 \in K(\{\sigma\}).$$

If char$(K) = 2$ then we represent the Newton polygon of λ_2 in Figure 1. It has only one edge of slope $\frac{3}{2}$ hence by Theorem 1 the t-module is abelian and pure of weight $\frac{2}{3}$.

The authors of [4] showed in the same paper the next result.

Theorem 2 (Namoijam, Papanikolas). *With the previous notation, an almost strictly pure t-module is pure of weight $\frac{s}{r}$.*

We now turn our interest to the reciprocal of the above result, and we answer negatively.

Theorem 3. *For any integer $d \geq 2$ there exists a pure but not almost strictly pure t-module of dimension d.*

Proof. We first consider the case $d = 2$. Let us note $\theta = \ell(t)$. Consider the t-module given by the matrix

$$D_2 = \begin{pmatrix} \theta & 0 \\ 1 & \theta \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ \theta & \theta \end{pmatrix} \cdot \tau \in M_d(K\{\tau\}).$$

Let us diagonalize the matrix $tI_2 - D_2$ (the diagonalization is being taken over $K(\{\sigma\})$):

$$\begin{pmatrix} t-\theta-\tau & -\tau \\ -1-\theta\tau & t-\theta-\theta\tau \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2} \begin{pmatrix} -1-\theta\tau & t-\theta-\theta\tau \\ t-\theta-\tau & -\tau \end{pmatrix} \xrightarrow{C_1 \to C_1\gamma} \begin{pmatrix} 1 & t-\theta-\theta\tau \\ (t-\theta-\tau)\gamma & -\tau \end{pmatrix}$$

$$\begin{pmatrix} 1 & t-\theta-\theta\tau \\ (t-\theta-\tau)\gamma & -\tau \end{pmatrix} \xrightarrow{L_2 \to L_2-(t-\theta-\tau)\gamma L_1} \begin{pmatrix} 1 & t-\theta-\theta\tau \\ 0 & -\tau-(t-\theta-\tau)\gamma(t-\theta-\theta\tau) \end{pmatrix}$$

$$\begin{pmatrix} 1 & t-\theta-\theta\tau \\ 0 & -\tau-(t-\theta-\tau)\gamma(t-\theta-\theta\tau) \end{pmatrix} \xrightarrow{C_2 \to C_2+C_1(-t+\theta+\theta\tau)} \begin{pmatrix} 1 & 0 \\ 0 & \lambda' \end{pmatrix} \xrightarrow{L_2 \to -\gamma^{-1}L_2} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where

$$\begin{aligned} \lambda_2 &= -\gamma^{-1}\lambda' \\ &= -\gamma^{-1}\left(-\tau-(t-\theta-\tau)\gamma(t-\theta-\theta\tau)\right) \\ &= t^2 + t\cdot(-\theta-\theta\tau-\gamma^{-1}\theta\gamma-\gamma^{-1}\tau\gamma) + \gamma^{-1}\tau + \gamma^{-1}\theta\gamma\theta + \gamma^{-1}\theta\gamma\theta\tau + \gamma^{-1}\tau\gamma\theta + \gamma^{-1}\tau\gamma\theta\tau \end{aligned}$$

and $\gamma = (-1-\theta\tau)^{-1}$.

We represent the Newton polygon of λ_2 in Figure 2. It has only one edge of positive slope equal to 1, hence according to Theorem 1 this t-module is pure of weight 1.

An immediate induction shows that for $n \geq 2$, the leading coefficient of D_2^n is given by the matrix

$$\prod_{k=1}^{n-1}(1+\theta^{q^k})\cdot A$$

where

$$A = \begin{pmatrix} 1 & 1 \\ \theta & \theta \end{pmatrix}$$

whose determinant is zero, so this t-module is not almost strictly pure. □

Now we consider the general case $d > 2$. We put $m := d-2 > 0$ and consider the t-module given by the matrix

$$D_{2+m} = \begin{pmatrix} D_2 & & & \\ & \theta+\tau & & \\ & & \ddots & \\ & & & \theta+\tau \end{pmatrix} \in M_{2+m}(K\{\tau\}).$$

This t-module is the direct sum of pure t-modules of weight 1 (the t-module associated to D_2 and $d-2$ copies of the Carlitz module), so we can prove it is a pure t-module of weight 1, but here we give a proof using Maurischat's algorithm.

For $n \geq 1$, the leading coefficient of the matrix D_{2+m}^n is the matrix

$$\begin{pmatrix} D_2^n & \\ & I_m \end{pmatrix}$$

whose determinant is zero, so this t-module is not almost strictly pure.

Consider (J_0) the algorithm that diagonalize as previously the matrix $tI_2 - D_2$. Applying (J_0) and exchanging row and columns, we get the matrix:

$$tI_{2+m} - D_{2+m} \longrightarrow S = \begin{pmatrix} 1 & & & & \\ & t-\theta-\tau & & & \\ & & \ddots & & \\ & & & t-\theta-\tau & \\ & & & & \lambda_2 \end{pmatrix} \in M_{2+m}(K\{\tau\}[t]).$$

Consider the euclidean division of λ_2 by $t-\theta-\tau$:

$$\lambda_2 = q(t-\theta-\tau)+r,\ r \neq 0 \text{ and } \deg_t(r) = 0.$$

Let

$$S' = \begin{pmatrix} t-\theta-\tau & \\ & \lambda_2 \end{pmatrix} \in M_2(K\{\tau\}[t]).$$

We apply the following operations to the matrix S' (and denote by (J_1) this algorithm):

$$\begin{aligned}
L_2 &\longrightarrow L_2 - qL_1 \\
C_2 &\longrightarrow C_2 + C_1 \\
L_2 &\longrightarrow r^{-1}L_2 \\
L_1 &\longrightarrow L_1 - (t-\theta-\tau)L_2 \\
C_2 &\longrightarrow C_2 - C_1 r^{-1}\lambda_2 \\
L_1 &\longleftrightarrow L_2 \\
L_2 &\longrightarrow -L_2.
\end{aligned}$$

We get the matrix:

$$\begin{pmatrix} \lambda_2 & \\ & (t-\theta-\tau)r^{-1}\lambda_2 \end{pmatrix}.$$

By successively applying the algorithm (J_1) to the matrices S' which appear from the matrix S, we obtain the matrix

$$S \longrightarrow \begin{pmatrix} 1 & & & & \\ & \lambda_2 & & & \\ & & (t-\theta-\tau)r^{-1}\lambda_2 & & \\ & & & \ddots & \\ & & & & (t-\theta-\tau)r^{-1}\lambda_2 \end{pmatrix}.$$

As λ_2 and $t-\theta-\tau$ have Newton polygons consisting of only one edge of slope 1, the Newton polygon of the last coefficient of the last matrix has only one edge of slope 1. It follows that the Newton polygon of the last invariant factor of D_{m+2} has only one edge of slope 1. Hence D_{m+2} is also a t-module which is pure of weight 1 but not almost strictly pure for all $m \geq 0$.

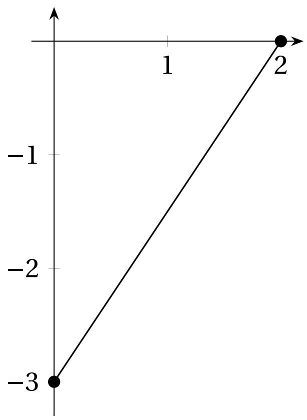

Figure 1. Newton polygon of the Anderson module M constructed by Maurischat when $\mathrm{char}(K) = 2$.

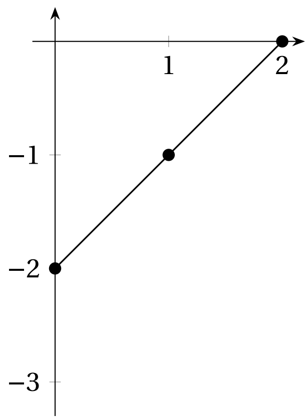

Figure 2. Newton polygon of the Anderson module D_2 in Theorem 3.

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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