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A decomposition theorem for $\mathbb{Q}$-Fano Kähler–Einstein varieties

Un théorème de décomposition pour les variétés $\mathbb{Q}$-Fano qui admettent une métrique de Kähler–Einstein

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It is an honor for us to dedicate this paper to the memory of Jean-Pierre Demailly, who always shared his tremendous insights and vision of complex geometry with kindness and generosity.

Abstract. Let X be a $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric. We prove that up to a finite quasi-étale cover, X splits isometrically as a product of Kähler–Einstein $\mathbb{Q}$-Fano varieties whose tangent sheaf is stable with respect to the anticanonical polarization. This relies among other things on a very general splitting theorem for algebraically integrable foliations. We also prove that the canonical extension of T_X by $\mathscr{O}_X$ is polystable with respect to the anticanonical polarization.

Résumé. Soit X une variété $\mathbb{Q}$-Fano admettant une métrique de Kähler–Einstein. Nous montrons, qu'à un revêtement fini quasi-étale près, X est un produit de variétés $\mathbb{Q}$-Fano admettant une métrique de Kähler–Einstein dont le fibré tangent est stable relativement au diviseur anticanonique. La démonstration repose notamment sur un théorème de décompostion pour les feuilletages algébriquement intégrables. Nous montrons également que l'extension canonique de T_X par $\mathscr{O}_X$ est polystable à nouveau relativement au diviseur anticanonique.

Keywords. $\mathbb{Q}$-Fano varieties, singular Kähler–Einstein metrics, stable reflexive sheaves, algebraically integrable foliations.

Mots-clés. Variétés $\mathbb{Q}$-Fano, métriques de Kähler–Einstein singulières, faisceaux réflexifs stables, feuilletages algébriquement intégrables.

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1. Introduction

Let (X,ω) be a Fano Kähler–Einstein manifold, i.e. X is a projective manifold with $-K_X$ ample and admitting a Kähler metric ω solving $\mathrm{Ric}\,\omega = \omega$. It follows from the (easy direction of the) Kobayashi–Hitchin correspondence that the tangent bundle of X splits as a direct sum of parallel subbundles

$$T_X = \bigoplus_{i\in I} F_i \tag{1}$$

such that F_i is stable with respect to $-K_X$. Since X is simply connected, de Rham's splitting theorem asserts that one can integrate the foliations arising in decomposition (1) and obtain an isometric splitting

$$(X,\omega) \simeq \prod_{i\in I} (X_i,\omega_i)$$

into Kähler–Einstein Fano manifolds which is compatible with (1).

Over the last few decades, a lot of attention has been drawn to projective varieties with mild singularities, in relation to the progress of the Minimal Model Program (MMP). In that context, the notion of $\mathbb{Q}$-Fano variety (cf. Definition 1) has emerged and played a central role in birational geometry.

On the analytic side, singular Kähler–Einstein metrics have been introduced and constructed in various settings (see e.g. [2, 4, 18] and Definition 2). They induce genuine Kähler–Einstein metrics on the regular part of the variety but are in general incomplete, preventing the use of most useful results in differential geometry (like the de Rham splitting theorem mentionned above) to analyze their behavior. However, these objects are well-suited to study (poly)-stability properties of the tangent sheaf as it was observed by [25], relying on earlier results by [17].

In the Ricci-flat case, the holonomy of the singular metrics was computed in [20]. Moreover, [15] provided an algebraic integrability result for foliations as well as a splitting result in that setting. Building upon those results, Höring and Peternell [26] could eventually prove the singular version of the Beauville–Bogomolov decomposition theorem.

In the positive curvature case, some simplifications appear (for instance, the algebraic integrability of foliations can be related to stability properties by [5]) but new difficulties also arise: the singularities are klt rather than canonical and Gorenstein, and one cannot regularize the singular Kähler–Einstein metrics with an equally good control on the Ricci curvature. In this paper, our main contribution is to single out and overcome those difficulties in order to prove the following structure theorem for $\mathbb{Q}$-Fano varieties that admit a Kähler–Einstein metric.

Theorem A. *Let X be a $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric ω. Then T_X is polystable with respect to $c_1(X)$. Moreover, there exists a quasi-étale cover $f\colon Y \to X$ such that $(Y, f^*\omega)$ decomposes isometrically as a product*

$$(Y, f^*\omega) \simeq \prod_{i\in I} (Y_i,\omega_i),$$

where Y_i is a $\mathbb{Q}$-Fano variety with stable tangent sheaf with respect to $c_1(Y_i)$ and ω_i is a Kähler–Einstein metric on Y_i.

Below are a few remarks about the result above.

- Theorem A shows that for all "practical aspects" the tangent sheaf of a $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric can always be assumed to be stable. Moreover, it can be expressed in a purely algebraic way using the notion of K-stability, cf. Remark 4 (this is the case for Theorem B below as well).
- The quasi-étale cover above is needed to split X even when T_X is already split, as we see by taking e.g. $X = (\mathbb{P}^1\times\mathbb{P}^1)/\langle \iota\times\iota\rangle$ where $\iota : \mathbb{P}^1 \to \mathbb{P}^1$ is the involution $\iota([u:v]) = [u:-v]$.

- It was proved very recently by Braun [6, Thm. 2] that the fundamental group of the regular locus of a $\mathbb{Q}$-Fano variety is finite. Relying on that result, one can refine Theorem A and obtain that the varieties Y_i satisfy the additional property: $\pi_1(Y_i^{\mathrm{reg}}) = \{1\}$.
- Semistability of T_X for a Kähler–Einstein $\mathbb{Q}$-Fano variety X was proved by Chi Li in [35, Prop. 3.7] in the case where X admits a resolution where all exceptional divisors have non-positive discrepancy, e.g. a crepant resolution.

Our second main result is the following generalisation of a theorem of Tian [37, Thm. 0.1], which is a way to express some "strong" polystability of T_X.

Theorem B. *Let X be a $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric. Then the canonical extension of T_X by $\mathscr{O}_X$ is polystable with respect to $c_1(X)$.*

We refer to Section 3.1 for the construction of the canonical extension. As we explain further below, at the end of the introduction (see paragraph on the strategy of proof of Theorem B), the generalization from the smooth to the singular case requires some non-trivial new input on top of the analytic techniques already developed for the proof of the semistability/polystability of the *tangent sheaf* T_X, i.e. Theorem 6.

In another direction, the semistability of the canonical extension has been proved in [35, Thm. 1.4] for K-semistable *log smooth* log Fano pairs. It is very likely that the proof of the above theorem will carry over *mutatis mutandis* to the more general setting of log Fano pairs, but we will not pursue this direction in this paper.

Our last main result is a very general splitting theorem for algebraicallly integrable foliations, which plays a key role in the proof of Theorem A, but is certainly of independent interest.

Theorem C. *Let X be a normal projective variety, and let*

$$T_X = \bigoplus_{i\in I} \mathscr{F}_i$$

be a decomposition of T_X into involutive subsheaves with algebraic leaves. Suppose that there exists a $\mathbb{Q}$-divisor Δ such that (X,Δ) is klt. Then there exists a quasi-étale cover $f\colon Y \to X$ as well as a decomposition

$$Y \simeq \prod_{i\in I} Y_i$$

of Y into a product of normal projective varieties such that the decomposition $T_X = \bigoplus_{i\in I}\mathscr{F}_i$ lifts to the canonical decomposition

$$T_{\prod_{i\in I} Y_i} = \bigoplus_{i\in I} \mathrm{pr}_i^* \, T_{Y_i}.$$

Theorem C can be seen as the generalization of the splitting result in [15] where additional assumptions are made, both on the singularities of X and the positivity of $K_{\mathscr{F}_i}$. More precisely, in [15] X is assumed to have canonical singularities, and the $K_{\mathscr{F}_i}$ are assumed to be $\mathbb{Q}$-linearly trivial. We also refer to [16, Thm. 1.5] for a somewhat related result. In comparison to [15, Prop. 4.10], the range of applications of Theorem C is significantly broader.

Strategy of proof of the main results

Theorem A. The first step is the object of Theorem 6 where one proves that T_X is the direct sum of stable subsheaves that are parallel with respect to the Kähler–Einstein metric ω on X_{reg}. This is achieved by computing slopes of subsheaves using the metric induced by the Kähler–Einstein metric and using Griffiths' well-known formula for the curvature of a subbundle. However, the presence of singularities (for X and ω) makes it hard to carry out the analysis directly on X. One has to work on a resolution using approximate Kähler–Einstein metrics as in [25]. Yet an additional error term appears in the Fano case, requiring to introduce some new ideas to deal with it as explained on page 99, cf "term (**I**)".

Once Theorem 6 is at hand, one can appeal to Theorem C where the foliations are induced by the Kähler–Einstein metric as showed in the first step. Note that the algebraic integrability of these foliations follows from the deep results of [5]. An easy induction allows one to split X as a product of $\mathbb{Q}$-Fano varieties with stable tangent sheaf. The isometric splitting follows from a suitable characterization of singular Kähler–Einstein metrics, cf. Claim 28.

Theorem B. The proof of Theorem B takes up most of Section 3. It relies largely on the computations carried out in Section 2 to prove the polystability of T_X, but on top of those, several new ideas are needed to overcome the presence of singularities.

First, one needs to reduce the statement to one on a resolution in order to use analytic methods. Then we use again the technique of working with approximate Kähler–Einstein metrics, but in the current context this has the effect of modifying the canonical extension as well. As a result, we cannot evaluate directly the slope of a subsheaf of the canonical extension corresponding to the initial Kähler–Einstein metric. Dealing with this difficulty is our main contribution in this framework. The rest of the proof uses a combination of the original idea of Tian and the computations of Section 2.

Theorem C. The starting point is the observation that since each foliation $\mathscr{F}_i$ admits a complement inside T_X, $\mathscr{F}_i$ is automatically weakly regular. It turns out that weakly regular foliations have many nice properties. The important fact which is established here is that an algebraically integrable, weakly regular foliation on a $\mathbb{Q}$-factorial projective variety with klt singularities is induced by a surjective, equidimensional morphism $X \to Y$, cf. Theorem 17. When combined with suitable generalisations of other techniques and results in [16], this leads to the proof of Theorem C.

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2. Polystability of the tangent sheaf

2.1. *Set-up*

2.1.1. *Notation*

Definition 1. *Let X be a projective variety of dimension n. We say that X is a $\mathbb{Q}$-Fano variety if X has klt singularities and $-K_X$ is an ample $\mathbb{Q}$-line bundle.*

We also recall the definition of (twisted) singular Kähler–Einstein metric, cf. [2].

Definition 2. *Let X be a $\mathbb{Q}$-Fano variety, let $\vartheta \in c_1(X)$ be a smooth representative and let $\gamma \in [0,1)$. A twisted Kähler–Einstein metric relatively to the couple (ϑ, γ) is a closed, positive current $\omega_{\mathrm{KE},\gamma} \in c_1(X)$ with bounded potentials, which is smooth on X_{reg} and satisfies*

$$\operatorname{Ric} \omega_{\mathrm{KE},\gamma} = (1-\gamma)\omega_{\mathrm{KE},\gamma} + \gamma\vartheta$$

on that open set. When $\gamma = 0$, we write $\omega_{\mathrm{KE}} := \omega_{\mathrm{KE},0}$ and we call it a Kähler–Einstein metric.

Remark 3. By [2, Prop. 3.8], a smooth Kähler metric $\omega \in c_1(X_{\mathrm{reg}})$ on X_{reg} satisfying $\mathrm{Ric}\,\omega = \omega$ extends to a Kähler–Einstein metric in the sense of Definition 2 if and only if $\int_{X_{\mathrm{reg}}} \omega^n = c_1(X)^n$. In particular, if $f : Y \to X$ is a (finite) quasi-étale cover between $\mathbb{Q}$-Fano varieties and ω_{KE} is a Kähler–Einstein metric on X, then $f^*\omega_{\mathrm{KE}}$ is a Kähler–Einstein metric on Y.

Let $\omega_X \in c_1(X)$ be a fixed Kähler metric on X. We will systematically make either one of the following assumptions:

Assumption A. *For any $\gamma \in (0,1)$ small enough, there exists a twisted Kähler–Einstein metric $\omega_{\mathrm{KE},\gamma}$ on X relatively to (ω_X, γ).*

Assumption B. *There exists a Kähler–Einstein metric ω_{KE} on X.*

Remark 4. One can rephrase the Assumptions A-B using the algebraic notion of K-stability. It follows from [36] (building upon results of [8–10], [38], [34], [3] in the smooth case) that

- X satisfies Assumption A if and only if X is K-semistable.
- X satisfies Assumption B if X is uniformly K-stable, and the converse holds provided $\mathrm{Aut}^\circ(X) = \{1\}$.

Notation 5. Let $\pi : \widehat{X} \to X$ be a resolution of singularities of X with exceptional divisor $E = \sum_{k\in I} E_k$ and discrepancies $a_k > -1$ given by

$$K_{\widehat{X}} = \pi^* K_X + \sum a_k E_k.$$

There exist numbers $\varepsilon_k \in \mathbb{Q}_+$ such that the cohomology class $\pi^* c_1(X) - \sum \varepsilon_k c_1(E_k)$ contains a Kähler metric $\omega_{\widehat{X}}$. We fix them for the rest of the paper. Next, we pick sections $s_k \in H^0(\widehat{X}, \mathscr{O}_{\widehat{X}}(E_k))$ such that $E_k = (s_k = 0)$, smooth hermitian metrics h_k on $\mathscr{O}_{\widehat{X}}(E_k)$ with Chern curvature $\vartheta_k := i\Theta_{h_k}(E_k)$ and a volume form $\mathrm{d}V$ on $\widehat{X}$ such that $\mathrm{Ric}\,\mathrm{d}V = \pi^*\omega_X - \sum_{k\in I} a_k \vartheta_k$. We set

$$h_E := \prod_{k\in I} h_k \tag{2}$$

which defines a smooth metric on $\mathscr{O}_{\widehat{X}}(E)$.

2.1.2. *The twisted Kähler–Einstein metric and its regularizations*

In this section, we assume that either Assumption A or Assumption B is fulfilled so that there exists a (twisted) Kähler–Einstein metric $\omega_{\mathrm{KE},\gamma}$

- either for any $\gamma \in [0,1)$ such that $0 < \gamma \ll 1$
- or for $\gamma = 0$.

For the time being, the parameter γ is *fixed.*

We denote by $\pi^*\omega_{\mathrm{KE},\gamma} = \pi^*\omega_X + \mathrm{dd}^c\varphi$ the singular metric solving

$$(\pi^*\omega_X + \mathrm{dd}^c\varphi)^n = e^{-(1-\gamma)\varphi} f\,\mathrm{d}V$$

where $f = \prod_{i\in I} |s_i|^{2a_i} \in L^p(\mathrm{d}V)$ for some $p > 1$. It is known that φ is bounded (even continuous) on $\widehat{X}$ and smooth outside E, cf. [2]. Note that φ depends on γ, but as notation will get quite heavy later, we choose not to highlight that dependence.

Next, we choose a family $\psi_\varepsilon \in \mathscr{C}^\infty(\widehat{X})$ of quasi-psh functions on $\widehat{X}$ such that:

- One has $\psi_\varepsilon \to \varphi$ in $L^1(\widehat{X})$ and in $\mathscr{C}^\infty_{\mathrm{loc}}(\widehat{X}\setminus E)$.
- There exists $C > 0$ such that $\|\psi_\varepsilon\|_{L^\infty(\widehat{X})} \le C$.
- There exists a continuous function $\kappa : [0,1] \to \mathbb{R}_+$ with $\kappa(0) = 0$ such that $\pi^*\omega_X + \mathrm{dd}^c\psi_\varepsilon \ge -\kappa(\varepsilon)\omega_{\widehat{X}}$.

This is a standard application of Demailly's regularization results ([11]). The smooth convergence outside E claimed in the first item follows from the explicit expression of the function ψ_ε, see e.g. [13, (3.3)].

For $\varepsilon, t \geq 0$, one introduces the unique function $\varphi_{t,\varepsilon} \in L^\infty(X) \cap \mathrm{PSH}(\widehat{X}, \pi^*\omega_X + t\omega_{\widehat{X}})$ solving

$$\begin{cases} (\pi^*\omega_X + t\omega_{\widehat{X}} + dd^c\varphi_{t,\varepsilon})^n = f_\varepsilon e^{-(1-\gamma)\psi_\varepsilon} e^{-c_t} dV \\ \sup_{\widehat{X}} \varphi_{t,\varepsilon} = 0 \end{cases}$$

where

- $f_\varepsilon := e^{a_\varepsilon} \prod (|s_i|^2 + \varepsilon^2)^{a_i}$,
- a_ε is a normalizing constant such that $\int_{\widehat{X}} f_\varepsilon e^{-(1-\gamma)\psi_\varepsilon} dV = c_1(X)^n$; it converges to 1 when $\varepsilon \to 0$.
- c_t is defined by $\{\pi^*\omega_X + t\omega_{\widehat{X}}\}^n = e^{c_t} \cdot c_1(X)^n$.

The existence and uniqueness of $\varphi_{t,\varepsilon}$ follows from Yau's theorem [39] when $t, \varepsilon > 0$ (in which case $\varphi_{t,\varepsilon}$ is actually smooth) while the general case is treated in [18]. It follows from *ibid.* that there exists a constant $C > 0$ such that

$$\|\varphi_{t,\varepsilon}\|_{L^\infty(X)} \leq C \tag{3}$$

for any $t, \varepsilon \in [0,1]$. Moreover, any weak limit $\widehat{\varphi}$ of a sequence $(\varphi_{t_k,\varepsilon_k})$ is bounded and is a smooth limit outside E. Therefore, it solves the equation

$$(\pi^*\omega_X + dd^c\widehat{\varphi})^n = e^{-(1-\gamma)\varphi} f dV$$

on $\widehat{X}$. By the uniqueness result [18, Thm. A], we have $\widehat{\varphi} = \varphi$. That is

$$\varphi_{t,\varepsilon} \underset{t,\varepsilon \to 0}{\longrightarrow} \varphi \quad \text{in } L^1(\widehat{X}) \text{ and in } \mathscr{C}^\infty_{\mathrm{loc}}(\widehat{X} \setminus E). \tag{4}$$

One sets

$$\omega_{t,\varepsilon} := \pi^*\omega_X + t\omega_{\widehat{X}} + dd^c\varphi_{t,\varepsilon} \tag{5}$$

which solves the equation

$$\mathrm{Ric}\,\omega_{t,\varepsilon} = \pi^*\omega_X + (1-\gamma) dd^c\psi_\varepsilon - \Theta_\varepsilon \tag{6}$$

where

$$\Theta_\varepsilon = \Theta(E, h_E^\varepsilon) = \sum a_i \vartheta_{i,\varepsilon} \tag{7}$$

is the curvature of

$$h_E^\varepsilon = \prod_i (|s_i|^2 + \varepsilon^2)^{-1} h_i \tag{8}$$

and $\vartheta_{i,\varepsilon} = \vartheta_i + dd^c \log(|s_i|^2 + \varepsilon^2)$ converges to the current of integration along E_i when $\varepsilon \to 0$.

2.2. *Stability of T_X.*

Setup and notation as in Section 2.1.

Let $\mathscr{F} \subset T_{\widehat{X}}$ be a subsheaf of positive rank r. We can assume that $\mathscr{F}$ is saturated in $T_{\widehat{X}}$, i.e. $T_{\widehat{X}}/\mathscr{F}$ is torsion-free. This is because saturating a subsheaf increases its slope.

From now on, we choose small numbers $t, \varepsilon > 0$ which we will later let go to zero. The Kähler metric $\omega_{t,\varepsilon}$ defined in (5) induces an hermitian metric $h_{t,\varepsilon}$ on $T_{\widehat{X}}$ which in turn induces a hermitian metric h_F on $F := \mathscr{F}|_W$, where $W \subset \widehat{X}$ is the maximal locus where $\mathscr{F}$ is a subbundle of $T_{\widehat{X}}$. Then, it is classical (see e.g. [29, Rem. 8.5]) that one can compute the slope of $\mathscr{F}$ by integrating the trace of the first Chern form of (F, h_F) over W, i.e.

$$\int_W c_1(F, h_F) \wedge \omega_{t,\varepsilon}^{n-1} = c_1(\mathscr{F}) \cdot \{\omega_{t,\varepsilon}\}^{n-1}. \tag{9}$$

On W, we have the following standard identity (cf. e.g. [12, Thm. 14.5])

$$i\Theta(F, h_F) = \mathrm{pr}_F\left(i\Theta(T_{\widehat{X}}, h_{t,\varepsilon})|_F\right) + \beta_{t,\varepsilon} \wedge \beta_{t,\varepsilon}^*,$$

where $\beta \in \mathscr{C}^\infty_{0,1}(W, \mathrm{Hom}(T_{\widehat{X}}, F))$ (i.e. β is a smooth $(0,1)$-form on W with values in $\mathrm{Hom}(T_{\widehat{X}}, F)$) and β^* is its adjoint with respect to $h_{t,\varepsilon}$ and h_F. Therefore, we get

$$c_1(F, h_F) \wedge \omega_{t,\varepsilon}^{n-1} = \mathrm{tr}_{\mathrm{End}}\left(\mathrm{pr}_F\left(i\Theta(T_{\widehat{X}}, h_{t,\varepsilon})|_F\right)\right) \wedge \omega_{t,\varepsilon}^{n-1} + \mathrm{tr}_{\mathrm{End}}(\beta_{t,\varepsilon} \wedge \beta_{t,\varepsilon}^* \wedge \omega_{t,\varepsilon}^{n-1}). \tag{10}$$

By (9), the integral of the left-hand side over W, yields r times the slope of $\mathscr{F}$ with respect to $\{\pi^*\omega_X + t\omega_{\widehat{X}}\}$. As for the right-hand side, one can simplify the first term using the formula

$$n \cdot i\Theta(T_{\widehat{X}}, h_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} = (\sharp\,\mathrm{Ric}\,\omega_{t,\varepsilon})\,\omega_{t,\varepsilon}^{n}. \tag{11}$$

Here we denote by $\sharp\,\mathrm{Ric}\,\omega_{t,\varepsilon}$ the endomorphism of $T_{\widehat{X}}$ induced by the Ricci curvature of $\omega_{t,\varepsilon}$.

The equation (6) is equivalent to

$$\mathrm{Ric}\,\omega_{t,\varepsilon} = (1-\gamma)\omega_{t,\varepsilon} + \gamma\pi^*\omega_X - t\omega_{\widehat{X}} + (1-\gamma)\mathrm{dd}^c(\psi_\varepsilon - \varphi_{t,\varepsilon}) - \Theta_\varepsilon. \tag{12}$$

Using the formula above, one gets

$$\begin{aligned}\mu_{\omega_{t,\varepsilon}}(\mathscr{F}) \le (1-\gamma)\mu_{\omega_{t,\varepsilon}}(T_X) &+ \frac{1-\gamma}{nr}\underbrace{\int_{\widehat{X}} \mathrm{tr}_{\mathrm{End}}\,\mathrm{pr}_F(\sharp\mathrm{dd}^c(\psi_\varepsilon - \varphi_{t,\varepsilon}))|_F\,\omega_{t,\varepsilon}^n}_{=:(\mathbf{I})} \\ &+ \underbrace{\frac{\gamma}{nr}\int_{\widehat{X}} \mathrm{tr}_{\mathrm{End}}\,\mathrm{pr}_F(\sharp\pi^*\omega_X)|_F\,\omega_{t,\varepsilon}^n}_{=:(\mathbf{II})} - \frac{1}{nr}\underbrace{\int_{\widehat{X}} \mathrm{tr}_{\mathrm{End}}\,\mathrm{pr}_F(\sharp\Theta_\varepsilon)|_F\,\omega_{t,\varepsilon}^n}_{=:(\mathbf{III})} \\ &+ \frac{1}{nr}\underbrace{\int_W \mathrm{tr}_{\mathrm{End}}(\beta_{t,\varepsilon}\wedge\beta_{t,\varepsilon}^*\wedge\omega_{t,\varepsilon}^{n-1})}_{=:(\mathbf{IV})}.\end{aligned}$$

We therefore have four terms to deal with. To deal with (**II**)–(**IV**), we will use the same computations as in [25], cf. explanations below. The main new term is (**I**), which we treat first.

The term (I). It arises from the fact that, say when $\gamma = 1$, we can not necessarily solve the perturbed equation $\mathrm{Ric}\,\omega_{t,\varepsilon} = \omega_{t,\varepsilon} - t\omega_{\widehat{X}} - \Theta_\varepsilon$ unlike in the case where K_X is ample or trivial. If all the discrepancies a_i were negative, one could likely still solve that equation using e.g. properness of Ding functional but we will not expand on that.

In order to deal with (**I**), one makes the following observations:

• Given $\delta > 0$, there exist $\eta = \eta(\delta) > 0$ and an open neighborhood U_δ of $E \subset \widehat{X}$ such that

$$\forall\, \varepsilon, t \le \eta, \quad \int_{U_\delta} (\omega_{\psi_\varepsilon} + \omega_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} \le \delta, \tag{13}$$

where $\omega_{\psi_\varepsilon} = \pi^*\omega_X + t\omega_{\widehat{X}} + \mathrm{dd}^c\psi_\varepsilon$. This inequality is a consequence of the Chern–Levine–Nirenberg inequality along with the bound of the potentials below

$$\exists\, C > 0, \forall\, \varepsilon, t, \quad \|\varphi_{t,\varepsilon}\|_{L^\infty(\widehat{X})} + \|\psi_\varepsilon\|_{L^\infty(\widehat{X})} \le C \tag{14}$$

that we infer from (3). Indeed, as explained in [25], one proceeds as follows. Let $(\Xi_\delta)_{\delta>0}$ be a family of functions defined on $\mathbb{R}_+$, such that $\Xi_\delta(x) = 0$ if $x \le \delta^{-1}$ and $\Xi_\delta(x) = 1$ if $x \ge 1 + \delta^{-1}$. Moreover we can assume that the derivative of Ξ_δ is bounded by a constant independent of δ. Then we evaluate the quantity

$$\int_{\widehat{X}} \Xi_\delta\left(\log\log\frac{1}{|s_E|^2}\right)(\omega_{\psi_\varepsilon} + \omega_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} \tag{15}$$

and the proof of the classical Chern–Levine–Nirenberg (see e.g. [12, III.3 (3.3)]) inequality shows that the integral in (13) is smaller than

$$\int_{U_\delta} \omega_E^n \tag{16}$$

up to a constant which is independent of t, ε. In (16) we denote by ω_E a metric with Poincaré singularities along the divisor E, and by U_δ the support of the truncation function $\Xi_\delta\big(\log\log\frac{1}{|s_E|^2}\big)$. Here the main point is that the norm of the Hessian of the truncation function is uniformly bounded when measured with respect to ω_E. The conclusion follows.

The hermitian endormorphism $\sharp \mathrm{dd}^c(\psi_\varepsilon - \varphi_{t,\varepsilon})$ is dominated (in absolute value) by the positive endomorphism

$$\sharp(\omega_{\psi_\varepsilon} + \omega_{t,\varepsilon})$$

whose endomorphism trace is nothing but $\mathrm{tr}_{\omega_{t,\varepsilon}}(\omega_{\psi_\varepsilon} + \omega_{t,\varepsilon})$. By (13), we are done with (**I**) on U_δ.

- The second observation is that given $K \Subset \widehat{X} \setminus E$, there exists $\eta = \eta(K) > 0$ such that

$$\forall\, \varepsilon, t \le \eta, \quad \|\psi_\varepsilon - \varphi_{t,\varepsilon}\|_{\mathscr{C}^2(K)} \le \delta. \tag{17}$$

This is a consequence of the fact that $(\varphi_{t,\varepsilon})$ and (ψ_ε) converge uniformly (in ε, t) to φ on K by stability of the Monge–Ampère operator, cf. e.g. [24, Thm. C], and have uniformly bounded $\mathscr{C}^p(K)$ norm for any p thanks to (14), Tsuji's trick and Evans–Krylov plus Schauder estimates.

Therefore, one has $\pm \sharp \mathrm{dd}^c(\psi_\varepsilon - \varphi_{t,\varepsilon}) \le \delta \omega_{\widehat{X}}$ hence (**I**) is controlled on K by $\delta \int_K \omega_{\widehat{X}} \wedge \omega_{t,\varepsilon}^n \le C\delta$.

Conclusion. Let $F_{t,\varepsilon} := \mathrm{tr}_{\mathrm{End}}\, \mathrm{pr}_F(\sharp \mathrm{dd}^c(\psi_\varepsilon - \varphi_{t,\varepsilon}))|_F\, \omega_{t,\varepsilon}^n$. One fixes $\delta > 0$. We get a neighborhood U_δ of E and a number $\eta' = \eta'(\delta) > 0$ such that $\int_{U_\delta} F_{t,\varepsilon} \le \delta$ for any $\varepsilon, t \le \eta'$. Applying the second observation to $K = \widehat{X} \setminus U_\delta$, we find $\eta'' = \eta''(\delta)$ such that $\int_{X \setminus U_\delta} F_{t,\varepsilon} \le C\delta$ for any $\varepsilon, t \le \eta''$. Choosing $\eta := \min\{\eta', \eta''\}$, we find that

$$\forall\, \varepsilon, t \le \eta, \quad \int_{\widehat{X}} F_{t,\varepsilon} \le C'\delta.$$

In short, the term (**I**) converges to zero when $\varepsilon, t \to 0$.

The term (II). As $\pi^* \omega_X \ge 0$, one has

$$\begin{aligned} \mathrm{tr}_{\mathrm{End}}\, \mathrm{pr}_F(\sharp \pi^* \omega_X)|_F \omega_{t,\varepsilon}^n &\le \mathrm{tr}_{\mathrm{End}}(\sharp \pi^* \omega_X) \omega_{t,\varepsilon}^n \\ &= \mathrm{tr}_{\omega_{t,\varepsilon}}(\pi^* \omega_X) = n \pi^* \omega_X \wedge \omega_{t,\varepsilon}^{n-1}. \end{aligned}$$

Integrating over X, one finds

$$(\mathbf{II}) \le \gamma r^{-1} (\pi^* c_1(X) \cdot \{\omega_{t,\varepsilon}\}^{n-1})$$

and the right-hand side converges to $\frac{\gamma n}{r} \mu(T_{\widehat{X}})$ when $t \to 0$, where the slope is taken with respect to $\pi^* c_1(X)$.

The term (III). As said above, the arguments to treat this term are borrowed from [25]. For the convenience of the reader, we will recall the important steps. To lighten notation, we will drop the index i. One can write $\Theta_\varepsilon = \frac{\varepsilon^2 |D's|^2}{(|s|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \cdot \vartheta$. Let us set $g_\varepsilon := \frac{\varepsilon^2}{|s|^2 + \varepsilon^2}$. Up to rescaling $\omega_{\widehat{X}}$, one can assume that $-\omega_{\widehat{X}} \le \vartheta \le \omega_{\widehat{X}}$ so that $\Theta_\varepsilon + g_\varepsilon \omega_{\widehat{X}} \ge 0$. Then one sees easily that

$$\begin{aligned} \mathrm{tr}_{\mathrm{End}}\, \mathrm{pr}_F(\sharp \Theta_\varepsilon)|_F\, \omega_{t,\varepsilon}^n &\le \mathrm{tr}_{\mathrm{End}} \Big(\sharp \Theta_\varepsilon + \sharp(g_\varepsilon \omega_X)\Big) \omega_{t,\varepsilon}^n \\ &= \Theta_\varepsilon \wedge \omega_{t,\varepsilon}^{n-1} + g_\varepsilon \omega_{\widehat{X}} \wedge \omega_{t,\varepsilon}^{n-1} \end{aligned}$$

and one obtains that the term (**III**) converges to zero when $\varepsilon, t \to 0$ since

- $\int_X \Theta_\varepsilon \wedge \omega_{t,\varepsilon}^{n-1} = c_1(E) \cdot \{\pi^* \omega_X + t \omega_{\widehat{X}}\}^{n-1}$ and E is exceptional,
- $\int_X g_\varepsilon \omega_{\widehat{X}} \wedge \omega_{t,\varepsilon}^{n-1} \to 0$ when $\varepsilon, t \to 0$ thanks to the smooth convergence to 0 outside E and the Chern–Levine–Nirenberg inequality combined with the bound (3) on the potentials, cf. first item in Part (**I**).

The term (IV). Note that the term $\beta_{t,\varepsilon} \wedge \beta_{t,\varepsilon}^*$ is pointwise negative in the sense of Griffiths on W. In particular, the term (**IV**) is non-positive. Since (**I**) and (**III**) converge to zero, this shows that

$$\mu(\mathscr{F}) \le \big(1 + \gamma\big(\frac{n}{r} - 1\big)\big) \cdot \mu(T_{\widehat{X}}), \tag{18}$$

where the slope is taken with respect to $\pi^* c_1(X)$.

Working under Assumption A, one obtains the inequality (18) above for any $\gamma > 0$ small enough. In particular, this shows that under Assumption A, $T_{\widehat{X}}$ is semistable with respect to $\pi^* c_1(X)$.

From now on, we assume that the stronger Assumption B holds; i.e. one can choose $\gamma = 0$. Assume additionally that there exists a subsheaf $\mathscr{F} \subset T_{\widehat{X}}$ with the same slope as $T_{\widehat{X}}$ and let $\mathscr{F}^{\mathrm{sat}}$ be its saturation in $T_{\widehat{X}}$; it is a subbundle in codimension one. As the slope has not increased by saturation, $\mathscr{F} = \mathscr{F}^{\mathrm{sat}}$ in codimension one on $\widehat{X} \setminus E$. Therefore, if we set $W^\circ := W \cap (\widehat{X} \setminus E)$, then $W^\circ \subset \widehat{X} \setminus E$ has codimension at least two and by the above computation, one has

$$\lim_{\varepsilon, t \to 0} \int_{W^\circ} (\beta_{t,\varepsilon} \wedge \beta^*_{t,\varepsilon} \wedge \omega^{n-1}_{t,\varepsilon}) = 0.$$

We know by (4) that $\beta_{t,\varepsilon} \to \beta_\infty$ locally smoothly on W° when $\varepsilon, t \to 0$ where β_∞ is the second fundamental form induced by the hermitian metric h_{KE} induced by $\pi^*\omega_{\mathrm{KE}}$ on $T_{\widehat{X}}|_{W^\circ}$ and on $\mathscr{F}|_{W^\circ}$ by restriction. By Fatou lemma, we have $\beta_\infty \equiv 0$ on W°, that is, we have a holomorphic decomposition $T_{\widehat{X}}|_{W^\circ} = \mathscr{F}|_{W^\circ} \oplus \mathscr{F}|^\perp_{W^\circ}$ where the orthogonal is taken with respect to h_{KE}.

We are now ready to prove

Theorem 6. *Let X be a $\mathbb{Q}$-Fano variety.*

(i) *If Assumption A is satisfied, then T_X is semistable with respect to $c_1(X)$.*

(ii) *If Assumption B is satisfied, then T_X is polystable with respect to $c_1(X)$. More precisely, we have:*

- *Any saturated subsheaf $\mathscr{F} \subset T_X$ with $\mu(\mathscr{F}) = \mu(T_X)$ is a direct summand of T_X and $\mathscr{F}|_{X_{\mathrm{reg}}} \subset T_{X_{\mathrm{reg}}}$ is a parallel subbundle with respect to ω_{KE}.*
- *There exists a decomposition*

$$T_X = \bigoplus_{i \in I} \mathscr{F}_i$$

such that $\mathscr{F}_i$ is stable with respect to $c_1(X)$, $\mathscr{F}_i|_{X_{\mathrm{reg}}} \subset T_{X_{\mathrm{reg}}}$ is a parallel subbundle with respect to ω_{KE}, and the decomposition $T_{X_{\mathrm{reg}}} = \oplus_{i \in I} \mathscr{F}_i|_{X_{\mathrm{reg}}}$ is orthogonal with respect to ω_{KE}.

Proof. Let $\mathscr{F} \subset T_X$ be a subsheaf and let $\alpha := c_1(X)$. The sheaf $\mathscr{F}$ induces a subsheaf $\mathscr{G}^\circ \subset T_{\widehat{X}}|_{\widehat{X} \setminus E}$ and we denote by $\mathscr{G} \subset T_{\widehat{X}}$ the saturation of $\mathscr{G}^\circ$ in $T_{\widehat{X}}$. By the arguments above (cf. inequality (18) and the comments below it), one has $\mu_{\pi^*\alpha}(\mathscr{G}) \le \mu_{\pi^*\alpha}(T_{\widehat{X}}) = c_1(X)^n/n = \mu_\alpha(T_X)$. Moreover, one has clearly $\mu_{\pi^*\alpha}(\mathscr{G}) = \mu_\alpha(\mathscr{F})$. This shows that T_X is semistable with respect to $c_1(X)$.

Now, assume that there exists a Kähler–Einstein metric ω_{KE}. If $\mathscr{F} \subset T_X$ satisfies $\mu_\alpha(\mathscr{F}) = 0$, then $\mu_{\pi^*\alpha}(\mathscr{G}) = 0$ and we have shown above that $\pi^*\omega_{\mathrm{KE}}$ induces a splitting $T_{\widehat{X}}|_W = \mathscr{G}|_W \oplus (\mathscr{G}|_W)^\perp$ over a Zariski open subset $W \subset \widehat{X} \setminus E$ whose complement in $\widehat{X} \setminus E$ has codimension at least two. Set $V := \pi(W) \subset X_{\mathrm{reg}}$ so that $\mathscr{F}|_V$ is a subbundle of T_X and we have a splitting $T_X|_V = \mathscr{F}|_V \oplus (\mathscr{F}|_V)^\perp$ induced by ω_{KE} and $\operatorname{codim}_X(X \setminus V) \ge 2$.

Let us denote by $j : V \hookrightarrow X$ the open immersion. As $\mathscr{F} \subset T_X$ is saturated, it is reflexive, hence $j_*(\mathscr{F}|_V) = \mathscr{F}$. Moreover, $(\mathscr{F}|_V)^\perp$ extends to a reflexive sheaf $\mathscr{F}^\perp := j_*((\mathscr{F}|_V)^\perp)$ on X satisfying $T_X = \mathscr{F} \oplus \mathscr{F}^\perp$ on the whole X. In particular, $\mathscr{F}$ is a direct summand of T_X and as such, it is subbundle of T_X over X_{reg}. By iterating this process and starting with $\mathscr{F}$ with minimal rank, one can decompose $T_X = \bigoplus_{i \in I} \mathscr{F}_i$ into reflexive sheaves which, over X_{reg}, are parallel (pairwise orthogonal) subbundles with respect to ω_{KE}. $\square$

3. Polystability of the canonical extension

In this section, we keep using the setup and notation of Section 2.1.

3.1. *The canonical extension*

Let $\mathscr{E}$ be a coherent sheaf on X sitting in the exact sequence below

$$0 \longrightarrow \Omega^{[1]}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X \longrightarrow 0. \tag{19}$$

The sheaf $\mathscr{E}$ is automatically torsion-free and it is locally free on X_{reg}.

Remark 7. Let $U \subset X$ be a non-empty Zariski open subset. As an extension of $\mathscr{O}_X$ by $\Omega_X^{[1]}$, $\mathscr{E}|_U$ is uniquely determined by the image of $1 \in H^0(U, \mathscr{O}_X)$ in $H^1(U, \Omega_X^{[1]})$ under the connecting morphism in the long exact sequence arising from $H^0(U, -)$.

From now on, one assumes that the extension class of $\mathscr{E}$ is the image of $c_1(X)$ in $H^1(X, \Omega_X^1)$ under the canonical map

$$\mathrm{Pic}(X) \otimes \mathbb{Q} \simeq H^1(X, \mathscr{O}_X^*) \otimes \mathbb{Q} \to H^1(X, \Omega_X^1) \to H^1(X, \Omega_X^{[1]}).$$

This is legitimate since K_X is $\mathbb{Q}$-Cartier.

Definition 8. *The dual $\mathscr{E}^*$ of the sheaf $\mathscr{E}$ sitting in the exact sequence* (19) *with extension class $c_1(X)$ is called the canonical extension of T_X by $\mathscr{O}_X$.*

The exact sequence (19) is locally splittable since for any affine $U \subset X$, one has $h^1(U, \Omega_U^{[1]}) = 0$. In particular, when one dualizes (19), one see that the canonical extension of T_X by $\mathscr{O}_X$ sits in the short exact sequence below

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E}^* \longrightarrow T_X \longrightarrow 0. \tag{20}$$

The goal of this section is to prove the following, cf. Theorem B.

Theorem 9. *Let X be a $\mathbb{Q}$-Fano variety. If Assumption A (resp. Assumption B) is satisfied, then the canonical extension $\mathscr{E}^*$ of T_X by $\mathscr{O}_X$ is semistable (resp. polystable) with respect to $c_1(X)$.*

The proof of Theorem 9 above is divided into three main steps corresponding to the next three sub-sections. First one can reduce the semistability statement above to a semistability property on the resolution $\widehat{X}$ thanks to Lemma 10, then we prove the said statement, cf. Theorem 11 and, finally, we prove polystability assuming the existence of a Kähler–Einstein metric.

3.2. *Reduction to a statement on the resolution*

Let $\widehat{\mathscr{E}}$ be the vector bundle on $\widehat{X}$ sitting in the exact sequence below

$$0 \longrightarrow \Omega^1_{\widehat{X}} \longrightarrow \widehat{\mathscr{E}} \longrightarrow \mathscr{O}_{\widehat{X}} \longrightarrow 0 \tag{21}$$

such that its extension class is $\pi^* c_1(X) \in H^1(\widehat{X}, \Omega^1_{\widehat{X}})$. Its dual sits in the exact sequence

$$0 \longrightarrow \mathscr{O}_{\widehat{X}} \longrightarrow \widehat{\mathscr{E}}^* \longrightarrow T_{\widehat{X}} \longrightarrow 0. \tag{22}$$

Lemma 10. *If the vector bundle $\widehat{\mathscr{E}}^*$ is semistable with respect to $\pi^* c_1(X)$, then the torsion-free sheaf $\mathscr{E}^*$ is semistable with respect to $c_1(X)$.*

Although slope stability is usually defined with respect to an ample polarization, the same definition actually makes sense with respect to an arbitrary nef class like $\pi^* c_1(X)$, cf e.g. [22].

Proof. Set $\alpha := c_1(X)$. Let $X^\circ \subseteq X_{\mathrm{reg}}$ be an open set with complement of codimension at least 2 in X such that the restriction $\pi_{|\widehat{X}^\circ}$ of π to $\widehat{X}^\circ := \pi^{-1}(X^\circ)$ induces an isomorphism $\widehat{X}^\circ \simeq X^\circ$. By Remark 7 we have

$$(\pi^* \mathscr{E}^*)_{|\widehat{X}^\circ} \simeq \widehat{\mathscr{E}}^*_{|\widehat{X}^\circ}. \tag{23}$$

Let $\mathscr{F} \subseteq \mathscr{E}^*$ be a subsheaf and let $\widehat{\mathscr{F}} \subseteq \widehat{\mathscr{E}}^*$ be the saturated subsheaf of $\widehat{\mathscr{E}}^*$ whose restriction to $\widehat{X}^\circ$ is $(\pi^* \mathscr{F})_{|\widehat{X}^\circ}$. By the projection formula together with the fact that $X \setminus X^\circ$ has codimension at least 2 in X, we have

$$\mu_\alpha(\mathscr{F}) = \mu_{\pi^*\alpha}(\widehat{\mathscr{F}}) \quad \text{and} \quad \mu_\alpha(\mathscr{E}^*) = \mu_{\pi^*\alpha}(\widehat{\mathscr{E}}^*).$$

The lemma follows easily. $\square$

3.3. *Statement on the resolution*

In this section, we prove that the vector bundle $\widehat{\mathscr{E}}^*$ from Section 3.2 is semistable with respect to $\pi^* c_1(X)$, cf. Theorem 11 below. In order to streamline the notation, we set $\mathscr{V} := \widehat{\mathscr{E}}^*$ and in the following we will not distinguish between the locally free sheaf $\mathscr{V}$ and the associated vector bundle. Recall that $\mathscr{V}$ fits into the exact sequence of locally free sheaves

$$0 \longrightarrow \mathscr{O}_{\widehat{X}} \longrightarrow \mathscr{V} \to T_{\widehat{X}} \longrightarrow 0. \tag{24}$$

We denote by $\beta \in H^1(\widehat{X}, T^\star_{\widehat{X}})$ the second fundamental form.

Our result in this section is a singular version of Theorem 0.1 in [37].

Theorem 11. *Let X be a $\mathbb{Q}$-Fano variety satisfying Assumption A. Let $\mathscr{V}$ be the vector bundle on $\widehat{X}$ appearing in* (24)*, whose extension class β coincides with the inverse image of the first Chern class of X by the resolution $\pi : \widehat{X} \to X$. Then $\mathscr{V}$ is semistable with respect to $\pi^\star c_1(X)$.*

Proof. The strategy of proof is as follows. We would like to compute the slope of $\mathscr{F}$ using an hermitian metric on $\mathscr{V}$ induced by the (twisted) Kähler–Einstein metric, using an approximation process as in Section 2.2. As the natural metric in the extension class of $\mathscr{V}$ is singular, we introduce an algebraic 1-parameter family $(\mathscr{V}_z)_{z\in\mathbb{C}}$ that can be endowed with natural smooth hermitian metrics for suitable $z \in \mathbb{R}$ close to zero and such that we have sheaf injections $\mathscr{V} \hookrightarrow \mathscr{V}_t \otimes \mathscr{O}_{\widehat{X}}(E)$. We then proceed to compute slopes following the strategy of Section 2.2.

Step 1. Deformations of $\mathscr{V}$. We pick an arbitrary subsheaf $\mathscr{F} \subseteq \mathscr{V}$ of the vector bundle $\mathscr{V}$ sitting in the exact sequence below

$$0 \to \mathscr{O}_{\widehat{X}} \to \mathscr{V} \to T_{\widehat{X}} \to 0$$

and corresponding to the extension class

$$\alpha = (a_{ij}) \in \operatorname{Ext}^1(T_{\widehat{X}}, \mathscr{O}_{\widehat{X}}) \simeq H^1(\widehat{X}, \mathscr{H}om(T_{\widehat{X}}, \mathscr{O}_{\widehat{X}}))$$

relatively to a covering by open subsets (U_i). The bundle $\mathscr{V}$ can be obtained as follows: on U_i, it is the trivial extension, $\mathscr{V}_{|U_i} = \mathscr{O}_{\widehat{X}|U_i} \oplus T_{\widehat{X}|U_i}$ and the transition functions are given by

$$\begin{pmatrix} \mathrm{Id}_{\mathscr{O}_{\widehat{X}}}|_{U_{ij}} & a_{ij} \\ 0 & \mathrm{Id}_{T_{\widehat{X}}}|_{U_{ij}} \end{pmatrix}.$$

The subsheaf $\mathscr{F}$ is given by two morphisms of sheaves $p_i \colon \mathscr{F}_{|U_i} \to \mathscr{O}_{\widehat{X}|U_i}$ and $q_i \colon \mathscr{F}_{|U_i} \to T_{\widehat{X}|U_i}$ satisfying

$$\begin{cases} p_i|_{U_{ij}} = p_j|_{U_{ij}} + a_{ij} \circ (q_j|_{U_{ij}}), \\ q_i|_{U_{ij}} = q_j|_{U_{ij}}. \end{cases}$$

Recall that we have a reduced divisor $E = E_1 + \cdots + E_r$. Up to refining the covering (U_i), one can assume that E_k is given by the equation $f_{ki} = 0$ on U_i. The transition functions of $\mathscr{O}_{\widehat{X}}(E_k)$ are $g_{k,ij} = \frac{f_{kj}}{f_{ki}}$.

Now, given complex numbers $z_1, \ldots, z_r \in \mathbb{C}$, one considers the extension $\mathscr{V}_{z_1,\ldots,z_r}$ of $T_{\widehat{X}}$ by $\mathscr{O}_{\widehat{X}}$ whose class is

$$\alpha + z_1 \left[\frac{\mathrm{d}g_{1,ij}}{g_{1,ij}} \right] + \cdots + z_r \left[\frac{\mathrm{d}g_{r,ij}}{g_{r,ij}} \right] = \alpha + \sum_k z_k c_1(E_k).$$

Set $\mathscr{V}_{z_1,\ldots,z_r}(E) := \mathscr{V}_{z_1,\ldots,z_r} \otimes \mathscr{O}_{\widehat{X}}(E)$. Then, there is an injection of sheaves

$$\mathscr{F} \subseteq \mathscr{V}_{z_1,\ldots,z_s}(E)$$

extending $\mathscr{F} \subseteq \mathscr{V} \subseteq \mathscr{V}(E)$ for (z_k) in a Zariski open neighborhood of $0 \in \mathbb{C}^r$.

Indeed, consider the morphism $\mathscr{F}_{|U_i} \to \mathscr{V}_{z_1,\dots,z_s}(E)|_{U_i}$ given by $p_i + \sum_k z_k \frac{\mathrm{d}f_{ki}}{f_{ki}} \circ q_i$ on the first factor and q_i on the second. Those morphisms can be glued since one has

$$\frac{\mathrm{d}f_{ki}}{f_{ki}} = \frac{\mathrm{d}g_{k,ij}}{g_{k,ij}} + \frac{\mathrm{d}f_{kj}}{f_{kj}},$$

for any index k. The induced map $\mathscr{F} \to \mathscr{V}_{z_1,\dots,z_s}(E)$ is obviously injective for (z_k) in a Zariski open neighborhood of $0 \in \mathbb{C}^r$.

Now, recall that $\alpha = \pi^* c_1(X)$ and that the Kähler metric $\omega_{\widehat{X}}$ lives in the class $\alpha - \sum \varepsilon_k c_1(E_k)$ for some $\varepsilon_k > 0$, so that the approximate Kähler–Einstein metric $\omega_{t,\varepsilon}$ belongs to $(1+t)\alpha_t$, where

$$\alpha_t := \alpha - \frac{t}{1+t} \sum_k \varepsilon_k c_1(E_k).$$

For any $t \in \mathbb{R}$, we set

$$\mathscr{V}_t := V_{z_1,\dots,z_r} \quad \text{and} \quad \mathscr{V}_t(E) := \mathscr{V}_t \otimes \mathscr{O}_{\widehat{X}}(E)$$

where $z_k := -\frac{t}{1+t} \cdot \varepsilon_k$ for $1 \le k \le r$. This vector bundle $\mathscr{V}_t$ is the extension of $T_{\widehat{X}}$ by $\mathscr{O}_{\widehat{X}}$ with extension class α_t and $\mathscr{V}_t(E)$ comes equipped with a sheaf injection

$$\mathscr{F} \subseteq \mathscr{V}_t(E). \tag{25}$$

Moreover, it is clear from the definition of $\mathscr{V}_{z_1,\dots,z_r}$ that we have

$$c_1(\mathscr{V}_t(E)) = c_1(\mathscr{V}) + c_1(E) \tag{26}$$

for any $t \in \mathbb{R}$.

Step 2. Metric properties of $\mathscr{V}_t(E)$. First of all, we pick one number $\gamma > 0$ as in Assumption A. It will be fixed until the very end of the argument.

We seek to endow $\mathscr{V}_t(E)$ with a suitable smooth hermitian metric, at least when $t > 0$ is small enough. Given that $\mathscr{V}_t(E) = \mathscr{V}_t \otimes \mathscr{O}_{\widehat{X}}(E)$ and that we have already fixed a smooth hermitian metric h_E on $\mathscr{O}_{\widehat{X}}(E)$ in (2), it is enough to construct a hermitian metric on $\mathscr{V}_t$.

Now, we can endow the bundles $\mathscr{O}_{\widehat{X}}$ and $T_{\widehat{X}}$ with the trivial metric and the hermitian metric $h_{t,\varepsilon}$ induced by $\omega_{t,\varepsilon}$, respectively. Now, we set

$$\beta_t = \frac{1}{1+t}\omega_{t,\varepsilon} \in \alpha_t$$

which we view as an element of $\mathscr{C}^\infty_{0,1}(\widehat{X}, T^*_{\widehat{X}})$. Relatively to a fixed $\mathscr{C}^\infty$ splitting of $\mathscr{V}_t$, the direct sum metric $h_{\mathscr{V}_t}$ induced on $\mathscr{V}_t$ has a Chern connection $D_{\mathscr{V}_t}$ which has the following expression

$$D_{\mathscr{V}_t} = \begin{pmatrix} d & -\beta_t \\ \beta_t^* & D_{T_{\widehat{X}}} \end{pmatrix}$$

or equivalently

$$D_{\mathscr{V}_t}(s_1, s_2) = \Big(ds_1 - \beta_t \cdot s_2, \beta_t^\star \cdot s_1 + D_{T_{\widehat{X}}} s_2\Big) \tag{27}$$

where $D_{T_{\widehat{X}}}$ is the Chern connections induced by $h_{t,\varepsilon}$ on $T_{\widehat{X}}$. Of course, it depends strongly on the parameters t, ε. We denote by $\beta_t^\star \in \mathscr{C}^\infty_{1,0}(\widehat{X}, T_{\widehat{X}})$ the adjoint of $\beta_t \in \mathscr{C}^\infty_{0,1}(\widehat{X}, T^*_{\widehat{X}})$. Moreover, the Chern curvature of $D_{\mathscr{V}_t}$ is given by

$$\Theta(\mathscr{V}_t, h_{\mathscr{V}_t}) = \begin{pmatrix} -\beta_t \wedge \beta_t^* & D'_{T^*_{\widehat{X}}} \beta_t \\ \bar\partial \beta_t^* & \Theta(T_{\widehat{X}}, h_{t,\varepsilon}) - \beta_t^* \wedge \beta_t \end{pmatrix},$$

where $D'_{T^*_{\widehat{X}}}$ is the $(1,0)$-part of the Chern connection of $(T^\star_{\widehat{X}}, h^*_{t,\varepsilon})$.

We analyze next several quantities which are playing a role in the evaluation of the curvature of $\mathscr{V}_t$.

• *The factor* β_t. The form β_t is given by

$$\beta_t = \frac{1}{1+t}\sum \omega_{p\overline{q}}\left(\frac{\partial}{\partial z_p}\right)^{\star} \otimes \mathrm{d}z_{\overline{q}}, \tag{28}$$

where $\omega_{p\overline{q}}$ are the coefficients of $\omega_{t,\varepsilon}$ with respect to the coordinates $(z_i)_{i=1,\dots,n}$. Its adjoint is computed by the formula

$$\langle \beta_t \cdot v, w\rangle + \langle v, \beta_t^{\star} \cdot w\rangle = 0, \tag{29}$$

where the first bracket is the standard hermitian product in $\mathbb{C}$ and the second one is the one induced by $(T_{\widehat{X}}, h_{t,\varepsilon})$. We have

$$\beta_t^{\star} = -\frac{1}{1+t}\sum \frac{\partial}{\partial z_i} \otimes \mathrm{d}z_i. \tag{30}$$

We have the following formulas

$$D'_{T^*_{\widehat{X}}}\beta_t = 0, \qquad \overline{\partial}\beta_t^{\star} = 0. \tag{31}$$

The first equality holds since $\omega_{t,\varepsilon}$ is a Kähler metric while the second one is obvious from (30).

Moreover, we have

$$(1+t)^2 \cdot \beta_t \wedge \beta_t^{\star} \wedge \omega_{t,\varepsilon}^{n-1} = -\frac{1}{n}\omega_{t,\varepsilon}^{n} \tag{32}$$

as well as

$$(1+t)^2 \cdot \beta_t^{\star} \wedge \beta_t = \omega_{t,\varepsilon} \otimes \mathrm{Id}_{T_{\widehat{X}}}. \tag{33}$$

• *The curvature of* $\mathscr{V}_t$. If we replace β_t by $(1+t)\sqrt{\mu}\beta_t$ for some positive number μ, this does not affect the complex structure of the bundles at stake but only the metrics. Moreover, we see from the identities (31)-(32)-(33) that the curvature becomes

$$\Theta(\mathscr{V}_t, h_{\mathscr{V}_t}) \wedge \omega_{t,\varepsilon}^{n-1} = \begin{pmatrix} \frac{\mu}{n}\omega_{t,\varepsilon}^{n} & 0 \\ 0 & \Theta(T_{\widehat{X}}, h_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} - \mu\omega_{t,\varepsilon}^{n} \otimes \mathrm{Id}_{T_{\widehat{X}}} \end{pmatrix}.$$

Now we choose μ so that $\frac{\mu}{n} = 1-\mu$, i.e. $\mu := \frac{n}{n+1}$. Recalling (11) and the expression of the Ricci curvature of $\omega_{t,\varepsilon}$ given in (12), we get that

$$\Theta(T_{\widehat{X}}, h_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} - \mu\omega_{t,\varepsilon}^{n} \otimes \mathrm{Id}_{T_{\widehat{X}}} = \frac{1}{n+1}\omega_{t,\varepsilon}^{n} \otimes \mathrm{Id}_{T_{\widehat{X}}} + A_{t,\varepsilon,\gamma}\omega_{t,\varepsilon}^{n},$$

where

$$A_{t,\varepsilon,\gamma} = -\gamma\,\mathrm{Id}_{T_{\widehat{X}}} + \sharp\left[\gamma\pi^*\omega_X - t\omega_{\widehat{X}} + (1-\gamma)\mathrm{dd}^c(\psi_\varepsilon - \varphi_{t,\varepsilon}) - \Theta_\varepsilon\right] \tag{34}$$

is such that the number

$$a_{t,\varepsilon,\gamma} := \frac{1}{n}\int_{\widehat{X}} \mathrm{tr}_{\mathrm{End}}\,\mathrm{pr}_F(A_{t,\varepsilon,\gamma})|_F\,\omega_{t,\varepsilon}^{n}$$

satisfies

$$\limsup_{\gamma\to 0}\limsup_{t\to 0}\limsup_{\varepsilon\to 0} a_{t,\varepsilon,\gamma} = 0 \tag{35}$$

thanks to the computations of Section 2.2.

• *The curvature of* $\mathscr{V}_t(E)$. Finally, we endow $\mathscr{V}_t(E)$ with the metric $h_{\mathscr{V}_t(E)} := h_{\mathscr{V}_t} \otimes h_E$. It satisfies

$$\Theta(\mathscr{V}_t(E), h_{\mathscr{V}_t(E)}) \wedge \omega_{t,\varepsilon}^{n-1} = \frac{1}{n+1}\omega_{t,\varepsilon}^{n} \otimes \mathrm{Id}_{\mathscr{V}_t} + A_{t,\varepsilon,\gamma}\omega_{t,\varepsilon}^{n} + (\Theta_E \wedge \omega_{t,\varepsilon}^{n-1}) \otimes \mathrm{Id}_{\mathscr{V}_t(E)}. \tag{36}$$

where $A_{t,\varepsilon,\gamma}$ is defined in (34) and satisfies (35).

Step 3. The slope inequality. Now, one wants to follow the strategy in Section 2.2 and compute the slope of $\mathscr{F}$ using the induced metric h_{F_t} from $(\mathscr{V}_t(E), h_{\mathscr{V}_t(E)})$ under the sheaf injection (25). The metric h_{F_t} is well-defined only on the locus $W \subset \widehat{X}$ where $F_t := \mathscr{F}|_W$ is a subbundle. As $\mathscr{F}$

may not be saturated in $\mathscr{V}_t(E)$, the complement of W may have codimension one. However, we have the formula

$$\begin{aligned}\mu_{\omega_{t,\varepsilon}}(\mathscr{F}) &= \frac{1}{r}\int_W c_1(F_t,h_{F_t})\wedge\omega_{t,\varepsilon}^{n-1} - c_1(D)\cdot\{\omega_{t,\varepsilon}\}^{n-1}\\ &\le \frac{1}{r}\int_W c_1(F_t,h_{F_t})\wedge\omega_{t,\varepsilon}^{n-1}\\ &\le \mu_{\omega_{t,\varepsilon}}(\mathscr{V}_t(E)) + a_{t,\varepsilon,\gamma} + c_1(E)\cdot\{\omega_{t,\varepsilon}\}^{n-1}\end{aligned}$$

where D is an effective divisor such that $\mathscr{O}_X(D) = \det((\mathscr{V}_t(E)/\mathscr{F})_{\mathrm{tor}})$. Since E is π-exceptional, the conclusion follows from the curvature formula (36) along with (35) and the two easy facts below

- $\mu_{\omega_{t,\varepsilon}}(\mathscr{F}) \to \mu_\alpha(\mathscr{F})$ when $t\to 0$,
- $\mu_{\omega_{t,\varepsilon}}(\mathscr{V}_t(E)) \to \mu_\alpha(\mathscr{V})$ when $t,\varepsilon\to 0$ since E is exceptional, cf. (26).

Theorem 11 is now proved. □

3.4. *Polystability*

In this paragraph, we work under the Assumption B and we aim to prove the second part of Theorem 9, i.e. that $\mathscr{E}^*$ is polystable with respect to $c_1(X)$.

By a standard inductive argument, it is enough to prove that if $\mathscr{F}\subset\mathscr{E}^*$ is any saturated subsheaf with $\mu_{c_1(X)}(\mathscr{F}) = \mu_{c_1(X)}(\mathscr{E}^*)$, then it is holomorphically complemented; i.e. there exists $\mathscr{G}\subset\mathscr{E}^*$ such that $\mathscr{E}^* = \mathscr{F}\oplus\mathscr{G}$.

Let $\mathscr{F}$ be such a subsheaf and let $\widehat{\mathscr{F}}\subset\mathscr{V}$ the induced sheaf on $\widehat{X}$, cf. Lemma 10; it satisfies $\mu_\alpha(\widehat{\mathscr{F}}) = \mu_\alpha(\mathscr{E}^*)$. The same arguments as in the end of Section 2.2 show the orthogonal complement $\widehat{\mathscr{G}}$ of $\widehat{\mathscr{F}}\subset\mathscr{V}_0(E)$ with respect to the well-defined hermitian metric $h_{\mathscr{V}_0(E)}$ on $\widehat{X}\setminus E$ is holomorphic. Note that $\mathscr{V}_0(E)\simeq\widehat{\mathscr{E}}^*$ on $\widehat{X}\setminus E$, hence $\pi_*(\mathscr{V}_0(E)|_{\widehat{X}\setminus E})\simeq\mathscr{E}^*$ by (23).

Now, define $\mathscr{G} := \pi_*\widehat{\mathscr{G}}$ on X_{reg}; this is a coherent subsheaf of $\mathscr{E}^*|_{X_{\mathrm{reg}}}$ by the observation above. We can extend it to a coherent saturated subsheaf $\mathscr{G}\subset\mathscr{E}^*$ across X_{sing}; in particular, $\mathscr{G}$ is reflexive. The injection $\mathscr{F}\oplus\mathscr{G}\hookrightarrow\mathscr{E}^*$ isomorphic over X_{reg}, hence everywhere by reflexivity of the sheaves involved. This concludes the proof of Theorem 9.

4. A splitting theorem

4.1. *Foliations*

In this section, we recollect some results about foliations that we will use later on for the reader's convenience. We refer to [16, §3 and 4] and the references therein for notions around foliations on normal varieties and their singularities.

Here we only recall the notion of weakly regular foliation. Let $\mathscr{F}$ be a foliation of positive rank r on a normal variety X. The r-th wedge product of the inclusion $\mathscr{F}\subseteq T_X$ gives a map

$$\mathscr{O}_X(-K_{\mathscr{F}})\hookrightarrow(\wedge^r T_X)^{**}.$$

We will refer to the dual map

$$\Omega_X^{[r]}\to\mathscr{O}_X(K_{\mathscr{F}})$$

as the *Pfaff field* associated to $\mathscr{F}$. The foliation $\mathscr{F}$ is called *weakly regular* if the induced map

$$(\Omega_X^r\otimes\mathscr{O}_X(-K_{\mathscr{F}}))^{**}\to\mathscr{O}_X$$

is surjective (see [16, §5.1]).

Examples of weakly regular foliations are provided by the following result (see [16, Lem. 5.8]).

Lemma 12. *Let X be a normal variety, and let $\mathscr{F}$ be a foliation on X. Suppose that there exists a distribution $\mathscr{G}$ on X such that $T_X = \mathscr{F} \oplus \mathscr{G}$. Then $\mathscr{F}$ is weakly regular.*

The following lemma says that a weakly regular foliation has mild singularities if its canonical divisor is Cartier and the ambient space has klt singularities (see [16, Lem. 5.9]).

Lemma 13. *Let X be a normal variety with klt singularities, and let $\mathscr{F}$ be a foliation on X. Suppose that $K_{\mathscr{F}}$ is Cartier. If $\mathscr{F}$ is weakly regular, then it has canonical singularities.*

Next, we recall the behaviour of weakly regular foliations with respect to finite covers (see [16, Prop. 5.13]).

Lemma 14. *Let X be a normal variety, let $\mathscr{F}$ be a foliation on X, and let $f\colon X_1 \to X$ be a finite cover. Suppose that each codimension 1 irreducible component of the branch locus of f is $\mathscr{F}$-invariant. Then $\mathscr{F}$ is weakly regular if and only if $f^{-1}\mathscr{F}$ is weakly regular.*

Finally, we recall the behaviour of foliations with canonical singularities with respect to finite covers and birational maps (see [16, Lem. 4.3]).

Lemma 15. *Let $f\colon X_1 \to X$ be a finite cover of normal varieties, and let $\mathscr{F}$ be a foliation on X with $K_{\mathscr{F}}$ $\mathbb{Q}$-Cartier. Suppose that each codimension 1 component of the branch locus of f is $\mathscr{F}$-invariant. If $\mathscr{F}$ has canonical singularities, then $f^{-1}\mathscr{F}$ has canonical singularities as well.*

Recall that $\mathbb{Q}$-divisors D_1 and D_2 are said to be $\mathbb{Q}$-linearly equivalent if there exists an integer $m > 0$ such that mD_1 and mD_2 are linearly equivalent. We write $D_1 \sim_{\mathbb{Q}} D_2$.

Lemma 16. *Let $q\colon Z \to X$ be a birational quasi-projective morphism of normal varieties, and let $\mathscr{F}$ be a foliation on X. Suppose that $K_{\mathscr{F}}$ is $\mathbb{Q}$-Cartier and that $K_{q^{-1}\mathscr{F}} \sim_{\mathbb{Q}} q^*K_{\mathscr{F}}$. If $\mathscr{F}$ has canonical singularities, then $q^{-1}\mathscr{F}$ has canonical singularities as well.*

Proof. By assumption, there exist a normal variety $\overline{Z} \supseteq Z$ and a projective birational morphism $\overline{q}\colon \overline{Z} \to X$ whose restriction to Z is q. The same argument used in the proof of [16, Lem. 4.2] shows that

$$a(E, \overline{Z}, \overline{q}^{-1}\mathscr{F}) = a(E, X, \mathscr{F})$$

for any exceptional prime divisor E over $\overline{Z}$ with non-empty center in Z. The lemma follows easily. □

4.2. *Weakly regular foliations with algebraic leaves*

This section contains a generalization of Theorem 6.1 in [16]. The following result is proved in [16] under the additional assumption that $\mathscr{F}$ has canonical singularities.

Theorem 17. *Let X be a normal projective variety with $\mathbb{Q}$-factorial klt singularities, and let $\mathscr{F}$ be a weakly regular foliation on X with algebraic leaves.*

(1) *Then $\mathscr{F}$ is induced by a surjective equidimensional morphism $p\colon X \to Y$ onto a normal projective variety Y.*
(2) *Moreover, there exists an open subset Y° with complement of codimension at least 2 in Y such that $p^{-1}(y)$ is irreducible for any $y \in Y^\circ$.*

Before we give the proof of Theorem 17, we need to prove a number of auxiliary statements. Throughout the present section, we will be working in the following setup.

Setup 18. Let X and Y be normal quasi-projective varieties, and let $p'\colon X \dashrightarrow Y$ be a dominant rational map with $r := \dim X - \dim Y > 0$. Let Z be the normalization of the graph of p', and let $p\colon Z \to Y$ and $q\colon Z \to X$ be the natural morphisms. Let $\mathscr{F}$ be the foliation induced by p'.

Proposition 19. *Let the setting and notation be as in 18, and assume that $K_{\mathscr{F}}$ is Cartier.*

(1) *Then the Pfaff field $\Omega_X^{[r]} \to \mathscr{O}_X(K_{\mathscr{F}})$ associated to $\mathscr{F}$ induces a map*

$$\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}})$$

which factors through the Pfaff field $\Omega_Z^{[r]} \to \mathscr{O}_Z(K_{q^{-1}\mathscr{F}})$ associated to $q^{-1}\mathscr{F}$. In particular, there exists an effective q-exceptional Weil divisor B on Z such that

$$K_{q^{-1}\mathscr{F}} + B \sim_{\mathbb{Z}} q^*K_{\mathscr{F}}.$$

(2) *Moreover, if E is a q-exceptional prime divisor on Z such that $p(E) = Y$, then $E \subseteq \operatorname{Supp} B$.*

Proof. Let $Z_0 \subseteq Y \times X$ be the graph of p', and denote by $n\colon Z \to Z_0$ the normalization map. Consider the foliation

$$\mathscr{G} := \mathrm{pr}_X^*\mathscr{F} \subseteq \mathrm{pr}_X^* T_X \subseteq \mathrm{pr}_Y^* T_Y \oplus \mathrm{pr}_X^* T_X.$$

Let $\Omega_X^r \to \mathscr{O}_X(K_{\mathscr{F}})$ be the map induced by the Pfaff field $\Omega_X^{[r]} \to \mathscr{O}_X(K_{\mathscr{F}})$. By construction, Z_0 is invariant under $\mathscr{G}$, and hence, there is a factorization:

$$\begin{tikzcd}
\Omega^r_{Y\times X}|_{Z_0} \arrow[r] \arrow[d, two heads] & \mathrm{pr}_X^*\Omega_X^r|_{Z_0} \arrow[r] & (\mathrm{pr}_X^*\mathscr{O}_X(K_{\mathscr{F}}))|_{Z_0} \arrow[d, equal] \\
\Omega^r_{Z_0} \arrow[rr] & & \mathscr{O}_{Y\times X}(K_{\mathscr{G}})|_{Z_0}.
\end{tikzcd}$$

Notice that the foliation induced by $\mathscr{G}$ on Z is $q^{-1}\mathscr{F}$. By [1, Prop. 4.5], the map $\Omega^r_{Z_0} \to (\mathrm{pr}_X^*\mathscr{O}_X(K_{\mathscr{F}}))|_{Z_0}$ extends to a map

$$\Omega_Z^r \to n^*((\mathrm{pr}_X^*\mathscr{O}_X(K_{\mathscr{F}}))|_{Z_0}) \simeq q^*\mathscr{O}_X(K_{\mathscr{F}}),$$

which gives a morphism

$$\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}}).$$

This map factors through the Pfaff field

$$\nu_Z\colon \Omega_Z^{[r]} \to \mathscr{O}_Z(K_{q^{-1}\mathscr{F}})$$

associated to $q^{-1}\mathscr{F}$ away from the closed set where ν_Z is not surjective, which has codimension at least 2 in Z. Hence, there exists an effective Weil divisor B on Z such that

$$K_{q^{-1}\mathscr{F}} + B \sim_{\mathbb{Z}} q^*K_{\mathscr{F}}.$$

Moreover, the morphism $\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}})$ identifies with the composition

$$\Omega_Z^{[r]} \to \mathscr{O}_Z(K_{q^{-1}\mathscr{F}}) \to q^*\mathscr{O}_X(K_{\mathscr{F}})$$

since $q^*\mathscr{O}_X(K_{\mathscr{F}})$ is torsion-free. Note that B is obviously q-exceptional, proving the first item.

The second item follows from [16, Lem. 4.19] by induction on the rank of $\mathscr{F}$ as in the proof of Proposition 4.17 in [16]. Notice that the assumption that the birational morphism is projective in the statement of Lemma 4.19 in [16] is not necessary. □

Corollary 20. *Setting and notation as in Setup 18. Suppose that X has klt singularities. Suppose in addition that $K_{\mathscr{F}}$ is Cartier and that $\mathscr{F}$ is weakly regular.*

(1) *Then the foliation $q^{-1}\mathscr{F}$ is weakly regular and $K_{q^{-1}\mathscr{F}} \sim_{\mathbb{Z}} q^* K_{\mathscr{F}}$.*
(2) *Moreover, if E is a prime q-exceptional divisor on Z, then $p(E) \subsetneq Y$.*

Proof. By Proposition 19 (1), the Pfaff field

$$\Omega_X^{[r]} \to \mathscr{O}_X(K_{\mathscr{F}})$$

associated to $\mathscr{F}$ induces a map

$$\Omega_Z^{[r]} \to q^* \mathscr{O}_X(K_{\mathscr{F}})$$

which factors through the Pfaff field $\Omega_Z^{[r]} \to \mathscr{O}_Z(K_{q^{-1}\mathscr{F}})$ associated to $q^{-1}\mathscr{F}$. On the other hand, by [28, Thm. 1.3], there exists a morphism of sheaves

$$q^*\Omega_X^{[r]} \to \Omega_Z^{[r]}$$

that agrees with the usual pull-back morphism of Kähler differentials wherever this makes sense. One then readily checks that we obtain a commutative diagram as follows:

$$\begin{tikzcd} q^*\Omega_X^{[r]} \arrow[r, two heads] \arrow[d] & q^*\mathscr{O}_X(K_{\mathscr{F}}) \arrow[d, equal] \\ \Omega_Z^{[r]} \arrow[r] & q^*\mathscr{O}_X(K_{\mathscr{F}}). \end{tikzcd}$$

This implies that the map $\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}})$ is surjective. Consequently, this map identifies with the Pfaff field associated to $q^{-1}\mathscr{F}$, proving item (2).

Finally, item (2) is an immediate consequence of item 1 together with Proposition 19 (2). □

As we will see, Theorem 17 is an easy consequence of Lemma 21 and Lemma 22 below.

Lemma 21. *Setting and notation as in 18. Suppose that X has klt singularities and that $\mathscr{F}$ is weakly regular. Then there exists an open subset Y° with complement of codimension at least 2 in Y such that, for any $y \in Y^\circ$, either $p^{-1}(y)$ is empty or any connected component of $p^{-1}(y)$ is irreducible.*

Proof. We argue by contradiction and assume that there exists a prime divisor $D \subset Y$ such that, for a general point $y \in D$, $p^{-1}(y)$ is non-empty and some connected component of $p^{-1}(y)$ is reducible. Let $S \subseteq p^{-1}(D)$ be a subvariety of maximal dimension and dominating D such that for a general point $z \in S$ there is at least two irreducible components of $p^{-1}(p(z))$ passing through z. We will show in Step 2 that S has codimension 2 in Z.

Step 1. Construction. Shrinking Y if necessary, we may assume without loss of generality that p is equidimensional. Replacing X by an open neighborhood of the generic point of $q(S)$, we may also assume that there exists a positive integer m such that

$$\mathscr{O}_X(mK_{\mathscr{F}}) \simeq \mathscr{O}_X.$$

Let $f : X_1 \to X$ be the associated cyclic cover, which is quasi-étale (see [33, Def. 2.52]), and let Z_1 be the normalization of the product $Z \times_X X_1$. The induced morphism $g : Z_1 \to Z$ is then a finite cover.

By [14, Lem. 4.2], there exists a finite cover $Y_2 \to Y$ with Y_2 normal and connected such that the following holds. If Z_2 denotes the normalization of the product $Y_2 \times_Y Z_1$, then the natural morphism $p_2\colon Z_2 \to Y_2$ has reduced fibers over codimension 1 points in Y_2. We may also assume that $Y_2 \to Y$ is a Galois cover. We obtain a commutative diagram as follows:

$$\begin{tikzcd}
Z_2 \arrow[r, "g_1"] \arrow[dd, "p_2"'] & Z_1 \arrow[r, "q_1"] \arrow[d, "g"] \arrow[dd, bend right, "p_1"'] & X_1 \arrow[d, "f"] \\
 & Z \arrow[r, "q"] \arrow[d, "p"] & X \\
Y_2 \arrow[r] & Y. &
\end{tikzcd}$$

Notice that $g \circ g_1\colon Z_2 \to Z$ is a finite Galois cover.

Step 2. Away from a closed subset of codimension at least 3, Z has quotient singularities and the foliation induced by p on Z is weakly regular. Moreover, S has codimension 2 in Z. Notice that X_1 has klt singularities by [30, Prop. 3.16], and that the foliation $\mathscr{F}_{X_1} := f^{-1}\mathscr{F}$ is weakly regular by Lemma 14. Observe now that the foliation $\mathscr{F}_{Z_1} := q_1^{-1}\mathscr{F}_{X_1}$ is given by p_1 and that Z_1 identifies with the normalization of the graph of the rational map $p_1 \circ q_1^{-1}$. Therefore, $\mathscr{F}_{Z_1}$ is weakly regular and

$$K_{\mathscr{F}_{Z_1}} \sim_{\mathbb{Z}} q_1^* K_{\mathscr{F}_{X_1}}$$

by Corollary 20 (1). On the other hand, $\mathscr{F}_{X_1}$ has canonical singularities (see Lemma 13). Applying Lemma 16, we conclude that $\mathscr{F}_{Z_1}$ has canonical singularities as well. This in turn implies that the foliation $\mathscr{F}_{Z_2} := g_2^{-1}\mathscr{F}_{Z_1}$ has also canonical singularities (see Lemma 15). From [14, Lem. 5.4], we conclude that Z_2 has canonical singularities over a big open set contained in Y_2, using the fact that p_2 has reduced fibers over codimension 1 points by construction. In particular, Z_2 has canonical singularities in codimension 2.

Since $g \circ g_1\colon Z_2 \to Z$ is a finite Galois cover, there exists an effective $\mathbb{Q}$-divisor Δ on Z such that

$$K_{Z_2} \sim_{\mathbb{Q}} (g \circ g_1)^*(K_Z + \Delta).$$

Moreover, away from a closed subset of codimension at least 3, $K_Z + \Delta$ is $\mathbb{Q}$-Cartier by [16, Lem. 2.6]), and the pair (Z, Δ) is klt by [30, Prop. 3.16] so that it has Cohen–Macaulay singularities. Then Harstshorne's connectedness theorem implies that S has codimension 2 in Z.

By construction, any irreducible codimension 1 component of the ramification locus of g is q_1-exceptional, and hence invariant under $\mathscr{F}_{Z_1}$ by Corollary 20 (2). It follows from Lemma 14 that $\mathscr{F}_Z := q^{-1}\mathscr{F}$ is weakly regular in codimension 2.

Step 3. End of proof. Let $z \in S$ be a general point. Recall from [21, Prop. 9.3] that z has an analytic neighborhood $U \subseteq Z$ that is biholomorphic to an analytic neighborhood of the origin in a variety of the form $\mathbb{C}^{\dim Z}/G$, where G is a finite subgroup of $\mathrm{GL}(\dim Z, \mathbb{C})$ that does not contain any quasi-reflections. In particular, if W denotes the inverse image of U in the affine space $\mathbb{C}^{\dim Z}$, then the quotient map

$$g_U\colon W \to W/G \simeq U$$

is étale outside of the singular set.

By Lemma 14 again, $\mathscr{F}_Z$ induces a regular foliation on W. Let F_1 and F_2 be irreducible components of $p^{-1}(p(z))$ passing through z with $F_1 \neq F_2$. Note that

$$g_U^{-1}(F_1 \cap U) \cap g_U^{-1}(F_2 \cap U) \neq \emptyset.$$

By general choice of z, F_1 and F_2 are not contained in the singular locus of $\mathscr{F}_Z$, and hence both $g_U^{-1}(F_1 \cap U)$ and $g_U^{-1}(F_2 \cap U)$ are a disjoint union of leaves. But then, any leaf passing through some point of $g_U^{-1}(F_1 \cap U) \cap g_U^{-1}(F_2 \cap U)$ is a connected component of both $g_U^{-1}(F_1 \cap U)$ and

$g_U^{-1}(F_2 \cap U)$. This in turn implies that $F_1 = F_2$, yielding a contradiction. This finishes the proof of the lemma. □

Lemma 22. *Setting and notation as in 18. Suppose that X has klt singularities and that $\mathscr{F}$ is weakly regular. Let E be a prime q-exceptional divisor on Z such that* $\dim p\,(E) \geq \dim Y - 1$.

(1) *Then* $\dim p\,(E) = \dim Y - 1$. *In particular, E is invariant under the foliation on Z induced by p.*
(2) *Moreover, if z is a general point in E, then there exists a curve $T \subseteq E$ passing through z with* $\dim p(T) = 1$ *such that $q(E_{p(t_1)}(t_1)) = q(E_{p(t_2)}(t_2))$ for general points t_1 and t_2 in T, where $E_{p(t)}(t)$ denotes the irreducible component of $E_{p(t)} \subseteq p^{-1}(p(t))$ passing through $t \in T \subset E$.*

Proof. For the reader's convenience, the proof is subdivided into a number of steps.

Step 1. Reduction to the case where $K_{\mathscr{F}}$ is Cartier and proof of (1). Replacing X by an open neighborhood of the generic point of $q(E)$, we may assume without loss of generality that there exists a positive integer m such that

$$\mathscr{O}_X(mK_{\mathscr{F}}) \simeq \mathscr{O}_X.$$

Let $f\colon X_1 \to X$ be the associated cyclic cover, which is quasi-étale (see [33, Def. 2.52]), and let Z_1 be the normalization of the product $Z \times_X X_1$. The induced morphism $g\colon Z_1 \to Z$ is then a finite cover. We obtain a commutative diagram as follows:

$$\begin{tikzcd} Z_1 \arrow[r, "q_1"] \arrow[d, "g"] \arrow[dd, bend right, "p_1"'] & X_1 \arrow[d, "f"] \\ Z \arrow[r, "q"] \arrow[d, "p"] & X \\ Y. & \end{tikzcd}$$

Notice that X_1 has klt singularities by [30, Prop. 3.16], and that the foliation $\mathscr{F}_{X_1} := f^{-1}\mathscr{F}$ is weakly regular by Lemma 14. Observe now that the foliation $\mathscr{F}_{Z_1} := q_1^{-1}\mathscr{F}_{X_1}$ is given by p_1 and that Z_1 identifies with the normalization of the graph of the rational map $p_1 \circ q_1^{-1}$. By item 1 in Corollary 20, $\mathscr{F}_{Z_1}$ is weakly regular. Let E_1 be a prime divisor on Z_1 such that $g(E_1) = E$. Notice that E_1 is q_1-exceptional and that $\dim p\,(E) = \dim p_1(E_1)$. Thus, replacing X by X_1, we may assume without loss of generality that

$$K_{\mathscr{F}} \sim_{\mathbb{Z}} 0.$$

Then, by Corollary 20 (2), we must have $p(E) \subsetneq Y$. It follows that $p(E)$ is a prime divisor on Y since $\dim p\,(E) \geq \dim Y - 1$ by assumption. In particular, E is invariant under the foliation $\mathscr{F}_Z := q^{-1}\mathscr{F}$.

Step 2. The foliation induced by $\mathscr{F}$ on $q(E)$. Set $B := q(E)$, and let $E^\circ \subseteq E \cap Z_{\mathrm{reg}}$ be a non-empty open set. We obtain a commutative diagram as follows:

$$\begin{tikzcd}
E^\circ \arrow[r, "a"] \arrow[d, hook] \arrow[dd, bend right, "j"'] & B \arrow[d, equal] \\
E \arrow[r, two heads] \arrow[d, hook] & B \arrow[d, hook, "i"] \\
Z \arrow[r, "q"] \arrow[d, "p"'] & X \\
Y. &
\end{tikzcd}$$

Shrinking X, if necessary, we may assume without loss of generality that B is smooth. By [28, Thm. 1.3 and Prop. 6.1], there is a factorization

$$\begin{tikzcd}
\Omega^r_X|_B \arrow[r] \arrow[rr, bend left, two heads, "di"] & \Omega^{[r]}_X|_B \arrow[r, "d_{\mathrm{refl}}\, i"'] & \Omega^r_B.
\end{tikzcd}$$

This implies that the map $\Omega^{[r]}_X|_B \to \Omega^r_B$ is surjective.

Claim 23. The foliation $\mathscr{F}_{E^\circ}$ on E° induced by $\mathscr{F}_Z$ is projectable under a.

Proof of Claim 23. Let

$$\nu_X \colon \Omega^{[r]}_X \twoheadrightarrow \mathscr{O}_X(K_{\mathscr{F}}) \quad \text{and} \quad \nu_Z \colon \Omega^{[r]}_Z \to \mathscr{O}_Z(K_{\mathscr{F}_Z})$$

be the Pfaff fields associated to $\mathscr{F}$ and $\mathscr{F}_Z$ respectively. Since E° is invariant by $\mathscr{F}_Z$, there is a factorization

$$\begin{tikzcd}
\Omega^r_Z|_{E^\circ} \arrow[r] \arrow[d, two heads] & \Omega^{[r]}_Z|_{E^\circ} \arrow[r, "\nu_Z|_{E^\circ}"] \arrow[d, "d_{\mathrm{refl}}\, j"'] & \mathscr{O}_Z(K_{\mathscr{F}_Z})|_{E^\circ} \arrow[d, equal] \\
\Omega^r_{E^\circ} \arrow[r, equal] & \Omega^r_{E^\circ} \arrow[r] & \mathscr{O}_Z(K_{\mathscr{F}_Z})|_{E^\circ}.
\end{tikzcd}$$

Recall from the proof of Corollary 1 that there is a commutative diagram

$$\begin{tikzcd}
q^*\Omega^{[r]}_X \arrow[r, two heads, "q^*\nu_X"] \arrow[d, "d_{\mathrm{refl}}\, q"'] & q^*\mathscr{O}_X(K_{\mathscr{F}}) \\
\Omega^{[r]}_Z \arrow[r, two heads, "\nu_Z"] & \mathscr{O}_Z(K_{\mathscr{F}_Z}). \arrow[u, "\wr"]
\end{tikzcd}$$

Finally, by [28, Prop. 6.1], the diagram

$$\begin{tikzcd}
(q^*\Omega^{[r]}_X)|_{E^\circ} \simeq a^*(\Omega^{[r]}_X|_B) \arrow[r, "a^* d_{\mathrm{refl}}\, i"] \arrow[d, "d_{\mathrm{refl}}\, q|_{E^\circ}"'] & a^*\Omega^r_B \arrow[d] \\
\Omega^{[r]}_Z|_{E^\circ} \arrow[r, "d_{\mathrm{refl}}\, j"'] & \Omega^r_{E^\circ}
\end{tikzcd}$$

is commutative as well. Therefore, we have a commutative diagramm as follows:

$$\begin{tikzcd}
(q^*\Omega_X^{[r]})|_{E^\circ} \simeq a^*(\Omega_X^{[r]}|_B) \arrow[rr, two heads, "a^* d_{\mathrm{refl}} i"] \arrow[dd, two heads, "(q^*\nu_X)|_{E^\circ}"'] & & a^*\Omega_B^r \arrow[d] \\
& & \Omega_{E^\circ}^r \arrow[d] \\
(q^*\mathscr{O}_X(K_{\mathscr{F}}))|_{E^\circ} & & \mathscr{O}_Z(K_{\mathscr{F}_Z})|_{E^\circ}. \arrow[ll, "\sim"]
\end{tikzcd}$$

This in turn implies that there is a factorization

$$\begin{tikzcd}
\Omega_X^{[r]}|_B \arrow[r, two heads, "d_{\mathrm{refl}} i"] \arrow[d, two heads, "\nu_X|_B"] & \Omega_B^r \arrow[d] \\
\mathscr{O}_X(K_{\mathscr{F}})|_B \arrow[r, equal] & \mathscr{O}_X(K_{\mathscr{F}})|_B
\end{tikzcd}$$

whose pull-back to E° gives the diagram above. It follows that the map

$$\Omega_B^r \twoheadrightarrow \mathscr{O}_X(K_{\mathscr{F}})|_B$$

is the Pfaff field associated to a weakly regular foliation $\mathscr{F}_B$ of rank r on B such that $da(\mathscr{F}_{E^\circ}) = \mathscr{F}_B$. This completes the proof of the claim. □

Then item (2) is an immediate consequence of Claim 23 above. □

We are now ready to prove Theorem 17.

Proof of Theorem 17. Let $p\colon Z \to Y$ be the family of leaves, and let $q\colon Z \to X$ be the natural morphism. Since p has connected fibers by construction, Lemma 21 applied to $p \circ q^{-1}$ implies that p has irreducible fibers over a big open set contained in Y. Hence, to prove Theorem 17, it suffices to show that $\operatorname{Exc} q$ is empty.

We argue by contradiction and assume that $\operatorname{Exc} q \neq \emptyset$. Let E be an irreducible component of $\operatorname{Exc} q$. Then E has codimension 1 since X is $\mathbb{Q}$-factorial by assumption. Recall from Lemma 21 that $p^{-1}(y)$ is irreducible for a general point y in $p(E)$. Therefore, by Lemma 22, we must have $E = p^{-1}(p(E))$. Moreover, if y is a general point in $p(E)$, then there exists a curve $T \subseteq p(E)$ passing through y such that $q(p^{-1}(t_1)) = q(p^{-1}(t_2))$ for general points t_1 and t_2 in T. Now, there exists a positive integer t such that the cycle theoretic fiber $p^{[-1]}(y)$ is $t[p^{-1}(y)]$ for a general point y in $p(E)$. It follows that the restriction of the map $Y \to \operatorname{Chow}(X)$ to $p(E)$ has positive dimensional fibers, yielding a contradiction. This finishes the proof of the theorem. □

Remark 24. In the setup of Theorem 17, let $p\colon Z \to Y$ be the family of leaves, and let $q\colon Z \to X$ be the natural morphism. If X is only assumed to have klt singularities, then the same argument used in the proof of the theorem shows that q is a small birational map. We have

$$K_{Z/Y} - R(p) \sim_{\mathbb{Q}} q^*K_{\mathscr{F}},$$

where $R(p)$ denotes the ramification divisor of p. In particular, if F denotes the normalization of the closure of a general leaf of $\mathscr{F}$, then

$$K_{\mathscr{F}}|_F \sim_{\mathbb{Q}} K_F.$$

4.3. *A splitting theorem*

The following theorem, advertised in the introduction as Theorem C, is the main result of this section.

Theorem 25. *Let X be a normal projective variety, and let*

$$T_X = \bigoplus_{i \in I} \mathscr{F}_i$$

be a decomposition of T_X into involutive subsheaves with algebraic leaves. Suppose that there exists a $\mathbb{Q}$-divisor Δ such that (X, Δ) is klt. Then there exists a quasi-étale cover $f\colon Y \to X$ as well as a decomposition

$$Y \simeq \prod_{i \in I} Y_i$$

of Y into a product of normal projective varieties such that the decomposition $T_X = \bigoplus_{i \in I} \mathscr{F}_i$ lifts to the canonical decomposition

$$T_{\prod_{i \in I} Y_i} = \bigoplus_{i \in I} \mathrm{pr}_i^* T_{Y_i}.$$

Proof. To prove the theorem, it is obviously enough to consider the case where $I = \{1, 2\}$. Set $\tau(i) = 3 - i$ for each $i \in \{1, 2\}$.

Step 1. Reduction to the case where X is $\mathbb{Q}$-factorial with klt singularities. Let $\pi\colon Z \to X$ be a $\mathbb{Q}$-factorialization, whose existence is established in [31, Cor. 1.37]. Recall that π is a small birational projective morphism and that Z is $\mathbb{Q}$-factorial with klt singularities. Then we have the decomposition

$$T_Z = \pi^{-1}\mathscr{F}_1 \oplus \pi^{-1}\mathscr{F}_2$$

into involutive subsheaves with algebraic leaves.

Suppose that there exist normal projective varieties W_1 and W_2 and a quasi-étale cover

$$g\colon W_1 \times W_2 \to Z$$

such that the decomposition $T_Z = \pi^{-1}\mathscr{F}_1 \oplus \pi^{-1}\mathscr{F}_2$ lifts to the canonical decomposition

$$T_{W_1 \times W_2} = \mathrm{pr}_1^* T_{W_1} \oplus \mathrm{pr}_2^* T_{W_2}.$$

The Stein factorization

$$f\colon Y \to X$$

of $\pi \circ g$ is then a quasi-étale cover, and the natural map

$$W_1 \times W_2 \to Y$$

is a small birational morphism. Moreover, by [30, Prop. 3.16], Y has klt singularities. In particular, it has rational singularities. Lemma 26 below applied to $Y \dashrightarrow W_1 \times W_2$ then implies that X satisfies the conclusion of Theorem 25.

Therefore, replacing X by Z, if necessary, we may assume without loss of generality that X is $\mathbb{Q}$-factorial with klt singularities.

Step 2. Covering construction. By Lemma 12, $\mathscr{F}_i$ is a weakly regular foliation. Therefore, by Theorem 17, $\mathscr{F}_i$ is induced by a surjective equidimensional morphism $p_i\colon X \to T_i$ onto a normal projective variety T_i. Moreover, p_i has irreducible fibers over a big open set contained in T_i. Let F_i be a general fiber of $p_{\tau(i)}$.

Let M_i denote the normalization of the product $F_i \times_{T_i} X$, and let $M_i \to N_i \to X$ denote the Stein factorization of the natural morphism $M_i \to X$. We will show that $N_i \to X$ is a quasi-étale cover. Notice that for any prime P on T_i, $p_i^* P$ is well-defined (see [16, §2.7]) and has irreducible support.

Write $p_i^* P = mQ$ for some prime divisor Q on X and some integer $m \geq 1$. Set $n := \dim X$, and $s := \dim T_i$. By general choice of F_i, we may assume that $F_i \setminus X_{\mathrm{reg}}$ has codimension at least 2 in F_i. In particular, $F_i \cap Q \cap X_{\mathrm{reg}} \neq \emptyset$. Let $x \in F_i \cap Q \cap X_{\mathrm{reg}}$ be a general point. Since $\mathscr{F}_1$ and $\mathscr{F}_2$ are regular foliations at x and $T_X = \mathscr{F}_1 \oplus \mathscr{F}_2$, there exist local analytic coordinates centered at x and $p_i(x)$ respectively such that p_i is given by

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1^m, x_2 \ldots, x_s),$$

and such that F_i is given by the equations

$$x_{s+1} = \cdots = x_n = 0.$$

A straightforward local computation then shows that $N_i \to X$ is a quasi-étale cover over the generic point of $p_i^{-1}(P)$. This immediately implies that $N_i \to X$ is a quasi-étale cover.

Let Y be the normalization of X in the compositum of the function fields $\mathbb{C}(N_i)$, and let $f\colon Y \to X$ be the natural morphism. Set $\mathscr{G}_i := f^{-1}\mathscr{F}_i$. By construction, f is a quasi-étale cover, and $\mathscr{G}_i$ is induced by a surjective equidimensional morphism $q_i\colon Y \to R_i$ with reduced fibers over a big open set contained in R_i. Moreover, there exists a subvariety $G_i \subseteq f^{-1}(F_i)$ such that the restriction $G_i \to R_i$ of q_i to G_i is a birational morphism.

Step 3. End of proof. Let R_i° denote the smooth locus of R_i, and set $Y_i^\circ := q_i^{-1}(R_i^\circ)$. Let $Z_i^\circ \subseteq Y_i^\circ$ be the open set where $q_i|_{Y_i^\circ}$ is smooth. Notice that Z_i° has complement of codimension at least 2 in Y_i° since q_i has reduced fibers over a big open set contained in R_i.

The restriction of the tangent map

$$Tq_i|_{Y_i^\circ}\colon T_{Y_i^\circ} \to \big(q_i|_{Y_i^\circ}\big)^* T_{R_i^\circ}$$

to $\mathscr{G}_{\tau(i)}|_{Z_i^\circ} \subseteq T_{Z_i^\circ}$ then induces an isomorphism $\mathscr{G}_{\tau(i)}|_{Z_i^\circ} \simeq \big(q_i|_{Z_i^\circ}\big)^* T_{R_i^\circ}$. Since $\mathscr{G}_{\tau(i)}|_{Y_i^\circ}$ and $\big(q_i|_{Y_i^\circ}\big)^* T_{R_i^\circ}$ are both reflexive sheaves, we finally obtain an isomorphism

$$\mathscr{G}_{\tau(i)}|_{Y_i^\circ} \simeq \big(q_i|_{Y_i^\circ}\big)^* T_{R_i^\circ}.$$

A classical result of complex analysis says that complex flows of vector fields on analytic spaces exist (see [27]). It follows that $q_i|_{Y_i^\circ}$ is a locally trivial analytic fibration for the analytic topology.

The morphism $q_1 \times q_2\colon Y \to R_1 \times R_2$ then induces an isomorphism

$$q_1^{-1}(R_1^\circ) \cap q_2^{-1}(R_2^\circ) \simeq R_1^\circ \times R_2^\circ$$

since $G_1 \cdot G_2 = 1$ and q_i is locally trivial over R_i°. In particular, $q_1 \times q_2$ is a small birational morphism. By [30, Prop. 3.16] again, Y has klt singularities. Hence, it has rational singularities. Lemma 26 below applied to $q_1 \times q_1$ then implies that X satisfies the conclusion of Theorem 25, completing the proof of the theorem. □

Lemma 26 ([32, Prop. 18]). *Let X, Y_1 and Y_2 be normal projective varieties, and let $\pi\colon X \dashrightarrow Y_1 \times Y_2$ be a birational map that does not contract any divisor. Suppose in addition that X has rational singularities. Then X decomposes as a product $X \simeq X_1 \times X_2$ and there exist birational maps $\pi_i\colon X_i \dashrightarrow Y_i$ such that $\pi = \pi_1 \times \pi_2$.*

5. Proof of Theorem A

The present section is devoted to the proof of Theorem A.

Proof of Theorem A. We have seen in Theorem 6 that the tangent sheaf of X is polystable. By definition it means that we have a decomposition

$$T_X = \bigoplus_{i \in I} \mathscr{F}_i$$

where the $\mathscr{F}_i$ are stable with respect to $c_1(X)$ and have the same slope. Moreover, each subsheaf $\mathscr{F}_i$ defines on X_{reg} a parallel subbundle of $T_{X_{\mathrm{reg}}}$ with respect to the Kähler–Einstein metric $\omega_{\mathrm{KE}}|_{X_{\mathrm{reg}}}$. This immediately implies that $\mathscr{F}_i|_{X_{\mathrm{reg}}}$ is involutive.

Claim 27. Each foliation $\mathscr{F}_i$ has algebraic leaves.

Proof. Let m be a positive integer such that $-mK_X$ is very ample, and let $C \subset X$ be a general complete intersection curve of elements in $|-mK_X|$. By general choice of C, we may assume that $C \subset X_{\mathrm{reg}}$ and that $\mathscr{F}_i$ is locally free in a neighborhood of C. If m is large enough, then the vector bundle $\mathscr{F}_i|_C$ is semistable by [19, Thm. 1.2]). We conclude that it is ample since it has positive slope. Then [5, Fact 2.1.1] says that $\mathscr{F}_i$ has algebraic leaves. Alternatively, one can apply [7, Thm. 1.1] to the foliation $\widehat{\mathscr{F}_i}$ on the resolution $\widehat{X}$ (cf. Notation 5) induced by $\mathscr{F}_i$ by pullback over X_{reg} and saturation inside $T_{\widehat{X}}$. □

Let $f : Y \to X$ be the quasi-étale cover and $Y = \prod_{i \in I} Y_i$ be the splitting that are both provided by Theorem 25. The decomposition

$$T_Y = \bigoplus_{i \in I} \mathrm{pr}_i^* T_{Y_i} \tag{37}$$

is a decomposition of T_Y into summands of maximal slope. If there exists $i \in I$ such that T_{Y_i} is not stable with respect to $c_1(Y_i)$, then it means that the polystable decomposition of T_Y provided by Theorem 6 via $f^*\omega_{\mathrm{KE}}$ refines strictly the decomposition (37). By applying Theorem 25 again, we can find another quasi-étale cover $Y' \to Y$ which splits according to the polystable decomposition of T_Y and one can then compare again the polystable decomposition of $T_{Y'}$ to the one coming from T_Y. After finitely many such steps, one can find a quasi-étale cover $g : Z \to X$ such that

(i) There exists a splitting $Z = \prod_{k \in K} Z_k$ into a product of $\mathbb{Q}$-Fano varieties.
(ii) For any $k \in K$, the tangent sheaf T_{Z_k} is stable with respect to $c_1(Z_k)$.
(iii) The variety Z admits a Kähler–Einstein metric given by $g^*\omega_{\mathrm{KE}}$.

Theorem A is a consequence of the Claim below.

Claim 28. There exists a Kähler–Einstein metric ω_k on each variety Z_k such that $g^*\omega = \sum_{k \in K} \mathrm{pr}_k^* \omega_k$.

Proof of Claim 28. We set $n_k := \dim Z_k$. As the saturated subsheaf $\mathscr{F}_k := \mathrm{pr}_k^* T_{Z_k} \subset T_Z$ is stable with maximal slope with respect to $c_1(Z)$, it has to coincide with one of the factors in the decomposition of T_Z provided by Theorem 6 (one can see that by looking at the projections on each factor and use stability). In particular, the $\mathscr{F}_k|_{Z_{\mathrm{reg}}}$ are mutually orthogonal with respect to $g^*\omega_{\mathrm{KE}}$, which enables one to define a smooth hermitian metric ω_k on Z_k^{reg} such that $g^*\omega_{\mathrm{KE}} = \sum_{k \in K} \mathrm{pr}_k^* \omega_k$ on Z_{reg}. Since $g^*\omega_{\mathrm{KE}}$ is closed and d commutes with pr_k^*, it follows that ω_k is a Kähler metric on Z_{reg}.

Clearly, one has $\mathrm{Ric}\,\omega_k = \omega_k$ on Z_k^{reg}. In order to check that ω_k defines a Kähler–Einstein metric on Z_k in the sense of Definition 2, it is sufficient to check that $\int_{Z_k^{\mathrm{reg}}} \omega_k^{n_k} = c_1(Z_k)^{n_k}$ by Remark 3. By [2, Prop. 3.8] we always have the inequality $\int_{Z_k^{\mathrm{reg}}} \omega_k^{n_k} \leq c_1(Z_k)^{n_k}$ and therefore

$$c_1(Z)^n = \int_{Z_{\mathrm{reg}}} g^*\omega_{\mathrm{KE}}^n = \prod_{k \in K} \int_{Z_k^{\mathrm{reg}}} \omega_k^{n_k} \leq \prod_{k \in K} c_1(Z_k)^{n_k}.$$

Since $c_1(Z)^n = \prod_{k \in K} c_1(Z_k)^{n_k}$, one must have $\int_{Z_k^{\mathrm{reg}}} \omega_k^{n_k} = c_1(Z_k)^{n_k}$ for all $k \in K$. □

Theorem A is now proved. □

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