ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Xing-Wang Jiang and Ya-Li Li

On the cardinality of subsequence sums II

Volume 362 (2024), p. 1279-1285

Online since: 14 November 2024

https://doi.org/10.5802/crmath.613

CENTRE MERSENNE
The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2024, Vol. 362, p. 1279-1285
https://doi.org/10.5802/crmath.613

Research article / *Article de recherche*
Number theory / *Théorie des nombres*

On the cardinality of subsequence sums II

Sur le cardinal de sommes de sous-suites II

Xing-Wang Jiang [a] **and Ya-Li Li** [b]

[a] Department of Mathematics, Luoyang Normal University, Luoyang 471934, P. R. China

[b] School of Mathematics and Statistics, Henan University, Kaifeng 475001, P. R. China

E-mails: xwjiangnj@sina.com, njliyali@sina.com

Abstract. Let $A = (\underbrace{a_1,\dots,a_1}_{r_1},\underbrace{a_2,\dots,a_2}_{r_2},\dots,\underbrace{a_k,\dots,a_k}_{r_k})$ be a finite sequence of integers with $a_1 < a_2 < \cdots < a_k$ and $r_i \ge 1$ $(1 \le i \le k)$. The sum of all terms of a subsequence B of A is called a subsequence sum of A and we denote it by $\sigma(B)$. For $0 \le \alpha \le \sum_{i=1}^{k} r_i$, let $\Sigma_\alpha(A) = \{\sigma(B) | B$ is a subsequence of A of length $\ge \alpha\}$. In this paper, we completely settle the two problems posed by Bhanja and Pandey about finding the optimal lower bound of $|\Sigma_\alpha(A)|$ and determining the structure of the sequence A for which the lower bound of $|\Sigma_\alpha(A)|$ is optimal.

Résumé. Soit $A = (\underbrace{a_1,\dots,a_1}_{r_1},\underbrace{a_2,\dots,a_2}_{r_2},\dots,\underbrace{a_k,\dots,a_k}_{r_k}))$ une suite finie d'entiers avec $a_1 < a_2 < \cdots < a_k$ et $r_i \ge 1$ $(1 \le i \le k)$. La somme de tous les termes d'une sous-suite B de A est appelée somme de sous-suite de A et nous la désignons par $\sigma(B)$. Pour $0 \le \alpha \le \sum_{i=1}^{k} r_i$, soit $\Sigma_\alpha(A) = \{\sigma(B) | B$ une sous-suite de A de longueur $\ge \alpha\}$. Dans cet article, nous résolvons complètement les deux problèmes posés par Bhanja et Pandey concernant la recherche de la borne inférieure de $|\Sigma_\alpha(A)|$ et la détermination de la structure de la suite A pour laquelle la borne inférieure de $|\Sigma_\alpha(A)|$ est atteinte.

Keywords. Subsequence sums, direct problem, inverse problem.

Mots-clés. Sommes de sous-suites, problème direct, problème inverse.

2020 Mathematics Subject Classification. 11P70, 11B25.

Funding. This work is supported by the National Natural Science Foundation of China (Grant Nos. 12201281, 12171243, 11901156) and the Natural Science Youth Foundation of Henan Province (Grant No. 222300420245).

Manuscript received 5 July 2023, revised 22 October 2023, accepted 15 January 2024.

1. Introduction

Let $k \ge 1$ and $\overline{r} = (r_1, r_2, \dots, r_k)$ with $r_i \ge 1$ $(1 \le i \le k)$. And let

$$A = (\underbrace{a_1,\dots,a_1}_{r_1},\underbrace{a_2,\dots,a_2}_{r_2},\dots,\underbrace{a_k,\dots,a_k}_{r_k})$$

be a finite sequence of integers, where $a_1 < a_2 < \cdots < a_k$ and a_i repeats r_i times. For convenience, we denote this sequence by $A = (a_1, a_2, \dots, a_k)_{\overline{r}}$. For a given subsequence B of A, the sum $\sigma(B) := \sum_{b \in B} b$ is called a subsequence sum of A. Here, we assume that $\sigma(\emptyset) = 0$. For $0 \le \alpha \le \sum_{i=1}^{k} r_i$, let $\Sigma_\alpha(A)$ be the set of all subsequence sums corresponding to the subsequences of A that are of the size at least α and $\Sigma^{\alpha}(A)$ be the set of all subsequence sums corresponding to the subsequences of A that are of the size at most $\sum_{i=1}^{k} r_i - \alpha$, that is,

$$\Sigma_{\alpha}(A) = \{\sigma(B) \mid B \text{ is a subsequence of } A \text{ of length} \geq \alpha\},$$

$$\Sigma^{\alpha}(A) = \left\{\sigma(B) \,\middle|\, B \text{ is a subsequence of } A \text{ of length} \leq \sum_{i=1}^{k} r_i - \alpha\right\}.$$

Briefly, we write $\Sigma_1(A)$ as $\Sigma(A)$. Obviously, $|\Sigma_{\alpha}(A)| = |\Sigma^{\alpha}(A)|$ and $0 \in \Sigma^{\alpha}(A)$. When $r_i = 1$ $(1 \leq i \leq k)$, A is a set of integers, and now we call the subsequence sums as subset sums. The subset and subsequence sums are fundamental objects in additive number which is very useful in some other combinatorial problems such as the zero-sum problems. The direct problem for the subsequence sums $\Sigma_{\alpha}(A)$ is to find the optimal lower bound of $|\Sigma_{\alpha}(A)|$ and the inverse problem for $\Sigma_{\alpha}(A)$ is to determine the structure of A for which $|\Sigma_{\alpha}(A)|$ is minimal.

In 1995, Nathanson [8] firstly studied the direct and inverse problems for $\Sigma(A)$ for sets A of integers. In 2015, Mistri, Pandey and Prakash [7] generalized Nathanson's results to the subsequence sums $\Sigma(A)$ for sequences A which contains either only nonnegative integers or only nonpositive integers. Later, the present author and Li [5] settled the remaining case, i.e., the sequence A contains positive integers, negative integers, and/or zero. In 2020, Bhanja and Pandey [2] considered the direct and inverse problems for subsequence sums $\Sigma_{\alpha}(A)$ when the sequence A contains nonnegative or nonpositive integers. Some conclusions are as follows.

Theorem A ([2, Theorem 3.1]). *Let $k \geq 1$ be an integer and $A = (a_1, a_2, \ldots, a_k)_{\overline{r}}$ be a finite sequence of integers, where $0 < a_1 < a_2 < \cdots < a_k$ and $\overline{r} = (r_1, r_2, \ldots, r_k)$ with $r_i > 0$ for $1 \leq i \leq k$. Let $0 \leq \alpha < \sum_{i=1}^{k} r_i$. Then there exists an integer $m \in [1, k]$ such that $\sum_{i=1}^{m-1} r_i \leq \alpha < \sum_{i=1}^{m} r_i$ and*

$$|\Sigma_{\alpha}(A)| \geq \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right) + 1.$$

Theorem B ([2, Theorem 3.2 and Remark 3.1]). *Let $k \geq 1$ and $\overline{r} = (r_1, r_2, \ldots, r_k)$ with $r_i > 0$ for $1 \leq i \leq k$. For $0 \leq \alpha \leq \sum_{i=1}^{k} r_i - 2$, let $m \in [1, k]$ be the integer such that $\sum_{i=1}^{m-1} r_i \leq \alpha < \sum_{i=1}^{m} r_i$. If $A = (a_1, a_2, \ldots, a_k)_{\overline{r}}$ is a finite sequence of integers with $0 < a_1 < a_2 < \cdots < a_k$ and*

$$|\Sigma_{\alpha}(A)| = \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha\right) + 1,$$

then

$$A = (a_1, 2a_1, \ldots, k a_1)_{\overline{r}},$$

except that $A = (a_1, a_2)_{\overline{r}}$ with $\overline{r} = (1, r_2)$ and $A = (a_1, a_2, a_1 + a_2)_{\overline{r}}$ with $\overline{r} = (1, 1, r_3)$, where a_1, a_2 are two arbitrary positive integers.

Recently, Bhanja and Pandey [3] posed the following open problems.

Problem 1 ([3, Problem (1)]). *It is important to find the optimal lower bound of $|\Sigma_{\alpha}(A)|$ for arbitrary finite sequence of integers.*

Problem 2 ([3, Problem (2)]). *It is also an important problem to study the structure of the sequence A for which the lower bound for $|\Sigma_{\alpha}(A)|$ is optimal.*

They [3] settled Problem 1 for the special case $r_i = r$ for all $1 \leq i \leq k$. For more related results, one may refer to [1, 4, 6, 9–11]. Problem 1 is the direct problem of the subsequence sums and Problem 2 is the inverse problem of the subsequence sums. In this paper, we solve completely the above two problems. In Section 2, Theorem 1 and Corollary 2 give the answer to Problem 1. In section 3, Theorem 3 and Corollary 4 give the answer to Problem 2.

Next, we give some definition and notations. Define $\sum_{i=u}^{v} f(i) = 0$ if $u > v$. For any sequences A and B, we denote by $A \cup B$ the sequence obtained by merging the sequences A and B and reordering the terms in nondecreasing order. For an integer a, $a + A$ and $a * A$ are defined respectively to be the sequences as follows:

$$a + A = (a + a_1, a + a_2, \dots, a + a_k)_{\overline{r}},\ a * A = (a * a_1, a * a_2, \dots, a * a_k)_{\overline{r}}.$$

For integers $l < k$, we write $(l, l+1, \dots, k)_{\overline{r}}$ as $[l, k]_{\overline{r}}$ briefly.

2. Direct problem

Firstly, we consider the direct problem of the sequence A does not contain zero terms.

Theorem 1. *Let $l, k \ge 1$ be two integers and $A = (a_{-l}, a_{-l+1}, \dots, a_{-1})_{\overline{r}} \cup (a_1, a_2, \dots, a_k)_{\overline{r'}}$ be a finite sequence of integers, where $a_{-l} < a_{-l+1} < \cdots < a_{-1} < 0 < a_1 < a_2 < \cdots < a_k$ and $\overline{r} = (r_{-l}, r_{-l+1}, \dots, r_{-1})$, $\overline{r'} = (r_1, r_2, \dots, r_k)$ with $r_i > 0$ for $i \in [-l, k] \setminus \{0\}$. For $0 \le \alpha < \sum_{i=1}^{l} r_{-i} + \sum_{i=1}^{k} r_i$, let*

$$L_\alpha = \begin{cases} \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i + 1, & \text{if } \alpha \le \min\left\{\sum_{i=1}^{l} r_{-i}, \sum_{i=1}^{k} r_i\right\}; \\ \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i \\ \qquad + m\left(\sum_{i=1}^{l} r_{-i} + \sum_{i=1}^{m} r_i - \alpha\right) + 1, & \text{if } \sum_{i=1}^{l} r_{-i} < \alpha \le \sum_{i=1}^{k} r_i; \\ \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m'} i r_{-i} \\ \qquad + m'\left(\sum_{i=1}^{k} r_i + \sum_{i=1}^{m'} r_{-i} - \alpha\right) + 1, & \text{if } \sum_{i=1}^{k} r_i < \alpha \le \sum_{i=1}^{l} r_{-i}; \\ \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{l} r_{-i} + \sum_{i=1}^{m} r_i - \alpha\right) \\ \qquad - \sum_{i=1}^{m'} i r_{-i} + m'\left(\sum_{i=1}^{k} r_i + \sum_{i=1}^{m'} r_{-i} - \alpha\right) + 1, & \text{if } \alpha > \max\left\{\sum_{i=1}^{l} r_{-i}, \sum_{i=1}^{k} r_i\right\}, \end{cases}$$

where m, m' are integers which satisfy $\sum_{i=1}^{m-1} r_i \le \alpha - \sum_{i=1}^{l} r_{-i} < \sum_{i=1}^{m} r_i$ and $\sum_{i=1}^{m'-1} r_{-i} \le \alpha - \sum_{i=1}^{k} r_i < \sum_{i=1}^{m'} r_{-i}$, respectively. Then

$$|\Sigma_\alpha(A)| \ge L_\alpha.$$

Moreover, the lower bound is optimal.

Proof of Theorem 1. Let

$$I = \sum_{i=1}^{l} r_{-i} a_{-i},\ J = \sum_{i=1}^{k} r_i a_i$$

and

$$A_1 = (a_{-l}, a_{-l+1}, \dots, a_{-1})_{\overline{r}},\ A_2 = (a_1, a_2, \dots, a_k)_{\overline{r'}}.$$

Clearly, $-\Sigma_\beta(A_1) = \Sigma_\beta(-A_1)$ for any nonnegative integer β, and the terms in $\Sigma_\beta(-A_1)$ are positive. Let

$$A_0 = [-l, -1]_{\overline{r}} \cup [1, k]_{\overline{r'}}.$$

If $\alpha \le \min\left\{\sum_{i=1}^{l} r_{-i}, \sum_{i=1}^{k} r_i\right\}$, then $I + J - \Sigma(A_1), I + J - \Sigma(A_2)$ and $\{I + J\}$ are included in $\Sigma_\alpha(A)$ and it is clear that they are pairwise disjoint. Thus,

$$\begin{aligned} |\Sigma_\alpha(A)| &\ge |I + J - \Sigma(A_1)| + |I + J - \Sigma(A_2)| + |\{I + J\}| \\ &= |\Sigma(-A_1)| + |\Sigma(A_2)| + 1 \\ &\ge \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i + 1 = L_\alpha, \end{aligned} \tag{1}$$

where the third inequality comes from Theorem A. Moreover,

$$\Sigma_\alpha(A_0) \subseteq \left[\sum_{i=1}^{l}(-i)r_{-i}, \sum_{i=1}^{k} i r_i\right].$$

Then $|\Sigma_\alpha(A_0)| \le L_\alpha$. Thus, $|\Sigma_\alpha(A_0)| = L_\alpha$. Therefore, at this point, L_α is optimal.

If $\sum_{i=1}^{l} r_{-i} < \alpha \le \sum_{i=1}^{k} r_i$, then $\sum_{i=1}^{m-1} r_i \le \alpha - \sum_{i=1}^{l} r_{-i} := \alpha_1 < \sum_{i=1}^{m} r_i$. At this point, it is clear that $I + \Sigma_{\alpha_1}(A_2)$ and $I + J - \Sigma(A_1)$ are included in $\Sigma_\alpha(A)$. Since

$$\max\{I + \Sigma_{\alpha_1}(A_2)\} = I + J < \min\{I + J - \Sigma(A_1)\} = I + J - a_{-1},$$

these two sets are disjoint. Thus,

$$\begin{aligned}
|\Sigma_\alpha(A)| &\ge |I + \Sigma_{\alpha_1}(A_2)| + |I + J - \Sigma(A_1)| \\
&= |\Sigma_{\alpha_1}(A_2)| + |\Sigma(-A_1)| \\
&\ge \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha_1\right) + 1 + \sum_{i=1}^{l} i r_{-i} = L_\alpha, \qquad (2)
\end{aligned}$$

where the third inequality comes from Theorem A. Moreover,

$$\Sigma_\alpha(A_0) \subseteq \left[\sum_{i=1}^{l}(-i)r_{-i} + \sum_{i=1}^{m} i r_i - m\left(\sum_{i=1}^{m} r_i - \alpha_1\right), \sum_{i=1}^{k} i r_i\right].$$

Then $|\Sigma_\alpha(A_0)| \le L_\alpha$. Thus, $|\Sigma_\alpha(A_0)| = L_\alpha$. Therefore, at this point, L_α is optimal.

If $\sum_{i=1}^{k} r_i < \alpha \le \sum_{i=1}^{l} r_{-i}$, then by $|\Sigma_\alpha(A)| = |\Sigma_\alpha(-A)|$ and similar to the above argument for $-A$, we can obtain that

$$|\Sigma_\alpha(A)| = |\Sigma_\alpha(-A)| \ge \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m'} i r_{-i} + m'\left(\sum_{i=1}^{k} r_i + \sum_{i=1}^{m'} r_{-i} - \alpha\right) + 1 = L_\alpha.$$

Moreover, we can get that $|\Sigma_\alpha(A_0)| = L_\alpha$ and so at this point, L_α is optimal.

If $\alpha > \max\{\sum_{i=1}^{l} r_{-i}, \sum_{i=1}^{k} r_i\}$, then there exist integers m, m' such that

$$\sum_{i=1}^{m-1} r_i \le \alpha - \sum_{i=1}^{l} r_{-i} := \alpha_1 < \sum_{i=1}^{m} r_i, \quad \sum_{i=1}^{m'-1} r_{-i} \le \alpha - \sum_{i=1}^{k} r_i := \alpha_2 < \sum_{i=1}^{m'} r_{-i}.$$

At this point, $I + \Sigma_{\alpha_1}(A_2)$ and $I + J - \Sigma^{\alpha_2}(A_1)$ are included in $\Sigma_\alpha(A)$. It follows from

$$\max\{I + \Sigma_{\alpha_1}(A_2)\} = I + J = \min\{I + J - \Sigma^{\alpha_2}(A_1)\}$$

that

$$\begin{aligned}
|\Sigma_\alpha(A)| &\ge |I + \Sigma_{\alpha_1}(A_2)| + |I + J - \Sigma^{\alpha_2}(A_1)| - |\{I + J\}| \\
&= |\Sigma_{\alpha_1}(A_2)| + |\Sigma_{\alpha_2}(-A_1)| - 1 \\
&\ge \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=1}^{m} r_i - \alpha_1\right) + \sum_{i=1}^{l} i r_{-i} - \sum_{i=1}^{m'} i r_{-i} + m'\left(\sum_{i=1}^{m'} r_{-i} - \alpha_2\right) + 1 \\
&= L_\alpha, \qquad (3)
\end{aligned}$$

where the third inequality comes from Theorem A. Moreover,

$$\Sigma_\alpha(A_0) \subseteq \left[\sum_{i=1}^{l}(-i)r_{-i} + \sum_{i=1}^{m} i r_i - m\left(\sum_{i=1}^{m} r_i - \alpha_1\right), \sum_{i=1}^{k} i r_i + \sum_{i=1}^{m'}(-i)r_{-i} - (-m')\left(\sum_{i=1}^{m'} r_{-i} - \alpha_2\right)\right].$$

Then $|\Sigma_\alpha(A_0)| \le L_\alpha$. Thus, $|\Sigma_\alpha(A_0)| = L_\alpha$. Therefore, at this point, L_α is optimal.

Thus, we complete the proof of Theorem 1. □

If zero is a term in A, one could easily get the following result.

Corollary 2. *Let $l, k \geq 1$ be two integers and $A = (a_{-l}, a_{-l+1}, \dots, a_k)_{\overline{r}}$ be a finite sequence of integers, where $a_{-l} < a_{-l+1} < \cdots < a_{-1} < 0 = a_0 < a_1 < a_2 < \cdots < a_k$ and $\overline{r} = (r_{-l}, r_{-l+1}, \dots, r_k)$ with $r_i > 0$ for $i \in [-l, k]$. Let $0 \leq \alpha < \sum_{i=-l}^{k} r_i$ and*

$$L'_\alpha = \begin{cases} \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i + 1, & \text{if } \alpha \leq \min\left\{\sum_{i=0}^{l} r_{-i},\ \sum_{i=0}^{k} r_i\right\}; \\ \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=0}^{l} r_{-i} + \sum_{i=1}^{m} r_i - \alpha\right) + 1, & \text{if } \sum_{i=0}^{l} r_{-i} < \alpha \leq \sum_{i=0}^{k} r_i; \\ \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m'} i r_{-i} + m'\left(\sum_{i=0}^{k} r_i + \sum_{i=1}^{m'} r_{-i} - \alpha\right) + 1, & \text{if } \sum_{i=0}^{k} r_i < \alpha \leq \sum_{i=0}^{l} r_{-i}; \\ \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i - \sum_{i=1}^{m} i r_i + m\left(\sum_{i=0}^{l} r_{-i} + \sum_{i=1}^{m} r_i - \alpha\right) & \\ \qquad - \sum_{i=1}^{m'} i r_{-i} + m'\left(\sum_{i=0}^{k} r_i + \sum_{i=1}^{m'} r_{-i} - \alpha\right) + 1, & \text{if } \alpha > \max\left\{\sum_{i=0}^{l} r_{-i},\ \sum_{i=0}^{k} r_i\right\}, \end{cases}$$

where m, m' are integers which satisfy $\sum_{i=1}^{m-1} r_i \leq \alpha - \sum_{i=0}^{l} r_{-i} < \sum_{i=1}^{m} r_i$ and $\sum_{i=1}^{m'-1} r_{-i} \leq \alpha - \sum_{i=0}^{k} r_i < \sum_{i=1}^{m'} r_{-i}$, respectively. Then

$$|\Sigma_\alpha(A)| \geq L'_\alpha.$$

Moreover, the lower bound is optimal.

Proof. The proof is clear. Here, we only give the outline of the proof. Let

$$A' = (a_{-l}, a_{-l+1}, \dots, a_{-1})_{\overline{r'}} \cup (a_1, a_2, \dots, a_k)_{\overline{r''}},$$

where $\overline{r'} = (r_{-l}, r_{-l+1}, \dots, r_{-1})$ and $\overline{r''} = (r_1, r_2, \dots, r_k)$. Since 0 repeats r_0 times, it is easy to see that when $0 \leq \alpha < r_0$, we have $|\Sigma_\alpha(A)| \geq |\Sigma_1(A')|$; when $\alpha \geq r_0$, we have $|\Sigma_\alpha(A)| \geq |\Sigma_{\alpha - r_0}(A')|$. When $0 \leq \alpha < r_0$, by Theorem 1 we have

$$|\Sigma_\alpha(A)| \geq |\Sigma_1(A')| \geq \sum_{i=1}^{l} i r_{-i} + \sum_{i=1}^{k} i r_i + 1.$$

When $\alpha \geq r_0$, by Theorem 1 for $|\Sigma_{\alpha - r_0}(A')|$ we can easily get the desired conclusion. □

3. Inverse problem

Corresponding to Theorem 1, we firstly deal with the case in which the sequence A does not contain zero terms.

Theorem 3. *Let A be a sequence which is the same as A defined in Theorem 1. Let $0 \leq \alpha \leq \sum_{i=1}^{l} r_{-i} + \sum_{i=1}^{k} r_i - 2$ and L_α, m, m' be the integers defined in Theorem 1. If $|\Sigma_\alpha(A)| = L_\alpha$, then*

$$A = a_1 * ([-l, -1]_{\overline{r}} \cup [1, k]_{\overline{r'}}).$$

Proof. Let A_1 and A_2 be the sequences defined in the proof of Theorem 1. Since $|\Sigma_\alpha(A)| = L_\alpha$, all inequalities of (1), (2) and (3) become equalities. By Theorem B for sequences $-A_1$ and A_2, we have

$$A_1 = -a_{-1} * [-l, -1]_{\overline{r}}$$

except that $A_1 = (a_{-2}, a_{-1})_{\overline{r}}$ with $\overline{r} = (r_{-2}, 1)$ and $A_1 = (a_{-2} + a_{-1}, a_{-2}, a_{-1})_{\overline{r}}$ with $\overline{r} = (r_{-3}, 1, 1)$;

$$A_2 = a_1 * [1, k]_{\overline{r'}}$$

except that $A_2 = (a_1, a_2)_{\overline{r'}}$ with $\overline{r'} = (1, r_2)$ and $A_2 = (a_1, a_2, a_1 + a_2)_{\overline{r'}}$ with $\overline{r'} = (1, 1, r_3)$.

Next, it only suffices to prove that $a_{-1} = -a_1$, $a_{-2} = -2a_1$ for $l \geq 2$ and $a_2 = 2a_1$ for $k \geq 2$. Since all inequalities of (1), (2) and (3) become equalities, we know that there is exactly one term $I+J$ in $\Sigma_\alpha(A)$ between $I+J-a_1$ and $I+J-a_{-1}$. Noting that $I+J-a_1-a_{-1} \in \Sigma_\alpha(A)$ and

$$I+J-a_1 < I+J-a_1-a_{-1} < I+J-a_{-1},$$

we have $I+J = I+J-a_1-a_{-1}$, and so $a_{-1} = -a_1$. If $l \geq 2$, then one could get similarly that $I+J-a_{-1}$ is the only one term in $\Sigma_\alpha(A)$ with

$$I+J < I+J-a_{-1} < I+J-a_{-2}.$$

However, in view of $a_{-1} = -a_1 > a_{-2}$, we have $I+J-a_1-a_{-2} \in \Sigma_\alpha(A)$ and

$$I+J < I+J-a_1-a_{-2} < I+J-a_{-2},$$

which shows that $I+J-a_1-a_{-2} = I+J-a_{-1}$, that is $a_{-2} = -a_1+a_{-1} = -2a_1$. If $k \geq 2$, then by similar argument, one could obtain that $a_2 = 2a_1$. Therefore,

$$A = a_1 * ([-l,-1]_{\overline{r}} \cup [1,k]_{\overline{r'}}).$$

This completes the proof. □

If zero is a term in A, by Theorem 3, one could get easily the following result.

Corollary 4. *Let A be a sequence which is the same as A defined in Corollary 2. Let $0 \leq \alpha \leq \sum_{i=-l}^{k} r_i - 2$ and L'_α, m, m' be the integers defined in Corollary 2. If $|\Sigma_\alpha(A)| = L'_\alpha$, then*

$$A = a_1 * [-l,k]_{\overline{r}}.$$

Remark 5. It is not difficult to see that for any sequence $A = (a_1, a_2, \ldots, a_k)_{\overline{r}}$ of integers with $a_1 < a_2 < \cdots < a_k$ and $\overline{r} = (r_1, r_2, \ldots, r_k)$, where $r_i \geq 1$ $(1 \leq i \leq k)$, $|\Sigma_\alpha(A)| = 1$ for $\alpha = \sum_{i=1}^{k} r_i$ and $|\Sigma_\alpha(A)| = k+1$ for $\alpha = \sum_{i=1}^{k} r_i - 1$.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] J. Bhanja, "A note on sumsets and restricted sumsets", *J. Integer Seq.* **24** (2021), no. 4, article no. 21.4.2 (9 pages).

[2] J. Bhanja and R. K. Pandey, "Inverse problems for certain subsequence sums in integers", *Discrete Math.* **343** (2020), no. 12, article no. 112148 (11 pages).

[3] J. Bhanja and R. K. Pandey, "On the minimum size of subset and subsequence sums in integers", *C. R. Math. Acad. Sci. Paris* **360** (2022), pp. 1099–1111.

[4] H. K. Dwivedi and R. K. Mistri, "Direct and inverse problems for subset sums with certain restrictions", *Integers* **22** (2022), article no. A112 (13 pages).

[5] X.-W. Jiang and Y.-L. Li, "On the cardinality of subsequence sums", *Int. J. Number Theory* **14** (2018), no. 3, pp. 661–668.

[6] R. K. Mistri and R. K. Pandey, "A generalization of sumsets of set of integers", *J. Number Theory* **143** (2014), pp. 334–356.

[7] R. K. Mistri, R. K. Pandey and O. Prakash, "Subsequence sums: direct and inverse problems", *J. Number Theory* **148** (2015), pp. 235–256.

[8] M. B. Nathanson, "Inverse theorems for subset sums", *Trans. Am. Math. Soc.* **347** (1995), no. 4, pp. 1409–1418.

[9] M. B. Nathanson, *Additive number theory. Inverse problems and the geometry of sumsets*, Springer, 1996, pp. xiv+293.

[10] V. H. Vu, "Some new results on subset sums", *J. Number Theory* **124** (2007), no. 1, pp. 229–233.

[11] Q.-H. Yang and Y.-G. Chen, "On the cardinality of general h-fold sumsets", *Eur. J. Comb.* **47** (2015), pp. 103–114.