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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 903-910

Online since: 7 October 2024

https://doi.org/10.5802/crmath.617

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Comptes Rendus. Mathématique
2024, Vol. 362, p. 903-910
https://doi.org/10.5802/crmath.617

Research article / *Article de recherche*
Complex analysis and geometry / *Analyse et géométrie complexes*

$\mathrm{SL}_4(\mathbf{Z})$ is not purely matricial field

$\mathrm{SL}_4(\mathbf{Z})$ *n'est pas purement MF*

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Abstract. We prove that every non-zero finite dimensional unitary representation of $\mathrm{SL}_4(\mathbf{Z})$ contains a non-zero $\mathrm{SL}_2(\mathbf{Z})$-invariant vector. As a consequence, there is no sequence of finite-dimensional representations of $\mathrm{SL}_4(\mathbf{Z})$ that gives rise to an embedding of its reduced C^*-algebra into an ultraproduct of matrix algebras.

Résumé. Nous montrons que toute représentation unitaire de dimension finie non nulle de $\mathrm{SL}_4(\mathbf{Z})$ a un vecteur $\mathrm{SL}_2(\mathbf{Z})$-invariant non nul. Il n'existe donc pas de suite de représentations de dimension finie de $\mathrm{SL}_4(\mathbf{Z})$ qui permettent de réaliser sa C^*-algèbre réduite dans un ultraproduit d'algèbres de matrices.

Keywords. Special linear groups, Finite dimensionnal unitary representations, Purely MF groups, MF C^*-algebra.

Mots-clés. Groupes spéciaux linéaires, représentations unitaires de dimension finie, groupes purement MF, C^*-algèbres MF.

2020 Mathematics Subject Classification. 20C15, 20C33, 22D25.

Funding. M. M. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1926686. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 949143). M. S. Research supported by the Charles Simonyi Endowment at the Institute for Advanced Study, and the ANR project ANCG Project-ANR-19-CE40-0002.

Manuscript received 6 December 2023, revised 19 January 2024, accepted 22 January 2024.

1. Statement of results

We view $\mathrm{SL}_2(\mathbf{Z})$ as the subgroup of $\mathrm{SL}_4(\mathbf{Z})$ consisting of matrices of the form $\left(\begin{smallmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. The point of this note is to prove the following theorem.

Theorem 1. *Every finite dimensional unitary representation of* $\mathrm{SL}_4(\mathbf{Z})$ *contains a non-zero* $\mathrm{SL}_2(\mathbf{Z})$*-invariant vector.*

We now explain some consequences of this theorem.

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ISSN (electronic): 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

Definition 2. *If* $\{\rho_i\}_{i=1}^{\infty}$ *is a sequence of finite dimensional unitary representations of a discrete group* Γ*, say* $\{\rho_i\}_{i=1}^{\infty}$ strongly converges to the regular representation *if for any* $z \in \mathbf{C}[\Gamma]$*,*

$$\lim_{i\to\infty} \|\rho_i(z)\| = \|\lambda_\Gamma(z)\|,$$

where $\lambda_\Gamma : \Gamma \to U(\ell^2(\Gamma))$ *is the left regular representation. The norms above are operator norms. We write* $\rho_i \xrightarrow{\text{strong}} \lambda_\Gamma$ *in this event.*[1]

If Γ is a discrete group, we say that Γ is purely matricial field if there is a sequence $\{\rho_i\}_{i=1}^{\infty}$ of finite dimensional unitary representations of Γ such that $\rho_i \xrightarrow{\text{strong}} \lambda_\Gamma$. In this case, if $\mathscr{U}$ is any free ultrafilter on $\mathbf{N}$, not only does the sequence $\{\rho_i : \Gamma \to U(N_i)\}_{i=1}^{\infty}$ induce an embedding

$$C_r^*(\Gamma) \xrightarrow{\varphi} \prod_{\mathscr{U}} \mathrm{Mat}_{N_i\times N_i}$$

into the C^*-ultraproduct of matrix algebras, in which case $C_r^*(\Gamma)$ is *matricial field* in the sense of Blackadar and Kirchberg [3], but also, there is a 'lifting' of the embedding restricted to the group algebra of the form

$$\begin{array}{ccc} \mathbf{C}[\Gamma] & \longrightarrow & \ell^{\infty}\left(\prod_{i\in\mathbf{N}} \mathrm{Mat}_{N_i\times N_i}\right) \\ & \searrow^{\varphi} & \downarrow \\ & & \prod_{\mathscr{U}} \mathrm{Mat}_{N_i\times N_i} \end{array}$$

See [6, Appendix A] for background on ultraproducts. Here $\ell^{\infty}\left(\prod_{i\in\mathbf{N}} \mathrm{Mat}_{N_i\times N_i}\right)$ is the collection of bounded sequences with respect to the C^*-norms. See Schafhauser [12] for a current overview of MF reduced C^*-algebras of groups.

Corollary 3. $\mathrm{SL}_4(\mathbf{Z})$ *is not purely matricial field.*

This appears to be the first example of a finitely generated residually finite group that is not purely matricial field. Groups that *are* known to be purely MF include free groups [7], limit groups and surface groups [10], and right-angled Artin groups, Coxeter groups, and hyperbolic three manifold groups [11].

It does not seem to be known whether $C_r^*(\mathrm{SL}_3(\mathbf{Z}))$ or $C_r^*(\mathrm{SL}_4(\mathbf{Z}))$ is MF in the sense of Blackadar and Kirchberg.

The property of a group being purely MF was historically relevant to the "$\mathrm{Ext}(C_r^*(F_2))$ is not a group" problem (see [14, Section 5.12]) and more recently a strong form of purely MF for free groups, due to Bordenave and Collins [5], was used to prove Buser's conjecture on the bottom of the spectrum of hyperbolic surfaces in two different ways [8, 10].

Proof of Corollary 3. Let S and T denote standard generators of $\mathrm{SL}_2(\mathbf{Z})$. Theorem 1 implies that for any finite dimensional representation ρ of $\mathrm{SL}_4(\mathbf{Z})$,

$$\|\rho(S+S^{-1}+T+T^{-1})\| = 4.$$

On the other hand, as an $\mathrm{SL}_2(\mathbf{Z})$-module, $\ell^2(\mathrm{SL}_4(\mathbf{Z}))$ breaks up into a direct sum of copies of $\ell^2(\mathrm{SL}_2(\mathbf{Z}))$. Since $\mathrm{SL}_2(\mathbf{Z})$ is not amenable, we have

$$\|\lambda_{\mathrm{SL}_4(\mathbf{Z})}(S+S^{-1}+T+T^{-1})\| = \|\lambda_{\mathrm{SL}_2(\mathbf{Z})}(S+S^{-1}+T+T^{-1})\| < 4. \qquad \square$$

[1] Some authors include weak convergence — that is, pointwise convergence of normalized traces to the canonical tracial state on the reduced group C^*-algebra — in the definition of strong convergence. In the case of $\mathrm{SL}_4(\mathbf{Z})$, these definitions agree.

Theorem 1 does not hold with 'four' replaced by 'three', since for primes p there are non-trivial irreducible representations of $\mathrm{SL}_3(\mathbf{Z}/p\mathbf{Z})$ without non-zero $\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})$-invariant vectors (P. Deligne, private communication, see Example 5). Nevertheless it could still be the case that $\mathrm{SL}_3(\mathbf{Z})$ is not purely MF and we would be very interested to know the answer of this question. It would perhaps clarify the relation between property (T) and purely MF — as far as we know there is no direct relation. Property (T) says that it is difficult to approach finite dimensional representations by arbitrary ones whereas the group not being purely matricial field says that it is difficult to approach the regular representation by finite-dimensional ones.

2. Proofs of results

It is an elementary consequence of work of Bass–Milnor–Serre on the congruence subgroup property [1] (e.g. [2, Section 5]) that every finite dimensional unitary representation of $\mathrm{SL}_4(\mathbf{Z})$ arises from a composition of homomorphisms

$$\mathrm{SL}_4(\mathbf{Z}) \longrightarrow \mathrm{SL}_4(\mathbf{Z}/N\mathbf{Z}) \xrightarrow{\phi} U(M)$$

for some $N \in \mathbf{N}$. To prove Theorem 1 it therefore suffices to prove the following.

Proposition 4. *For all $N \in \mathbf{N}$, every non-trivial finite dimensional representation ϕ of $\mathrm{SL}_4(\mathbf{Z}/N\mathbf{Z})$ has a non-zero $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$-invariant vector.*

As before $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ is the collection of matrices of the form $\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ in $\mathrm{SL}_4(\mathbf{Z}/N\mathbf{Z})$. The rest of the paper proves Proposition 4. We may assume that ϕ is irreducible and moreover that it is *new*, meaning that it does not factor through reduction modulo N'

$$\mathrm{SL}_4(\mathbf{Z}/N\mathbf{Z}) \longrightarrow \mathrm{SL}_4(\mathbf{Z}/N'\mathbf{Z})$$

for any $N' < N$ dividing N. (Or else we replace N by N'.)

2.1. *Reduction to prime powers*

Let

$$N = \prod_{p \text{ prime}} p^{e(p)}$$

be the prime factorization of N. By the Chinese remainder theorem

$$\mathrm{SL}_4(\mathbf{Z}/N\mathbf{Z}) \cong \prod_{p \text{ prime}, e(p)>0} \mathrm{SL}_4(\mathbf{Z}/p^{e(p)}\mathbf{Z})$$

and this induces a splitting

$$\phi \cong \bigotimes_{p \text{ prime}, e(p)>0} \phi_p$$

where ϕ_p are irreducible representations of $\mathrm{SL}_4(\mathbf{Z}/p^{e(p)}\mathbf{Z})$. The assumption that ϕ is new implies that each ϕ_p is new. If we can prove all the ϕ_p have non-zero $\mathrm{SL}_2(\mathbf{Z}/p^{e(p)}\mathbf{Z})$-invariant vectors v_p, then

$$v = \bigotimes_{p \text{ prime}, e(p)>0} v_p$$

will be the required non-zero invariant vector for $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}) \cong \prod_{p \text{ prime}, e(p)>0} \mathrm{SL}_2(\mathbf{Z}/p^{e(p)}\mathbf{Z})$ – the inclusion of SL_2 in SL_4 that we use commutes with our applications of the Chinese remainder theorem.

The strategy of the proof is the following:

Step 1. We prove the representation is non-trivial when restricted to all elementary cyclic subgroups of level p^{r-1}.

Step 2. We use Step 1 to prove that on restriction to a particular copy of the Heisenberg group modulo p^r, we find a particular type of character, namely, the one described in (4).

Step 3. We take a non-zero vector in the isotypic subspace of the character of the Heisenberg group found in Step 2. By averaging this vector over a copy of $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ we find a non-zero $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$-invariant vector. Here, the form of the Heisenberg group character we found in the previous step is important to make sure that this average is non-zero.

2.2. *Prime powers: step 1*

It therefore now suffices to prove Proposition 4 when $N = p^r$, $r \geq 1$. Let ϕ denote the irreducible representation. For $1 \leq i \neq j \leq 4$ let ε_{ij} denote the matrix with one in the i, j entry and zeros elsewhere. The first step is to find a non-trivial subrepresentation of some

$$C_{ij} \stackrel{\text{def}}{=} \langle I + p^{r-1}\varepsilon_{ij}\rangle.$$

As C_{ij} is abelian, by further passing to a subrepresentation, we may assume the non-trivial subrepresentation is irreducible and hence a character.

If $r = 1$ $\mathrm{SL}_4(\mathbf{Z}/p\mathbf{Z})$ is generated by such cyclic subgroups. So suppose for this step that $r > 1$.

We could proceed by using a result of Bass–Milnor–Serre [1, Corollary 4.3.b] — stating that the principal congruence subgroup of level p^r in $\mathrm{SL}_4(\mathbf{Z})$ is normally generated by elementary matrices. For completeness, below we give a simple self-contained proof of what we need.

Let $G(p^{r-1})$ denote the kernel of reduction mod p^{r-1} on $\mathrm{SL}_4(\mathbf{Z}/p^r\mathbf{Z})$. Since we assume ϕ is new, we know $G(p^{r-1})$ is not contained in the kernel of ϕ. Let $\mathrm{Mat}^0_{4\times 4}(\mathbf{Z}/p\mathbf{Z})$ denote the four by four matrices with entries in $\mathbf{Z}/p\mathbf{Z}$ and zero trace. The map

$$A \in \mathrm{Mat}^0_{4\times 4}(\mathbf{Z}/p\mathbf{Z}) \longmapsto I + p^{r-1}A \in G(p^{r-1}) \tag{1}$$

is easily seen to be an isomorphism of groups, where the group law on $\mathrm{Mat}^0_{4\times 4}(\mathbf{Z}/p\mathbf{Z})$ is addition.

We want to first show that some C_{ij} acts non-trivially in the representation.

Suppose for a contradiction that we do not find a non-trivial irreducible subrepresentation of some C_{ij}, so that all $1+p^{r-1}B$ with B zero on the diagonal are in $\ker(\phi)$. Using (1), this assumption implies that ϕ restricted to $G(p^{r-1})$ is equivalent to a non-trivial representation of

$$\mathrm{Mat}^0_{4\times 4}(\mathbf{Z}/p\mathbf{Z})/\{\text{elements of } \mathrm{Mat}^0_{4\times 4}(\mathbf{Z}/p\mathbf{Z}) \text{ that are zero on the diagonal}\}.$$

But this is spanned by equivalence classes of diagonal elements. Thus there is necessarily a diagonal matrix A such that $I+p^{r-1}A$ is not in the kernel of ϕ, without loss of generality (choosing a basis for the diagonal trace zero matrices) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

We calculate

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(I + p^{r-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I + p^{r-1} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \ker(\phi).$$

Then also

$$\left(I + p^{r-1} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \left(I + p^{r-1} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = I + p^{r-1} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \ker(\phi),$$

a contradiction. The conclusion of this step is no matter $r \geq 1$, we find $i \neq j$ such that $C_{ij} \notin \ker(\phi)$. But in fact, since all C_{ij} are conjugate in $\mathrm{SL}_4(\mathbf{Z}/p\mathbf{Z})$, this means that:

No C_{ij} is contained in the kernel of ϕ.

2.3. *Prime powers: step 2*

Let U_1 denote the group

$$U_1 \stackrel{\text{def}}{=} \left\{ \Upsilon(u_1,u_2,u_3) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \leq \mathrm{SL}_4(\mathbf{Z}/p^r\mathbf{Z}).$$

The group U_1 is isomorphic to $(\mathbf{Z}/p^r\mathbf{Z},+)^3$ so the restriction of ϕ to U_1 breaks into a direct sum of one-dimensional subspaces where U_1 acts by a character. Moreover, $\mathrm{SL}_3(\mathbf{Z}/p^r\mathbf{Z})$ normalizes U_1 so it acts on the characters of U_1 appearing like this by $g\chi = \chi(g^{-1}\cdot g)$. This action is called the dual action. Every such character is of the form

$$\chi\colon \Upsilon(u_1,u_2,u_3) \longmapsto \exp\left(2\pi \mathrm{i} \frac{(\xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3)}{p^r}\right) \tag{2}$$

for $(\xi_1,\xi_2,\xi_3) \in (\mathbf{Z}/p^r\mathbf{Z})^3$ and the dual action corresponds to $(\xi_1,\xi_2,\xi_3) \mapsto (\xi_1,\xi_2,\xi_3)g^{-1}$. If $(\xi_1,\xi_2,\xi_3) \equiv 0 \bmod p$ then all $\Upsilon(u_1,u_2,u_3)$ with $p^{r-1}|u_1,u_2,u_3$ are in the kernel of the character. If all obtained characters satisfy this condition, then ϕ restricted to U_1 has $U_1 \cap G(p^{r-1})$ in its kernel. But by Step 1, C_{14} is not contained in the kernel of ϕ. Hence in the restriction of ϕ to U_1 there must be a character of the form (2) where $(\xi_1,\xi_2,\xi_3) \not\equiv 0 \bmod p$. Since $\mathrm{SL}_3(\mathbf{Z}/p^r\mathbf{Z})$ acts transitively on the vectors in $(\mathbf{Z}/p^r\mathbf{Z})^3$ satisfying $(\xi_1,\xi_2,\xi_3) \not\equiv 0 \bmod p$, by considering the dual action we may assume

$$(\xi_1,\xi_2,\xi_3) = (0,0,1).$$

Let V_χ be the χ-isotypic space for the restriction of ϕ to U_1, where χ and ξ are as above.

The group

$$G_1 \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \leq \mathrm{SL}_4(\mathbf{Z}/p^r\mathbf{Z})$$

normalizes U_1 and fixes χ under the dual action. Hence V_χ is an invariant subspace for G_1. Now restrict V_χ to the group

$$U_2 \stackrel{\text{def}}{=} \left\{ [v_1;v_2] \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & v_1 & 0 \\ 0 & 1 & v_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \leq G_1$$

and we will decompose this into characters θ of U_2; let $V_{\chi,\theta}$ denote the corresponding isotypic subspace.

Consider now the group

$$H \stackrel{\text{def}}{=} \left\{ [x;y;z] \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & z \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \leq \mathrm{SL}_4(\mathbf{Z}/p^r\mathbf{Z}).$$

As we already mentioned, G_1 preserves V_χ. Obviously U_1 fixes all its characters under the dual action induced by conjugation, hence all $\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$ fix our chosen χ under the dual action, or in other words, leave V_χ invariant. Hence the space V_χ is invariant by H.

We have $[0;0;z] = \Upsilon(0,z,0) \in U_1$ and for $v \in V_\chi$

$$\Upsilon(0,z,0)\, v = \exp\left(2\pi \mathrm{i} \frac{(0\cdot 0 + 0\cdot z + 1\cdot 0)}{p^r}\right) v = v.$$

Hence the action of H on V_χ has kernel that contains the subgroup with $x = y = 0$, which is isomorphic to $\mathbf{Z}/p^r\mathbf{Z}$. Hence the action of H on V_χ factors through an action of

$$H/(\mathbf{Z}/p^r\mathbf{Z}) \cong (\mathbf{Z}/p^r\mathbf{Z})^2.$$

We want to find a particular character of H and to do so we split into the following cases.

Case 1. V_χ restricted to U_2 is trivial. Then obviously H acts on all of V_χ by

$$[x;y;z] \longmapsto \exp\left(2\pi \mathrm{i} \frac{y}{p^r}\right). \tag{3}$$

Case 2. Otherwise, we find a character θ in V_χ of the form

$$\theta : [v_1; v_2] \longmapsto \exp\left(2\pi \mathrm{i} \frac{(\zeta_1 v_1 + \zeta_2 v_2)}{p^r}\right)$$

with $(\zeta_1, \zeta_2) \not\equiv (0,0) \bmod p^r$. Write $(\zeta_1,\zeta_2) = p^R(z_1,z_2)$ with $(z_1,z_2) \neq (0,0) \bmod p$. By conjugation in $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z}) \le G_1$ — which normalizes U_2 — we can find a new θ' with corresponding $z_1 = 1, z_2 = 0$ so that

$$\theta' : [v_1; v_2] \longmapsto \exp\left(2\pi \mathrm{i} \frac{v_1}{p^{r-R}}\right).$$

In particular, on $V_{\chi,\theta'}$ H acts by the character (3).

To summarize, in any case, there exists a non-zero vector $v \in V_\chi$ such that

$$\phi([x;y;z])\, v = \exp\left(2\pi \mathrm{i} \frac{y}{p^r}\right) v. \tag{4}$$

2.4. *Prime powers: step 3*

Now let

$$G_2 \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \le \mathrm{SL}_4(\mathbf{Z}/p^r\mathbf{Z}).$$

From (4), v is fixed by the subgroup

$$N \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \le G_2.$$

This implies that if W denotes the representation of $G_2 \cong \mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ generated by v, that W is a quotient of the induced representation

$$\mathrm{Ind}_N^{G_2} \mathrm{triv} = \mathbf{C}[G_2] \otimes_N \mathbf{C}.$$

Suppose $g \in G_2$, with a, b, c, d as above in $\mathbf{Z}/p^r\mathbf{Z}$. We have

$$\begin{aligned}\phi([0;y;z])\phi(g^{-1})v &= \phi(g^{-1})\phi(g[0;y;z]g^{-1})v \\ &= \phi(g^{-1})\phi([0;cz+dy;az+by])v \\ &= \phi(g^{-1})\exp\left(2\pi \mathrm{i}\frac{(dy+cz)}{p^r}\right)v.\end{aligned}$$

This means, in this co-adjoint action of G_2 on characters of the group $\langle [0;y;z]\rangle$, N is precisely the stabilizer of the character of v, and hence

$$\dim W = |G_2|/|N| = \dim \operatorname{Ind}_N^{G_2} \mathrm{triv},$$

so in fact, $W \cong \operatorname{Ind}_N^{G_2}\mathrm{triv}$ as a G_2 representation. By Frobenius reciprocity, this contains the trivial representation of G_2. Finally, G_2 and the upper left copy of $\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})$ are conjugate in $\mathrm{SL}_4(\mathbf{Z}/p^r\mathbf{Z})$. This concludes the proof.

2.5. *Representations of* $\mathrm{SL}_3(\mathbf{Z}/p\mathbf{Z})$

The character tables of $\mathrm{SL}_3(\mathbf{F})$ for finite fields $\mathbf{F}$ have been computed in [13]. In particular, if we view $\mathrm{SL}_2(\mathbf{F})$ as the subgroup of $\mathrm{SL}_3(\mathbf{F})$ consisting of matrices of the form $\left(\begin{smallmatrix} * & * & 0\\ * & * & 0\\ 0 & 0 & 1\end{smallmatrix}\right)$, we obtain the following example, explained to us by Deligne:

Example 5. For every prime power q, $\mathrm{SL}_3(\mathbf{F}_q)$ has an irreducible representation such that, for every $g \in \mathrm{SL}_2(\mathbf{F}_q)$,

$$\mathrm{Tr}(\pi(g)) = \begin{cases} (q-1)(q^2-1) & \text{if } g = 1 \\ 1-q & \text{if } (g-1)^2 = 0 \neq g-1 \\ 0 & \text{if } (g-1)^2 \neq 0. \end{cases}$$

This representation does not have a non-zero $\mathrm{SL}_2(\mathbf{F}_q)$-invariant vector.

The representations are any of those denoted $\chi_{r^2s}(u)$ in [13, Table 1b] (that are associated with tori of split rank 0 in the Deligne–Lusztig theory [9]). The properties of $\mathrm{Tr}(\pi(g))$ follow readily from this table and the description of the conjugacy classes in $\mathrm{SL}_2(\mathbf{F}_q)$ (e.g. [4, Section 1.3]).

Such a representation does not have non-zero $\mathrm{SL}_2(\mathbf{F}_q)$-invariant vectors because, using that there are exactly q^2-1 unipotent matrices in $\mathrm{SL}_2(\mathbf{F}_q)\setminus\{1\}$ [4, Section 1.3], we can compute that the trace of the projection on the $\mathrm{SL}_2(\mathbf{F}_q)$-invariant vectors is 0:

$$\mathrm{Tr}\left(\sum_{g\in \mathrm{SL}_2(\mathbf{F}_q)} \pi(g)\right) = 1\cdot(q-1)(q^2-1) + (q^2-1)\cdot(1-q) = 0.$$

Acknowledgments

We thank Pierre Deligne for explaining to us the above mentioned fact about representations of $\mathrm{SL}_3(\mathbf{Z}/p\mathbf{Z})$. We thank Kevin Boucher, Yves de Cornulier and Olivier Dudas for comments and conversations about this project.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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