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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
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Deformations over non-commutative base

Déformations sur une base non-commutative

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Dedicated to the Memory of Professor Jean-Pierre Demailly

Abstract. We make some remarks on deformation theory over non-commutative base. We describe the base algebra of semi-universal non-commutative deformations using vector spaces T^1 and T^2.

Résumé. Nous faisons quelques remarques sur la théorie des déformations sur des bases non commutatives. Nous décrivons l'algèbre de base des déformations non commutatives semi-universelles à l'aide des espaces vectoriels T^1 et T^2.

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We will consider deformation theory over non-commutative (NC) base algebras. Such a theory is interesting because there are more deformations than the usual deformations over commutative bases. The deformations over commutative base can possibly be regarded as the "first order" approximation of more general "higher order" deformations. The formal theories of deformations over commutative and non-commutative bases are parallel and the extension to the non-commutative case is simple, but some new phenomena and invariants appear.

We make some remarks on NC deformations. The first remark is that the deformations over NC base is natural. This is because the differential graded algebras (DGA) which govern the deformations of sheaves are naturally non-commutative. Hence it is natural to consider deformations parametrized by NC base algebras. We will also consider the problem of convergence of formal NC deformations and the moduli space. The second remark is that we obtain "higher order invariants" because there are more NC deformations than commutative ones by slightly generalizing results of [12] and [4]. The last remark is that a description of the base algebra using the tangent space T^1 and the obstruction space T^2 is possible.

We use the abbreviation NC for "not necessarily commutative". In Section 1, we recall the definition of NC deformations, and explain how the base algebra of semi-universal NC deformations is described by a minimal A^∞-algebra arising from DGA in the case of deformations of coherent sheaves. In Section 2, we consider the problem of convergence and the existence of moduli space by taking an example of deformations of linear subspaces in a linear space. In

Section 3, we consider another example of flopping contractions of 3-dimensional manifolds, and show how invariants appear beyond those obtained by commutative deformations. We will give a description of the base algebra of the semi-universal NC deformation by using the tangent space and the obstruction space in Section 4.

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1. Multi-pointed non-commutative deformations

We recall the non-commutative deformation theory developed by [9] (see also [3], [6]). We use NC as “not necessarily commutative”. This is a generalization of the formal commutative deformation theory of [10] to the case where the base algebras are allowed to be NC.

Let k^r be the direct product ring of a field k, and let (Art_r) be the category of *augmented associative* k^r-algebras R which are finite dimensional as k-modules and such that the two-sided ideal $M = \mathrm{Ker}(R \to k^r)$ is nilpotent. We assume that the composition of the structure homomorphisms $k^r \to R \to k^r$ is the identity. (Art_r) is the category of the base spaces for r-*pointed* NC deformations.

Let $k_i \cong k$ be the i-th direct factor of the product ring k^r for $1 \le i \le r$. k_i is generated by $e_i = (0,\dots,1,\dots,0) \in k^r$, where 1 is placed at the i-th entry. A left k^r-module F has a direct sum decomposition $F = \bigoplus_{i=1}^r F_i$ as k-modules by $F_i = e_i F$, and k^r-bimodule has a further decomposition $F = \bigoplus_{i,j=1}^r F_{ij}$ by $F_{ij} = e_i F e_j$.

$R \in (\mathrm{Art}_r)$ is an NC Artin semi-local algebra with maximal two-sided ideals $M_i = \mathrm{Ker}(R \to k_i)$. NC deformation is *multi-pointed* because an NC semi-local algebra is not necessarily a direct product of local algebras unlike the case of a commutative algebra.

The model case is a deformation of a direct sum of coherent sheaves $F = \bigoplus_{i=1}^r F_i$ (r-pointed sheaf). The sheaves F_i interact each other and there are more NC deformations of F than those of the individual sheaves F_i.

Let F be something defined over k^r which will be deformed over $R \in (\mathrm{Art}_r)$. An *NC deformation* of F over R is a pair $(\widetilde{F}, \phi)$ where $\widetilde{F}$ is “flat” over R and $\phi : F \to R/M \otimes_R \widetilde{F}$ is an isomorphism. The definition depends on the cases what kind of F we are considering. The set of isomorphism classes of deformations of F over R gives an *NC deformation functor* $\Phi = \mathrm{Def}_F : (\mathrm{Art}_r) \to (\mathrm{Set})$.

More concretely, an r-*pointed NC deformation functor* $\Phi : (\mathrm{Art}_r) \to (\mathrm{Set})$ in this paper is a covariant functor which satisfies the conditions $(H_0), (H_f), (H_e), (\widetilde{H})$ stated below following [10] (see also [11, Chapter 2]).

We define an object $R_e \in (\mathrm{Art}_r)$ as a generalization of the ring of dual numbers $k[\epsilon]/(\epsilon^2)$. Let R_e be the trivial extension $k^r \oplus \mathrm{End}(k^r)$, where $\mathrm{End}(k^r)$ is a square zero two-sided ideal, and the multiplication of k^r and $\mathrm{End}(k^r)$ is induced from the embedding to diagonal matrices $k^r \to \mathrm{End}(k^r)$. As a k-module,

$$R_e = k^r \oplus \bigoplus_{i,j=1}^r k e_{ij}.$$

The multiplication is defined by $e_i e_{jk} = \delta_{ij} e_{jk}$, $e_{ij} e_k = \delta_{jk} e_{ij}$ and $e_{ij} e_{kl} = 0$ for all i, j, k, l. The augmentation $R_e \to k^r$ is given by $e_{ij} \mapsto 0$.

Now we state the conditions $(H_0),(H_f),(H_e),(\widetilde{H})$. For ring homomorphisms $R' \to R$ and $R'' \to R$ in (Art_r), let $\alpha : \Phi(R' \times_R R'') \to \Phi(R') \times_{\Phi(R)} \Phi(R'')$ be the map naturally defined by Φ.

(H_0) $\Phi(k^r)$ consists of one element.
(H_f) $\Phi(R_e)$ is finite dimensional as a k-module.
(H_e) The natural map α is bijective if $R = k^r$ and $R'' = R_e$.
$(\widetilde{H})$ The natural map α is surjective if $R'' \to R$ is surjective.

The *tangent space* T^1 of the functor Φ is defined by $T^1 = \Phi(R_e)$. The k^r-bimodule structure of the ideal $\mathrm{End}(k^r) \subset R_e$ induces a k^r-bimodule structure on T^1, so we can write $T^1 = \bigoplus_{i,j=1}^r T^1_{ij}$. We have $T^1_{ij} = \Phi(k^r \oplus ke_{ij})$. Indeed $T^1_{ij} = e_i T^1 e_j = \Phi(e_i R_e e_j) = \Phi(k^r \oplus ke_{ij})$.

An element $\xi \in \Phi(R)$ for $R \in (\mathrm{Art}_r)$ is called an *r-pointed NC deformation* over R of the unique element of $\Phi(k^r)$.

Let $T_R = (M/M^2)^*$ be the Zariski tangent space of R. It is a k^r-bimodule. The *Kodaira–Spencer map* $KS_\xi : T_R \to T^1$ associated to the deformation ξ is defined as follows. A tangent vector $v \in (T_R)_{ij} = (M/M^2)^*_{ij}$ induces a ring homomorphism $v_* : R \to k^r \oplus ke_{ij}$, hence $\Phi(v_*) : \Phi(R) \to \Phi(k^r \oplus ke_{ij}) = T^1_{ij}$. Then we define $KS_\xi(v) = \Phi(v_*)(\xi)$.

Let $\widehat{R} := \varprojlim R_i \in (\widehat{\mathrm{Art}}_r)$ be a pro-object of (Art_r), and let $\widehat{\xi} := \varprojlim \xi_i \in \widehat{\Phi}(\widehat{R}) := \varprojlim \Phi(R_i)$ be an element of a projective limit. Then $\widehat{\xi}$ is called a *formal r-pointed NC deformation* over $\widehat{R}$. The Kodaira–Spencer map $KS_{\widehat{\xi}} : T_{\widehat{R}} \to T^1$ is similarly defined.

A formal deformation $\widehat{\xi} \in \widehat{\Phi}(\widehat{R})$ is called a *versal* NC deformation if the following holds: for any NC deformation $\xi' \in \Phi(R')$, there exists a morphism $h : \widehat{R} \to R'$ such that $\xi' = \widehat{\Phi}(h)(\widehat{\xi})$.

In this case, the Kodaira–Spencer map $KS_{\widehat{\xi}} : T_{\widehat{R}} \to T^1$ is surjective. Indeed, let $v' \in T^1_{ij} = \Phi(k^r \oplus ke_{ij})$ be any element. Then there is a morphism $h : \widehat{R} \to k^r \oplus ke_{ij}$ such that $v' = \widehat{\Phi}(h)(\widehat{\xi})$. Let $v : (\widehat{M}/\widehat{M}^2)_{ij} \to ke_{ij}$ be the homomorphism induced from h. Then $v_* = h$ and $KS_{\widehat{\xi}}(v) = v'$.

A versal NC deformation is said to be *semi-universal* if the Kodaira–Spencer map is bijective. In this case, we have $\widehat{M}/\widehat{M}^2 \cong (T^1)^*$. We note that it is called "versal" in some literatures. The existence of the semi-universal NC deformation is proved in a similar way to [10] from the conditions $(H_0),(H_f),(H_e),(\widetilde{H})$.

In the case $r = 1$, if we take the abelianization $\widehat{R}^{ab} = \widehat{R}/[\widehat{R},\widehat{R}]$ of the base ring of the semi-universal deformation, then we obtain a usual semi-universal commutative deformation $\widehat{\xi}^{ab}$ over $\widehat{R}^{ab}$ given by $\widehat{\xi}^{ab} = \Phi(q)(\widehat{\xi})$, where $q : \widehat{R} \to \widehat{R}^{ab}$ is the quotient map.

We recall a description of the semi-universal NC deformation in the case of deformations of a coherent sheaf using an A^∞-algebra formalism ([8]). Let X be an algebraic variety over k and let $F = \bigoplus_{i=1}^r F_i$ be a coherent sheaf with proper support. Then the infinitesimal deformations of F are controlled by a *differential graded algebra (DGA)* $\mathrm{RHom}_X(F,F)$. The tangent space and the obstruction space are given by k^r-bimodules $T^i = \mathrm{Ext}^i_X(F,F)$ for $i = 1,2$ (cf. Section 4).

It is also controlled by an A^∞-algebra structure $\{m_d\}_{d\ge 2}$ of the cohomology group $A = \bigoplus_{p\ge 0} A_p := \bigoplus_{p\ge 0} \mathrm{Ext}^p(F,F) = \bigoplus_{p,i,j} \mathrm{Ext}^p(F_i,F_j)$;

$$m_d : T^d_{k^r} A := A \otimes_{k^r} \cdots \otimes_{k^r} A \longrightarrow A(2-d)$$

are the higher multiplications of degree $2-d$, where the left hand side is a tensor product with d factors over k^r and the right hand side has degree shift $2-d$. In particular, we have

$$m_d : T^d_{k^r} A_1 := A_1 \otimes_{k^r} \cdots \otimes_{k^r} A_1 \longrightarrow A_2$$

for $d \ge 2$.

In general, for a k^r-bimodule E, we have $E = \bigoplus_{i,j=1}^r E_{ij}$ with $E_{ij} = e_i E e_j$. We define a completed tensor algebra $\widehat{T}_{k^r} E = \prod_{d\ge 0} T^d_{k^r} E$ by

$$T^d_{k^r} E = E \otimes_{k^r} E \otimes_{k^r} \cdots \otimes_{k^r} E$$

where there are d-times E on the right hand side. We apply this construction to $E = (T^1)^*$. If $\{x^s_{ij}\}_s$ is a basis of E_{ij}, then we have

$$\widehat{T}_{k^r} E = k^r \langle\langle x^s_{ij} \rangle\rangle / (e_i x^s_{i'j}, x^s_{ij} e_{j'}, x^s_{i'j'} x^{s'}_{i''j''} \mid i \ne i', j \ne j', j' \ne i'').$$

Thus the set of monomials

$$x^{s_1}_{i_0 i_1} x^{s_2}_{i_1 i_2} \dots x^{s_d}_{i_{d-1} i_d}$$

with $i = i_0$ and $j = i_d$ is a k-basis of $(\widehat{T}_{k^r} E)_{ij}$.

Let

$$m^* = \sum_{d \ge 2} m^*_d : \operatorname{Ext}^2(F,F)^* \longrightarrow \widehat{T}_{k^r}(\operatorname{Ext}^1(F,F)^*)$$

be the formal sum of dual maps of m_d. Then the base algebra $\widehat{R}$ of the semi-universal NC deformation $\widehat{F}$ is determined as an augmented k^r-algebra to be

$$\widehat{R} = \widehat{T}_{k^r}(\operatorname{Ext}^1(X,X)^*) / (m^*(\operatorname{Ext}^2(X,X)^*))$$

([8]). Thus the Taylor coefficients of the equations of the formal NC moduli space are determined by A^∞-multiplications.

There is another way of describing a semi-universal r-pointed NC deformation of a direct sum of coherent sheaves with proper support $F = \bigoplus_{i=1}^r F_i$. The semi-universal NC deformation $\widehat{F}$ of F is given by a tower $\{F^{(n)}\}$ of universal extensions (cf. [6]):

$$0 \longrightarrow \operatorname{Ext}^1(F^{(n)}, F)^* \otimes_{k^r} F \longrightarrow F^{(n+1)} \longrightarrow F^{(n)} \longrightarrow 0$$

with $F^{(0)} = F$ and $\widehat{F} = \varprojlim F^{(n)}$. We have direct sum decompositions $F^{(n)} = \bigoplus_i F_i^{(n)}$, and we can write

$$0 \longrightarrow \bigoplus_{i,j} \operatorname{Ext}^1(F_i^{(n)}, F_j)^* \otimes_k F_j \longrightarrow \bigoplus_i F_i^{(n+1)} \longrightarrow \bigoplus_i F_i^{(n)} \longrightarrow 0.$$

If $\operatorname{End}(F) \cong k^r$, i.e., if $\operatorname{End}(F_i) \cong k$ and $\operatorname{Hom}(F_i, F_j) \cong 0$ for $i \ne j$, then F is called a *simple collection* ([6]). The deformation theory of a simple collection is particularly nice. In this case, $F^{(n)}$ is flat over $R^{(n)} = \operatorname{End}(F^{(n)})$, and the parameter algebra $\widehat{R}$ of the semi-universal deformation $\widehat{F}$ is given by $\widehat{R} = \varprojlim R^{(n)}$ ([6, Theorem 4.8]).

Remarks 1.

(1) We do not consider deformation theory of *varieties* over non-commutative base in this paper, because such a theory seems to be difficult by the following reason. Suppose that there is an infinitesimal deformation X_R of a variety X over an NC ring R. Then the structure sheaf $\mathcal{O}_{X_R}$ should be NC too. When we consider a base change over a ring homomorphism $R \to R'$, it seems necessary that the base rings should be commutative in order for the tensor product $\mathcal{O}_{X_R} \otimes_R R'$ to have a ring structure. Indeed the DGLie algebra which controls the deformations of X is NC but its non-commutativity is restricted.

But when X is a subvariety of an ambient variety Y, then we can consider a deformation of X inside Y over an NC base as a deformation of the structure sheaf $\mathcal{O}_X$ as a sheaf on Y (see Section 2).

(2) The deformation functor is pro-representable when there is a universal deformation. But a universal deformation does not exist in general (see [6, Remark 4.10]).

2. Convergence and moduli

The above described semi-universal NC deformation is a formal deformation, and the question on the convergence is important. We will make some remarks on the convergence of the formal NC deformations and the relationship with the moduli space of commutative deformations. We consider only 1-pointed NC deformations, and we take an example of the moduli space of linear subspaces in a fixed linear space. We consider NC deformations of the structure sheaves of linear subspaces.

We would like to say that the formal semi-universal NC deformation is convergent if the corresponding semi-universal commutative deformation is convergent. This is because the numbers of commutative monomials and non-commutative ones on n variables of degree d grow similarly to n^d. Maybe we should require that the growth of the Taylor coefficients of the non-commutative power series are bounded in a similar way as the commutative power series.

Any k-algebra homomorphism $R \to k$ for any associative k-algebra R factors through the abelianization $R \to R^{ab}$. Therefore we can think that the set of closed points of the moduli spaces are the same for commutative and NC deformation problems. In other words, when we observe points, then the moduli space of NC deformations is reduced to the usual moduli space. We can say that the NC deformations give an additional infinitesimal or formal structure at each point of the commutative moduli space. And the formal structure is usually convergent. However, a compactification is another problem, and it seems that it does not exists.

As an example, we consider NC deformations of linear subspaces in a finite dimensional vector space. As explained in Remark 1.1, we consider the NC deformations of the structure sheaf of the subspace instead of the subspace as a variety. The following is a slight generalization of [8, Example 7.8]. The commutative deformations are unobstructed and yield a compact moduli space, a Grassmann variety. But we will see that NC deformations are obstructed.

Let $V \cong k^n$ be an n-dimensional linear space with coordinate linear functions $x_1,\dots,x_n$, and let W be an m-dimensional linear subspace defined by an ideal $I = (x_{m+1},\dots,x_n)$. The commutative moduli space $G(m,n)$ has an affine open subset $\mathrm{Hom}(W, V/W) \cong k^{m(n-m)}$ with coordinates $a_{i,j}$ $(1 \le i \le m,\ m+1 \le j \le n)$. We consider NC deformations of W as a linear subspace of V, i.e., the NC deformations of the ideal sheaves generated by linear functions.

Proposition 2. *Let $V \cong k^n$ with coordinate linear functions $x_1,\dots,x_n$, and let $W \cong k^m$ be defined by $x_{m+1} = \cdots = x_n = 0$. Then the formal semi-universal NC deformation of W as a linear subspace of V has the parameter algebra $\widehat{R}$ and the ideal $\widehat{I}$ given as follows:*

$$\widehat{R} = k\langle\langle a_{ij} \mid 1 \le i \le m < j \le n\rangle\rangle/\widehat{J}$$

$$\widehat{J} = \big(a_{ij_1}a_{ij_2} - a_{ij_2}a_{ij_1},\ a_{i_1j_1}a_{i_2j_2} - a_{i_2j_2}a_{i_1j_1} + a_{i_1j_2}a_{i_2j_1} - a_{i_2j_1}a_{i_1j_2} \\ \big|\, 0 \le i \le m, 1 \le i_1 < i_2 \le m < j_1 < j_2 \le n\big)$$

$$\widehat{I} = \left(x_j + \sum_{i=1}^m a_{ij}x_i \,\middle|\, m+1 \le j \le n\right).$$

Proof. This is almost the same as [8, Example 7.8]. Let $Y = \mathbf{P}(W^*) \subset X = \mathbf{P}(V^*)$ be the corresponding projective spaces. We consider NC deformations of a coherent sheaf $F = \mathcal{O}_Y$ on X. The normal bundle of Y in X is given by $N_{Y/X} \cong \mathcal{O}_Y(1)^{\oplus n-m}$. Hence $T^1 = \mathrm{Ext}^1(F,F) \cong H^0(Y, N_{Y/X}) \cong k^{\oplus m(n-m)}$ and $T^2 = \mathrm{Ext}^2(F,F) \cong H^0(Y, \bigwedge^2 N_{Y/X}) \cong k^{\oplus \binom{m+1}{2}\binom{n-m}{2}}$.

Let $I' = \mathcal{O}_X(-Y)$ be the ideal sheaf of $Y \subset X$ generated by the homogeneous coordinates $x_{m+1},\dots,x_n$. By [8, Lemma 7.6], the semi-universal NC deformation of F is given in the form

$$\widehat{F} = \varprojlim (R_n \otimes \mathcal{O}_X)/I'_n$$

where $(R_n, M_n) \in (\mathrm{Art}_1)$ such that $M_n^{n+1} = 0$. By the flatness, the ideal sheaf I'_n is generated by linear forms $x_j + \sum_{i=1}^m a_{ij}x_i$ for $m+1 \le j \le n$, where $a_{ij} \in M_n$.

Since the x_i are commutative variables in $R_n \otimes \mathcal{O}_X$, we have $x_jx_l = x_lx_j$ for $m+1 \le j,l \le n$. Hence equalities

$$\sum_{i,k=1}^m a_{ij}a_{kl}x_ix_k = \sum_{i,k=1}^m a_{kl}a_{ij}x_ix_k$$

hold in $F_n = (R_n \otimes \mathcal{O}_X)/I'_n$ for such j, l. It follows that

$$a_{ij}a_{il} - a_{il}a_{ij} = 0 \quad (1 \le i \le m < j < l \le n),$$
$$a_{ij}a_{kl} - a_{kl}a_{ij} + a_{kj}a_{il} - a_{il}a_{kj} = 0 \quad (1 \le i < k \le m < j < l \le n)$$

in $\widehat{R} = \varprojlim R_n$. The above relations are non-commutative polynomials which are linearly independent quadratic forms, and their number is equal to

$$m\binom{n-m}{2} + \binom{m}{2}\binom{n-m}{2} = \binom{m+1}{2}\binom{n-m}{2}.$$

This is equal to the dimension of the obstruction space. Therefore there are no more independent relations contained in $\widehat{J}$. □

The above deformation is "algebraizable". There is an NC deformation of ideals $\widetilde{I}$ over a parameter algebra $\widetilde{R}$ which is a quotient algebra of an NC polynomial algebra:

$$\widetilde{R} = k\langle a_{ij} \mid 1 \le i \le m < j \le n\rangle/\widetilde{J}$$
$$\widetilde{J} = \big(a_{ij_1}a_{ij_2} - a_{ij_2}a_{ij_1},\ a_{i_1j_1}a_{i_2j_2} - a_{i_2j_2}a_{i_1j_1} + a_{i_1j_2}a_{i_2j_1} - a_{i_2j_1}a_{i_1j_2} \\ \big| \, 1 \le i \le m, 1 \le i_1 < i_2 \le m < j_1 < j_2 \le n\big)$$
$$\widetilde{I} = \left(x_j + \sum_{i=1}^m a_{ij}x_i \,\middle|\, m+1 \le j \le n\right)$$

The meaning of this formula is that it induces a semi-universal NC deformation at every closed point of an affine open subset $\mathrm{Spec}(\widetilde{R}^{ab}) \subset G(m+1, n+1)$ with $\widetilde{R}^{ab} = k[a_{ij} \mid 0 \le i \le m < j \le n]$. Indeed we have

$$(a_{ij} - a^0_{ij})(b_{kl} - b^0_{kl}) - (b_{kl} - b^0_{kl})(a_{ij} - a^0_{ij}) = a_{ij}b_{kl} - b_{kl}a_{ij}$$

for NC variables a_{ij}, b_{kl} and $a^0_{ij}, b^0_{kl} \in k$.

Hilbert schemes and Quot schemes are constructed from Grassmann varieties. We wonder if their NC deformations are also semi-globalizable.

Examples 3.

(1) $n = 3$ and $m = 1$. We have $G(1,3) \cong \mathbf{P}^2$. Then $\widetilde{R} \cong k\langle a, b\rangle/(ab - ba) = k[a, b]$.

(2) $n = 3$ and $m = 2$. We have $G(2,3) \cong \mathbf{P}^2$. Then $\widetilde{R} = k\langle a, b\rangle$ is not Noetherian. Indeed a two-sided ideal $(ab^k a \mid k > 0)$ is not finitely generated.

$\widetilde{R}$ has a following quotient algebra, which corresponds to an NC deformation which is not semi-universal:

$$R_\epsilon = k\langle a, b\rangle/(ab - ba - \epsilon)$$

where $\epsilon \in k$. For example, if $\epsilon = 1$, then $R_1 \cong k[t, d/dt]$.

(3) $n = 4$ and $m = 2$. We have $G(2,4)$. Then we have

$$\widetilde{R} = k\langle a, b, c, d\rangle/(ab - ba,\ cd - dc,\ ad - da - bc + cb).$$

$\widetilde{R}$ has a following quotient algebra:

$$R_{\epsilon_1,\epsilon_2} = k\langle a, b, c, d\rangle/(ab - ba,\ cd - dc,\ ad - da - 1,\ bc - cb - 1,\ ac - ca - \epsilon_1,\ bd - db - \epsilon_2)$$

where $\epsilon_i \in k$. For example, if $\epsilon_i = 0$, then $R_1 \cong k[t_1, t_2, \partial/\partial t_1, \partial/\partial t_2]$.

3. Flopping contractions of 3-folds

As a typical example of multi-pointed NC deformations, we will consider NC deformations of exceptional curves of a flopping contraction from a smooth 3-fold $f : Y \to X$ over $k = \mathbf{C}$. [2] observed that there are more NC deformations than commutative ones, and the base algebra of NC deformations gives an important invariant of the flopping contraction called the contraction algebra. Indeed Donovan and Wemyss conjectured that the contraction algebra, which is a finite dimensional associative algebra, determines the complex analytic type of the singularity of X. [12] and [4] proved that the dimension count of the contraction algebra yields Gopakumar-Vafa invariants of rational curves defined in [5]. We will consider slight generalizations where there are more than one exceptional curves.

Let $f : Y \to X = \mathrm{Spec}(B)$ be a projective birational morphism defined over $k = \mathbf{C}$ from a smooth 3-dimensional variety Y whose exceptional locus C is 1-dimensional. Let $C = \bigcup_{i=1}^r C_i$ be a decomposition into irreducible components. We assume that f is crepant, i.e., $(K_Y, C_i) = 0$ for all i. It is known that $C_i \cong \mathbf{P}^1$, the dual graph of the C_i is a tree, and X has only isolated hypersurface singularities of multiplicity 2.

The *contraction algebra* R for f is defined to be the base algebra of the semi-universal r-pointed NC deformation of the sheaf $F = \bigoplus_{i=1}^r \mathscr{O}_{C_i}(-1)$.

We consider commutative one parameter deformation of the contraction morphism $f : Y \to X$, and investigate the behavior of the contraction algebras under deformation. Let $p : \mathscr{X} \to \Delta$ be a one parameter flat deformation of X over a disk Δ, and assume that there is a flat deformation $\tilde{f} : \mathscr{Y} \to \mathscr{X}$ of the flopping contraction $f : Y \to X$. We assume that there are Cartier divisors $\mathscr{L}_1, \dots, \mathscr{L}_r$ on $\mathscr{Y}$ such that $(\mathscr{L}_i, C_j) = \delta_{i,j}$. This is always achieved when we replace X by its complex analytic germ containing $f(C)$ and Δ by a smaller disk.

Let $C^t = \bigcup_{j=1}^{s_t} C_j^t$ be the exceptional curves with decomposition to irreducible components for the flopping contraction $f_t : Y_t \to X_t$ for $t \ne 0$, where $Y_t = (p\tilde{f})^{-1}(t)$ and $X_t = p^{-1}(t)$. It is not necessarily connected even if C is connected. We may assume that $s = s_t$ is constant on $t \ne 0$. We define integers $m_{j,i}$ by the degeneration of 1-cycles $C_j^t \to \sum m_{j,i} C_i$ when $t \to 0$. This means that $\mathscr{O}_{C_j^t}$ degenerates in a flat family to $\mathscr{O}_{\sum_i m_{j,i} C_i}$. We have $(\mathscr{L}_i, C_j^t) = m_{j,i}$.

If the deformation $\tilde{f}$ is generic, then C^t is a disjoint union of $(-1,-1)$-curves, i.e., smooth rational curves whose normal bundles are isomorphic to $\mathscr{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$. In this case, we denote

$$m_j = \sum_i m_{j,i}, \quad n_d = \#\{j \mid m_j = d\}.$$

The numbers n_d should be called the *Gopakumar-Vafa invariants* ([5] for the case $r = 1$). In the case $r = 1$, [12] proved that n_1 is equal to the dimension of the abelianization of the contraction algebra $n_1 = \dim R^{ab}$, while higher terms n_d for $d \ge 2$ contribute to $\dim R$ (see Theorem 4 (3)).

We consider NC deformations of $F = \bigoplus_{i=1}^r F_i$ for $F_i = \mathscr{O}_{C_i}(-1)$ on Y and $\mathscr{Y}$. The set $\{F_i\}$ is called a *simple collection* on Y and $\mathscr{Y}$ in the terminology of [6] in the sense that $\mathrm{Hom}_Y(F,F) \cong \mathrm{Hom}_{\mathscr{Y}}(F,F) \cong k^r$. The NC deformations of a simple collection behave particularly nice.

Let $\hat{\Delta} = \mathrm{Spec}(k[\![t]\!])$ be the completion of Δ at the origin. By the flat base change $\hat{\Delta} \to \Delta$, we define $\widehat{\mathscr{X}} = \mathscr{X} \times_\Delta \hat{\Delta}$ and $\widehat{\mathscr{Y}} = \mathscr{Y} \times_\Delta \hat{\Delta}$. Let $\hat{f} : \widehat{\mathscr{Y}} \to \widehat{\mathscr{X}}$ and $\hat{p} : \widehat{\mathscr{X}} \to \hat{\Delta}$ be natural morphisms.

Let $\widehat{\mathscr{F}} = \bigoplus_{i=1}^r \mathscr{F}_i$ and $\tilde{F}^0 = \bigoplus_{i=1}^r \tilde{F}_i^0$ be the semi-universal NC deformations of F on $\widehat{\mathscr{Y}}$ and Y, respectively, and let $\widehat{\mathscr{R}}$ and R be the base algebras of these semi-universal deformations. We note that $\tilde{F}^0$ is obtained by finite number of extensions of the F_i while $\widehat{\mathscr{F}}$ may not. This is because C is isolated in Y while C may move inside $\mathscr{Y}$. Hence we have $\dim R < \infty$ as k-modules. We will see that $\dim \widehat{\mathscr{R}} = \infty$ (see Theorem 4 (1)).

$\widehat{\mathscr{F}}$ is also a semi-universal NC deformation of F on $\mathscr{Y}$. We will see that there is also a "convergent version" $\mathscr{F}$ on $\mathscr{Y}$, and $\widehat{\mathscr{F}}$ is its completion.

By [6, Theorem 4.8], the base algebras coincide with the endomorphism algebras:

$$\widehat{\mathscr{R}} = \mathrm{End}_{\hat{\mathscr{Y}}}(\widehat{\mathscr{F}}), \quad R = \mathrm{End}_Y(\widetilde{F}).$$

$\widehat{\mathscr{F}}$ and $\widetilde{F}^0$ can be described explicitly in the following way ([2, 6, 7]). In particular, there exists a sheaf $\mathscr{F}$ on $\mathscr{Y}$ such that

$$\widehat{\mathscr{F}} \cong \mathscr{F} \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\hat{\mathscr{Y}}} \tag{1}$$

i.e., the semi-universal NC deformation $\widehat{\mathscr{F}}$ is convergent when we replace Δ by a smaller disk if necessary.

By [13], we construct extensions of locally free sheaves on $\mathscr{Y}$:

$$0 \longrightarrow \mathscr{O}_{\mathscr{Y}}^{s_i} \longrightarrow M_i \longrightarrow \mathscr{L}_i \longrightarrow 0$$

with some integers s_i such that $R^1\tilde{f}_*M_i^* = 0$, where M_i^* is the dual sheaf. Let $M = \bigoplus_{i=1}^r M_i$ and $M^0 = M \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_Y$. We also denote $\widehat{M} = M \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\hat{\mathscr{Y}}}$. We have an exact sequence

$$0 \longrightarrow M^* \longrightarrow M^* \longrightarrow (M^0)^* \longrightarrow 0.$$

Since the dimensions of fibers of $\tilde{f}$ are at most 1, we obtain $R^1f_*(M^0)^* = 0$ from $R^1\tilde{f}_*M_i^* = 0$. Then semi-universal NC deformations $\widehat{\mathscr{F}} = \bigoplus \widehat{\mathscr{F}}_i$ and $\widetilde{F}^0$ are given as the kernels of natural homomorphisms ([7, Theorem 1.2]):

$$0 \longrightarrow \widehat{\mathscr{F}} \longrightarrow \hat{f}^*\hat{f}_*\widehat{M} \longrightarrow \widehat{M} \longrightarrow 0,$$
$$0 \longrightarrow \widetilde{F}^0 \longrightarrow f^*f_*M^0 \longrightarrow M^0 \longrightarrow 0.$$

We define $\mathscr{F}$ by an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \tilde{f}^*\tilde{f}_*M \longrightarrow M \longrightarrow 0$$

and let $\mathscr{R} = \mathrm{End}_{\mathscr{Y}}(\mathscr{F})$. By the flat base change, we obtain (1) and

$$\widehat{\mathscr{R}} \cong \mathscr{R} \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\hat{\mathscr{Y}}}.$$

We denote $\widetilde{F}^t = \mathscr{F} \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{Y_t}$ and $R^t = \mathscr{R} \otimes_{\mathscr{O}_\Delta} k_t$, where $Y_t = (p\tilde{f})^{-1}(t)$ and k_t is the residue field at $t \in \Delta$.

The following is a slight generalization of results in [4] and [12]:

Theorem 4.

(1) *$\mathscr{F}$ is flat over Δ, and $\widetilde{F}^0 = \mathscr{F} \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_Y$.*
(2) *([4, Conjecture 4.3]). $\mathscr{R}$ is a flat $\mathscr{O}_\Delta$-module, and $R \cong \mathscr{R} \otimes_{\mathscr{O}_\Delta} k$, where k is the residue field of $\mathscr{O}_\Delta$ at 0.*
(3) *Assume in addition that C^t is a disjoint union of $(-1,-1)$-curves C_j^t for $t \ne 0$. Then*

$$\widetilde{F}^t \cong \bigoplus_j \mathscr{O}_{C_j^t}(-1)^{m_j},$$
$$R^t \cong \prod_j \mathrm{Mat}(m_j \times m_j),$$
$$\dim R = \sum_j m_j^2 = \sum_d n_d d^2.$$

Proof. **(1).** We have an exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow M^0 \longrightarrow 0$$

where the first arrow is the multiplication by t. Because $R^1\tilde{f}_*M = 0$, there is an exact sequence

$$0 \longrightarrow \tilde{f}_*M \longrightarrow \tilde{f}_*M \longrightarrow \tilde{f}_*M^0 \longrightarrow 0.$$

Because $L_1\widetilde{f}^*\widetilde{f}_*M^0 = 0$ by [1] Lemma 3.4, we obtain the first row of the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{f}^*\widetilde{f}_*M & \longrightarrow & \widetilde{f}^*\widetilde{f}_*M & \longrightarrow & \widetilde{f}^*\widetilde{f}_*M^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & M^0 & \longrightarrow & 0. \end{array}$$

By snake lemma, we obtain

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \widetilde{F}^0 \longrightarrow 0$$

hence the flatness.

(2). Since $t : \mathscr{F} \to \mathscr{F}$ is injective, $\mathscr{R}$ has no t-torsion. Thus it is sufficient to prove that the natural homomorphism $\mathrm{Hom}_{\mathscr{Y}}(\mathscr{F},\mathscr{F}) \to \mathrm{Hom}_Y(\widetilde{F}^0,\widetilde{F}^0)$ is surjective. By the flat base change, it is also sufficient to prove that $\mathrm{Hom}_{\widehat{\mathscr{Y}}}(\widehat{\mathscr{F}},\widehat{\mathscr{F}}) \to \mathrm{Hom}_Y(\widetilde{F}^0,\widetilde{F}^0)$ is surjective, i.e., $\widehat{\mathscr{R}} \to R$ is surjective. Then the assertion follows from the fact that $\widehat{\mathscr{R}}$ and R are the base algebras of NC semi-universal deformations of the same sheaf F with $Y \subset \mathscr{Y}$.

(3). This is proved in [4] and [12] when $r = 1$. Let $x_j^t = \widetilde{f}(C_j^t) \in X_t = p^{-1}(t)$ for $t \ne 0$. Since C_j^t is a $(-1,-1)$-curve, x_j^t is an ordinary double point on a 3-fold. We take a small complex analytic neighborhood $x_j^t \in U_j^t \subset X_t$, and let $V_j^t = \widetilde{f}^{-1}(U_j^t)$.

Let L_j^t be a Cartier divisor on V_j^t such that $(L_j^t, C_j^t) = 1$. We know that $(\mathscr{L}_i, C_j^t) = m_{j,i}$ and $R^1\widetilde{f}_*M_i^* = 0$. Since $C_j^t \cong \mathbf{P}^1$ and M_i is relatively generated, $M_i|_{V_j^t}$ is a direct sum of line bundles whose degrees are non-negative but at most 1. Since the total degree is equal to $m_{j,i}$, it follows that $M_i|_{V_j^t} = (L_j^t)^{\oplus m_{j,i}} \oplus \mathscr{O}_{V_j^t}^{\oplus(\mathrm{rank}(M_i)-m_{j,i})}$.

We will prove that $\mathrm{Ker}(\widetilde{f}^*\widetilde{f}_*L_j^t \to L_j^t) \cong \mathscr{O}_{C_j^t}(-1)$. Indeed there is a commutative diagram

$$\begin{array}{ccccccccc} & & \widetilde{f}^*\widetilde{f}_*(L_j^t)^* & \longrightarrow & \mathscr{O}_{V_j^t}^{\oplus 2} & \longrightarrow & \widetilde{f}^*\widetilde{f}_*L_j^t & \longrightarrow & 0 \\ & & {\scriptstyle h_1}\downarrow & & {\scriptstyle\cong}\downarrow & & {\scriptstyle h_2}\downarrow & & \\ 0 & \longrightarrow & (L_j^t)^* & \longrightarrow & \mathscr{O}_{V_j^t}^{\oplus 2} & \longrightarrow & L_j^t & \longrightarrow & 0. \end{array}$$

Hence $\mathrm{Ker}(h_2) \cong \mathrm{Coker}(h_1)$. Since $(L_j^t)^* \otimes_{\mathscr{O}_{V_j^t}} I_{C_j^t}$ for the ideal sheaf $I_{C_j^t}$ of $C_j^t \subset V_j^t$ is generated by global sections, we have $\mathrm{Coker}(h_1) \cong (L_j^t)^* \otimes \mathscr{O}_{C_j^t} \cong \mathscr{O}_{C_j^t}(-1)$.

Therefore $\mathscr{F}_i|_{V_j^t} = \mathscr{O}_{C_j^t}(-1)^{\oplus m_{j,i}}$. Hence $\mathscr{F}|_{V_j^t} = \mathscr{O}_{C_j^t}(-1)^{\oplus m_j}$, and $\widetilde{F}^t \cong \bigoplus_j \mathscr{O}_{C_j^t}(-1)^{m_j}$. Thus $\mathrm{End}_{Y_t}(\widetilde{F}^t) \cong \prod_j \mathrm{Mat}(m_j \times m_j)$, and the assertion is proved. □

4. Abstract description using T^1 and T^2

We will describe the base algebra of the semi-universal NC deformation of a deformation functor Φ which has the tangent space T^1 and the obstruction space T^2, which is defined below.

Let $\Phi : (\mathrm{Art}_r) \to (\mathrm{Set})$ be an NC deformation functor which has a formal semi-universal deformation $\widehat{\xi} \in \widehat{\Phi}(\widehat{R})$. A k^r-bimodule $T^2 = \bigoplus_{i,j=1}^r T_{ij}^2$ is said to be the *obstruction space* if the following condition is satisfied. Let $\xi \in \Phi(R)$ be an NC deformation over $(R, M) \in (\mathrm{Art}_r)$, and let $(R', M') \in (\mathrm{Art}_r)$ be an extension of R by a two-sided ideal J:

$$0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$$

such that $M'J = 0$, so that J is a left k^r-module. Then there is an obstruction class $o_\xi \in T^2 \otimes_{k^r} J$ such that ξ extends to an NC deformation $\xi' \in \Phi(R')$ if and only if $o_\xi = 0$.

We assume that the obstruction class is functorial in the following sense. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & R' & \longrightarrow & R & \longrightarrow & 0 \\ & & {\scriptstyle g}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \\ 0 & \longrightarrow & J_1 & \longrightarrow & R'_1 & \longrightarrow & R_1 & \longrightarrow & 0 \end{array} \tag{2}$$

be a commutative diagram of such extensions. Let $\xi \in \Phi(R)$ be an NC deformation, and let $\xi_1 = \Phi(f)(\xi) \in \Phi(R_1)$. Let $o_\xi \in T^2 \otimes_{k^r} J$ and $o_{\xi_1} \in T^2 \otimes_{k^r} J_1$ be the obstructions classes of extending ξ and ξ_1 over R' and R'_1, respectively. Then $o_{\xi_1} = g(o_\xi)$.

Theorem 5. *Let $\Phi : (\mathrm{Art}_r) \to (\mathrm{Set})$ be an NC deformation functor. Assume that the obstruction space T^2 is finite dimensional. Then there is a k^r-linear map $m : (T^2)^* \to \widehat{T}_{k^r}(T^1)^*$ such that $\widehat{R} \cong \widehat{T}_{k^r}(T^1)^*/(m((T^2)^*))$, a quotient algebra of the completed tensor algebra by a two-sided ideal generated by the image of m.*

Proof. Denote $\widehat{A} = \widehat{T}_{k^r}(T^1)^* = k^r \oplus \widehat{M}$. Then the base algebra of the semi-universal NC deformation $\widehat{R}$ is a quotient algebra $\widehat{A}/\widehat{I}$ by some two-sided ideal $\widehat{I}$. Let $\{z_l\}_{l=1}^N$ be a k-basis of T^2.

Let $R_k = \widehat{A}/(\widehat{I} + \widehat{M}^{k+1})$. We define a sequence of two-sided ideals $I_k \subset \widehat{A}/\widehat{M}^{k+1}$ by $R_k = \widehat{A}/(I_k + \widehat{M}^{k+1})$. By definition of the semi-universal deformation, there is an NC deformation $\xi_k \in \Phi(R_k)$. We will prove that I_k is generated by elements $\{s_{k,l}\}_{l=1}^N \in \widehat{A}/\widehat{M}^{k+1}$ such that $s_{k+1,l} \mapsto s_{k,l}$ by the natural map $\widehat{A}/\widehat{M}^{k+2} \to \widehat{A}/\widehat{M}^{k+1}$ inductively as follows.

We set $s_{1,l} = 0$ for all l, because $I_1 = 0$ and $R_1 = \widehat{A}/\widehat{M}^2$.

Let k be an arbitrary integer, and let $R = R_k$, $R' = \widehat{A}/(\widehat{M}\widehat{I} + \widehat{M}^{k+1})$ and $J = (\widehat{I} + \widehat{M}^{k+1})/(\widehat{M}\widehat{I} + \widehat{M}^{k+1})$. Then $R = R'/J$ and $M'J = 0$ for $M' = \widehat{M}/(\widehat{M}\widehat{I} + \widehat{M}^{k+1})$. We write the obstruction of extending ξ_k to R' as $o_{\xi_k} = \sum_l z_l \otimes s_{k,l} \in T^2 \otimes_{k^r} J$, where $s_{k,l} \in J$.

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & R' & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & =\downarrow & & \\ 0 & \longrightarrow & J/(s_{k,l}) & \longrightarrow & R'/(s_{k,l}) & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

By the functoriality of the obstruction class, the obstruction class of the lower sequence vanishes, and ξ_k is extendible to $R'/(s_{k,l})$. By the semi-universality, it follows that

$$\widehat{I} + \widehat{M}^{k+1} = (s_{k,l}) + \widehat{M}\widehat{I} + \widehat{M}^{k+1}.$$

By Nakayama's lemma, we have $\widehat{I} + \widehat{M}^{k+1} = (s_{k,l}) + \widehat{M}^{k+1}$. Thus we can write $I_k = (s_{k,l})_{l=1}^N$ as a two-sided ideal in $\widehat{A}/\widehat{M}^{k+1}$.

Here we use a following version of Nakayama's lemma. Let $(A, M) \in (\mathrm{Art}_r)$ and I a two-sided ideal. Assume that there are elements $h_i \in I$ such that $I = MI + (h_i)$. Then $I = (h_i)$. Indeed let $\bar{I} = I/(h_i) \subset \bar{A} = A/(h_i)$. Then $\bar{I} = M\bar{I}$. Since M is nilpotent, $\bar{I} = M\bar{I} = \cdots = M^m\bar{I} = 0$ for some m.

Now we have a commutative diagram

$$\begin{array}{ccc} \widehat{A}/(\widehat{M}\widehat{I} + \widehat{M}^{k+2}) & \longrightarrow & \widehat{A}/(\widehat{I} + \widehat{M}^{k+2}) \\ \downarrow & & \downarrow \\ \widehat{A}/(\widehat{M}\widehat{I} + \widehat{M}^{k+1}) & \longrightarrow & \widehat{A}/(\widehat{I} + \widehat{M}^{k+1}) \end{array}$$

Then the obstruction for the extension on the first line $o_{\xi_{k+1}} = \sum z_l \otimes s_{k+1,l}$ for $s_{k+1,l} \in (\widehat{I} + \widehat{M}^{k+2})/(\widehat{M}\widehat{I} + \widehat{M}^{k+2})$ is mapped to $o_{\xi_k} = \sum z_l \otimes s_{k,l}$. Hence we have $s_{k+1,l} + \widehat{M}^{k+1} = s_{k,l} + \widehat{M}^{k+1}$. Thus we can define $s_l \in \widehat{I}$ such that $s_l + \widehat{M}^{k+1} = s_{k,l} + \widehat{M}^{k+1}$ for all k. Then the s_l generate $\widehat{I}$. □

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