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Multiplicities of Representations in Algebraic Families

Multiplicités des représentations dans les familles algébriques

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Abstract. In this short notes, we consider multiplicities of representations in general algebraic families, especially the upper semi-continuity of homological multiplicities and the locally constancy of Euler–Poincaré numbers. This generalizes the main result of Aizenbud–Sayag for unramified twisting families.

Résumé. Dans cette courte note, nous considérons les multiplicités de représentations dans des familles algébriques générales,en particulier la semi-continuité supérieure des multiplicités homologiques et la constance locale des nombres d'Euler–Poincaré. Ceci généralise le résultat principal d'Aizenbud–Sayag pour les familles obtenues en induisant une représentation fixée que l'on tord par des caractères non ramifiés.

Keywords. Branching laws, Homological multiplicities, Spherical varieties.

Mots-clés. Lois de branchement, multiplicités homologiques, variétés sphériques.

2020 Mathematics Subject Classification. 22E45, 20G25.

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1. Introduction

Let G be a reductive group over a p-adic field F and $H \subset G$ be a closed *spherical* reductive subgroup, i.e. H admits an open orbits the flag variety of G. Let $\mathrm{Rep}(G,\mathbb{C})$ be the category of complex smooth $G(F)$-representations. In the *relative Langlands program* (see [18] etc.), it is central to study the *multiplicity* $m(\sigma) := \dim \mathrm{Hom}_{H(F)}(\sigma,\mathbb{C})$ for smooth admissible $\sigma \in \mathrm{Rep}(G,\mathbb{C})$.

As suggested in [16], to study $m(\sigma)$, it is more convenient to consider the homological multiplicities $m^i(\sigma) := \dim \mathrm{Ext}^i_{H(F)}(\sigma,\mathbb{C})$ and the Euler-Poincare number $\mathrm{EP}(\sigma) := \sum_{i\geq 0}(-1)^i m^i(\sigma)$ simultaneously. Usually, the Euler Poincare number $\mathrm{EP}(\sigma)$ is easier to control and in many circumstances, one may expect to deduce results on $m(\sigma)$ from those of $\mathrm{EP}(\sigma)$. For example, it is

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conjectured in [16] (see also [20, Conjectures 6.4, 6.5]) that when the pair (G,H) is *strongly tempered*, i.e. the matrix coefficients of tempered $G(F)$-representations are absolutely integrable on $H(F)$, then $m(\sigma)=\mathrm{EP}(\sigma)$ for any irreducible tempered $\sigma\in\mathrm{Rep}(G,\mathbb{C})$. Actually the stronger result $m^i(\sigma)=0$ for $i>0$ is known for the $(\mathrm{GL}_{n+1}\times\mathrm{GL}_n,\mathrm{GL}_n)$-case (see [7]), Bessel model for classical groups (see [8]), the triple product case (see [5]) and when $H(F)$ is compact (see [2, Thorem 2.14] etc.) or σ is supercuspidal (see [20, Remark 6.6]).

In this paper, we shall consider variations of $m^i(\sigma)$ and $\mathrm{EP}(\sigma)$ in families. Throughout this paper, we assume the following working hypothesis:

the multiplicity $m(\sigma)$ is *finite* for all irreducible $\sigma\in\mathrm{Rep}(G,\mathbb{C})$.

This implies that $m^i(\sigma)$ and $\mathrm{EP}(\sigma)$ are all well-defined and finite for arbitrary finite length $\sigma\in\mathrm{Rep}(G,\mathbb{C})$ (see the discussion at the beginning of Section 3). Note that this hypothesis is already known in many cases [18, Theorem 5.1.5] and conjectured to hold for all spherical pairs.

To explain the flavor of the main result, let us start with the *unramified twisting family*.

Unramified twisting family. *Let $P\subset G$ be a parabolic subgroup with Levi factor M and take $\sigma\in\mathrm{Rep}(M,\mathbb{C})$ of finite length. Attached to the data (P,M,σ), one has the unramfied twisting family $\left\{I_P^G(\sigma\chi)\middle|\chi\in\widehat{M}\right\}$ where $\widehat{M}$ is the complex torus parameterizing unramified characters of $M(F)$. Then as functions on the complex torus $\widehat{M}$,*

- *$m(I_P^G(\sigma\chi))$ is* upper semi-continuous*, i.e. for each $n\in\mathbb{N}$, the set $\{\chi\in\widehat{M}\mid m(I_P^G\sigma\chi)\le n\}$ is open (see [12, Appendix D]);*
- *$\mathrm{EP}(I_P^G(\sigma\chi))$ is constant (see [3, Theorem E]).*

For arithmetic applications such as p-adic special value formulae on eigenvarieties (see [10] etc.), we are motivated to consider the following setting (following [9, 11]): Fix a subfield $E\subset\mathbb{C}$ and let R be a finitely generated reduced E-algebra. Let π be a finitely generated smooth admissible torsion-free $R[G(F)]$-module, namely a finitely generated $R[G(F)]$-module π such that

- any $v\in\pi$ is fixed by an open compact subgroup of $G(F)$;
- the submodule $\pi^K\subset\pi$ of K-fixed elements is finitely generated over R for any compact open subgroup $K\subset G(F)$;
- π is torsion-free as a R-module.

For any point $x\in\mathrm{Spec}(R)$, let $k(x)$ be the residue field and denote the category of smooth $G(F)$-representations over $k(x)$ by $\mathrm{Rep}(G,k(x))$. For $\pi|_x:=\pi\otimes_R k(x)\in\mathrm{Rep}(G,k(x))$, set

$$m^i(\pi|_x):=\dim_{k(x)}\mathrm{Ext}^i_{H(F)}(\pi|_x,k(x)),\quad \mathrm{EP}(\pi|_x):=\sum_{i\ge0}(-1)^i m^i(\pi|_x)$$

where the Ext-groups are computed in $\mathrm{Rep}(G,k(x))$. Note that by Proposition 15 below, all the numbers $m^i(\pi|_x)$ and $\mathrm{EP}(\pi|_x)$ are well-defined and finite under our running hypothesis.

Supported by the results for unramified twisting families, we propose the following conjecture:

Conjecture 1. *With respect to the Zariski topology on $\mathrm{Spec}(R)$, $m^i(\pi|_x)$ is upper semi-continuous for each $i\in\mathbb{N}$ and $\mathrm{EP}(\pi|_x)$ is locally constant.*

Remark 2. The following example in [5] shows that the upper-semicontinuity is optimal to expect in general. Let K/F be a quadratic field extension and $\theta\in\mathrm{Gal}(K/F)$ be the non-trivial element. The spherical pair $(G:=\mathbb{G}_m\backslash\mathrm{Res}_{K/F}\mathrm{GL}_2, H:=\mathbb{G}_m\backslash\mathrm{GL}_2)$ is not strongly tempered. Consider the $G(F)$-representation $I_P^G(\chi)$ where P is the parabolic subgroup consisting of upper triangular matrices and $\chi=(\chi_1,\chi_2)$ is a character of the Levi quotient $M(F)\cong(K^\times)^2/F^\times$. Then $m^i(I_P^G\chi)=0$, $i\ge2$ and

- $m(I_P^G\chi)\le1$ with the equality holds iff $\chi_1|_{F^\times}=\chi_2|_{F^\times}=1$ or $\mu(\chi):=\chi_1\cdot(\chi_2\circ\theta)=1$;
- $m^1(I_P^G\chi)\le1$ with the equality holds iff $\chi_1|_{F^\times}=\chi_2|_{F^\times}=1$ and $\mu(\chi)\neq1$;
- $\mathrm{EP}(I_P^G\chi)\le1$ with the equality holds iff $\mu(\chi)=1$.

In particular, consider the family $I_P^G(\sigma\chi_\lambda)$ where $\sigma = (\xi, 1)$ with $\xi:\ F^\times\backslash K^\times \to \mathbb{C}^\times$ is a non-trivial character and $\chi_\lambda = (|\cdot|^\lambda, |\cdot|^{-\lambda})$, $\lambda \in \mathbb{C}$. Then as functions of λ, $m^0(I_P^G(\sigma\chi_\lambda))$ and $m^1(I_P^G(\sigma\chi_\lambda))$ both jump at $\lambda = 0$ while $\mathrm{EP}(I_P^G(\sigma\chi_\lambda))$ is constant.

To state the main result, we need to introduce more notations. As the local analogue of classical points in eigen-varieties, we fix a Zariski dense subset $\Sigma \subset \mathrm{Spec}(R)$ of closed points. We say *the fiber rank of π is locally constant on* Σ if for any open compact subgroup $K \subset G(F)$, the function $\dim_{k(x)} \pi^K|_x$ is locally constant on Σ. For any $x \in \mathrm{Spec}(R)$, denote by $(\pi|_x)^\vee$ the smooth dual of $\pi|_x$.

Theorem 3. *Let π be a finitely generated torsion-free smooth admissible $R[G(F)]$-module whose fiber rank is locally constant on Σ. Assume moreover there exists a finitely generated torsion-free admissible R[G(F)]-module $\widetilde{\pi}$ such that for any $x \in \Sigma$, $\widetilde{\pi}|_x \cong (\pi|_x)^\vee$. Then Conjecture 1 holds for π.*

Before explaining the proof, we make several remarks.

Remark 4. For the unramified twisting family π,

- the fiber rank is locally constant by construction;
- the underlying space $\widehat{M}$ is connected and smooth;
- the family $\widetilde{\pi}$ can be taken as $\{I_P^G(\sigma^\vee\chi^{-1})\,|\,\chi \in \widehat{M}\}$.

Thus Theorem 3 covers unramified twisting families.

Remark 5. The existence of the "dual" module $\widetilde{\pi}$ and the locally constancy of fiber rank are necessary for our approach (see the paragraphs after the remark). But these conditions do not impose very serious restrictions:

- the locally constancy of fiber rank may holds for general finitely generated torsion-free smooth admissible $R[G(F)]$-modules. If $\pi|_x$ is absolutely irreducible for all $x \in \Sigma$ and $G = \mathrm{GL}_n$, one can deduce the localy constancy of fiber rank from the theory of co-Whittaker modules in [11];
- if $\pi|_x$ is absolutely irreducible for all $x \in \Sigma$ and G is classical, one can construct the $R[G(F)]$-module $\widetilde{\pi}$ from π by the MVW involution (see [17]).

Now we explain our approach to Theorem 3. We shall use the language of derived categories (see Section 2 for the basics). Let $i_H^G E$ be the compact induction of the trivial representation E of $H(F)$ and $\mathscr{H}(K, E)$ be the level-K Hecke algebra over E. Then by the Frobenius reciprocity law and Bernstein's decomposition theorem (see [3, Theorem 2.5(1)] etc.), for properly chosen open compact subgroup $K \subset G(F)$ (see Proposition 15 below)

$$m^i(\pi|_x) = \dim_{k(x)} \mathrm{Ext}^i_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K\Big|_x\right) = \dim_{k(x)} H^i\left(\mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K\Big|_x\right)\right).$$

By the following upper semi-continuous theorem, this simple observation reduces Theorem 3 to

- the perfectness of $(i_H^G E)^K \in D(\mathscr{H}(K,E))$, which we show in Proposition 16 using the projective resolutions of $G(F)$-representations in [13, Appendix];
- the projectiveness of the R-module $\widetilde{\pi}^K$ by our assumption on the locally constancy of fiber rank (up to shrinking R, see Lemma 13 below).

Here $D(\mathscr{H}(K,E))$ is the derived category of $\mathscr{H}(K,E)$-modules.

Proposition 6 (Upper semi-continuous theorem). *For any complex $M \in D(R)$,*

(i) *the function $\dim_{k(x)} H^i(M \otimes_R^L k(x))$ is upper semicontinuous for each i if M is pseudo-coherent, i.e. quasi-isomorphic to a bounded above complex of finite free R-modules;*
(ii) *the Euler Poincare number $\sum_i (-1)^i \dim_{k(x)} H^i(M \otimes_R^L k(x))$ is locally constant if M is perfect i.e. quasi-isomorphic to a bounded above and below complex of finite projective R-modules.*

Proof. Item (i) is [1, Lemma 0BDI] and Item (ii) is [1, Lemma 0BDJ] □

More precisely, Theorem 3 holds if

(a) $\mathrm{RHom}_{\mathscr{H}(K,E)}\left((i_H^G E)^K, \widetilde{\pi}^K\right)$ is perfect and
(b) there is an isomorphism

$$\mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K\right) \otimes_R^L k(x) \cong \mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K\Big|_x\right).$$

As $\widetilde{\pi}^K$ is projective, the isomorphism in (b) is equivalent to

$$\mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K\right) \otimes_R^L k(x) \cong \mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K \otimes_R^L k(x)\right).$$

By the perfectness of $(i_H^G E)^K \in D(\mathscr{H}(K,E))$, the isomorphism above holds by standard homological algebra (see Lemma 10).

With (b) at hand, by the general criterion of perfectness given in Lemma 12 below, the complex $\mathrm{RHom}_{\mathscr{H}(K,E)}\left((i_H^G E)^K, \widetilde{\pi}^K\right)$ is perfect as $\mathrm{Ext}^i_{\mathscr{H}(K,E)}\left((i_H^G E)^K, \widetilde{\pi}^K\right)$ is finitely generated over R (see Lemma 14) and there is a positive integer N such that for any closed point x, $m^i(\pi|_x) = 0$ for any $i \geq N$ (see Proposition 15).

Remark 7. We briefly compare our approach with that in [3], which deals with unramfied twisting families associated with (P, M, σ). By Frobenius reciprocity law,

$$\mathrm{Ext}^i_{H(F)}\left(I_P^G(\sigma\chi), \mathbb{C}\right) \cong \mathrm{Ext}^i_M\left(r_M^G\left(i_H^G \mathbb{C}\right), \sigma^\vee \chi^{-1}\right)$$

where r_M^G is the normalized Jacquet module functor from $\mathrm{Rep}(G,\mathbb{C})$ to $\mathrm{Rep}(M,\mathbb{C})$. In *loc.cit*, the authors work over $\widehat{M}$ and make full advantage of the theory of Bernstein center and Bernstein decomposition to show that there is a perfect complex $\mathscr{G}(M,\sigma)$ over $\widehat{M}$ associated to $(M,\sigma^\vee)$ such that

$$\mathrm{Ext}^i_M\left(r_M^G\left(i_H^G \mathbb{C}\right), \sigma^\vee \chi^{-1}\right) = H^i\left(\mathrm{RHom}_{\mathbb{C}[\widehat{M}]}(\mathscr{G}(M,\sigma), \delta_\chi)\right)$$

where δ_χ is the skyscraper sheaf at χ^{-1}. Then the locally constancy of Euler-Poincare numbers holds by the semicontinuity theorem for coherent sheaves over smooth varieties. In comparison, our approach seems more direct:

- it works over G and does not depend on the special form of the family;
- it requires less results from representation theory (while more results from homological algebra).

We conclude the introduction by the local constancy of $m(\pi|_x)$ for a finitely generated smooth admissible torsion-free $R[G(F)]$-module π.

Corollary 8. *Assume the pair (G,H) is strongly tempered and*

- *the fiber rank of π is locally constant on Σ and there exists a finitely generated smooth admissible torsion-free $R[G(F)]$-module $\widetilde{\pi}$ such that $\widetilde{\pi}|_x \cong (\pi|_x)^\vee$ for any $x \in \Sigma$,*
- *for any $x \in \Sigma$, $(\pi|_x) \otimes_{k(x),\tau} \mathbb{C}$ is irreducible and tempered for some field embedding $\tau : k(x) \hookrightarrow \mathbb{C}$.*

Then $m(\pi|_x)$ is locally constant on Σ if $m(\sigma) = \mathrm{EP}(\sigma)$ holds for all irreducible tempered $\sigma \in \mathrm{Rep}(G,\mathbb{C})$.

We remark that

- when $H(F)$ is compact, the upper semi-continuity of multiplicities holds under weaker assumptions (see Proposition 18 below);
- when (G,H) is strongly tempered and *Gelfand*, i.e. $m(\sigma) \leq 1$ for all $\sigma \in \mathrm{Rep}(G,\mathbb{C})$ irreducible, the local constancy of multiplicities can be deduced from the upper semi-continuity and the meromorphy property of canonical local periods considered in [6] (see also [4] for analytic families).

2. Homological algebra

For any (unital but possibly noncommutative) ring A, denote by Mod_A (resp. $K(A)$) the category of left A-modules (resp. complexes of A-modules). The derived category $(D(A), q)$ consists a category $D(A)$ with a functor $q : K(A) \to D(A)$ such that any functor $F : K(A) \to \mathscr{C}$ (to any category) which sends quasi-isomorphisms to isomorphisms factors uniquely through $q : K(A) \to D(A)$ (see [14, Chapter III, Section 2]). We usually denote the derived category by $D(A)$. The tensor product and Hom functor on Mod_A admit derived version on $D(A)$ (see [1, Chapter 15] for A commutative and [21] for general A). In particular, we record that

- for any A-algebra A', viewed as a left A'-module and right A-module, the tensor product functor
$$A' \otimes_A - : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{A'}, \quad M \longmapsto A' \otimes_A M$$
has the derived version
$$A' \otimes^L_A - : D(A) \longrightarrow D(A').$$
Note that if $M \in D(A)$ is represented by a bounded above complex $P^\bullet \in K(A)$ of projective A-modules, $A' \otimes^L_A M$ is represented by $A' \otimes_A P^\bullet$;
- for any $N \in \mathrm{Mod}_A$, the functor
$$\mathrm{Hom}_A(-, N) : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}; \quad M \longmapsto \mathrm{Hom}_A(M, N)$$
has the derived version
$$\mathrm{RHom}_A(-, N) : D(A) \longrightarrow D(\mathbb{Z}).$$
Note that if $M \in D(A)$ is represented by a bounded above complex $P^\bullet \in K(A)$ of projective A-modules, $\mathrm{RHom}_A(M, N)$ is represented by the complex $\mathrm{Hom}(P^\bullet, N)$. Moreover, if $N \in \mathrm{Mod}_A$ admits a compatible left R-module structure for some commutative ring R, the functor $\mathrm{RHom}_A(-, N)$ admits a natural lifting, which we denote by the same notation,
$$\mathrm{RHom}_A(-, N) : D(A) \longrightarrow D(R).$$

The following results on base change morphisms in derived category are crucial to our approach. Recall that a complex $M \in D(A)$ is called *pseudo-coherent* (resp. *perfect*) if it is quasi-isomorphic to a bounded above (resp. above and below) complex of finite free (resp. projective) A-modules. Let R be a commutative ring.

Lemma 9. *Let A' be a flat A-algebra and take $N \in \mathrm{Mod}_{A'}$. Then for any pseudo-coherent $M \in D(A)$, there is a canonical isomorphism*
$$\mathrm{RHom}_A(M, N) \cong \mathrm{RHom}_{A'}(A' \otimes^L_A M, N).$$

Proof. Assume M is represented by the bounded above complex $P^\bullet \in K(A)$ of finite projective A-modules. Then $\mathrm{RHom}_A(M, N)$ is represented by $\mathrm{Hom}_A(P^\bullet, N)$ and $\mathrm{RHom}_{A'}(A' \otimes^L_A M, N)$ is represented by $\mathrm{Hom}_{A'}(A' \otimes_A P^\bullet, N)$. The desired result follows from the canonical isomorphism
$$\mathrm{Hom}_A(P, Q) \cong \mathrm{Hom}_{A'}(A' \otimes_A P, Q)$$
for any A-module P and A'-module Q and the exactness of $A' \otimes_A$. □

Lemma 10. *Assume that R is commutative and $N \in \mathrm{Mod}_A$ admits a compatible left R-module structure. Then for any perfect $M \in D(A)$ and $P \in D(R)$, one has the canonical isomorphism*
$$\mathrm{RHom}_A(M, N) \otimes^L_R P \longrightarrow \mathrm{RHom}_A(M, N \otimes^L_R P).$$

Proof. Represent M by a bounded above and below complex $Q^\bullet$ of finite projective modules and P by any complex $P^\bullet$. Then $\mathrm{RHom}_A(M,N)\otimes^L_R P$ is represented by the total complex $\mathrm{Tot}(\mathrm{Hom}_A(Q^\bullet,N)\otimes_R P^\bullet)$ and $\mathrm{RHom}_A(M,N\otimes^L_R P)$ is represented by the complex $\mathrm{Hom}^\bullet_A(Q^\bullet,N\otimes_R P^\bullet)$ with

$$\mathrm{Hom}^n_A(Q^\bullet,N\otimes_R P^\bullet):=\prod_{n=p+q}\mathrm{Hom}_A(Q^{-p},N\otimes_R P^q).$$

Note that for any $W\in\mathrm{Mod}_A$ finite projective and $V\in\mathrm{Mod}_R$, one has

$$\mathrm{Hom}_A(W,N)\otimes_R V\cong\mathrm{Hom}_A(W,N\otimes_R V).$$

Thus the complexes $\mathrm{Tot}(\mathrm{Hom}_A(Q^\bullet,N)\otimes_R P^\bullet)$ and $\mathrm{Hom}^\bullet_A(Q^\bullet,N\otimes_R P^\bullet)$ are isomorphic and we are done. □

Lemma 11. *Assume A is an algebra over R and let R' be a flat commutative R-algebra. Then for $M\in D(A)$ pseudo-coherent and $N\in\mathrm{Mod}_A$, the canonical morphism*

$$\mathrm{RHom}_A(M,N)\otimes^L_R R'\longrightarrow\mathrm{RHom}_{A\otimes_R R'}(M\otimes^L_R R',N\otimes^L_R R')$$

is an isomorphism. In particular if A is Noetherian, one has natural isomorphism

$$\mathrm{Hom}_A(M,N)\otimes_R R'\longrightarrow\mathrm{Hom}_{A\otimes_R R'}(M\otimes_R R',N\otimes_R R')$$

for any finitely generated $M\in\mathrm{Mod}_A$.

Proof. Take a bounded above complex $P^\bullet$ of finite free A-modules representing $M\in D(A)$. Then $\mathrm{RHom}_A(M,N)\otimes^L_R R'$ is represented by $\mathrm{Hom}_A(P^\bullet,N)\otimes_R R'$ and $\mathrm{RHom}_{A\otimes_R R'}(M\otimes^L_R R',N\otimes^L_R R')$ is represented by $\mathrm{Hom}_{A\otimes_R R'}(P^\bullet\otimes_R R',N\otimes_R R')$. The desired result follows from the canonical isomorphism

$$\mathrm{Hom}_A(P,Q)\otimes_R R'\cong\mathrm{Hom}_{A\otimes_R R'}(P\otimes_R R',Q\otimes_R R')$$

for any finite free A-module P and arbitrary A-module Q.

By [1, Lemma 064T], any finitely generated A-module is pseudo-coherent when A is Noetherian and the 'in particular' part follows. □

Now we turn to perfect complexes over commutative rings. Let R be a commutative Noetherian ring. For any $x\in\mathrm{Spec}(R)$, let $k(x)$ be the residue field.

Lemma 12. *A complex $M\in D(R)$ is perfect if the following conditions holds:*

(i) *the R-module $H^i(M)$ is finitely generated for each $i\in\mathbb{Z}$;*

(ii) *there exists $a<b\in\mathbb{Z}$ such that for all closed point $x\in\mathrm{Spec}(R)$, $H^i(M\otimes^L_R k(x))=0$ if $i\notin[a,b]$.*

Proof. By [1, Lemma 068W], when M is pseudo-coherent, Item (ii) implies M is perfect. By [1, Lemma 064T] the assumption R is Noetherian implies that all $H^i(M)$ is pseudo-coherent. Thus by [1, Lemma 066B], M is pseudo-coherent if $H^i(M)=0$ for all $i>b+1$. To see this, consider the exact sequence

$$0\longrightarrow\mathfrak{p}^n/\mathfrak{p}^{n+1}\longrightarrow R/\mathfrak{p}^{n+1}\longrightarrow R/\mathfrak{p}^n\longrightarrow 0$$

for any maximal ideal $\mathfrak{p}\subset R$. By induction, one finds that $H^i(M\otimes^L_R R/\mathfrak{p}^n)=0$ for all $i>b$. By [1, Lemma 0CQE] and [1, Proposition 0922], one has the short exact sequence

$$0\longrightarrow R^1\lim H^{i-1}\left(M\otimes^L_R R/\mathfrak{p}^n\right)\longrightarrow H^i\left(R\lim M\otimes^L_R R/\mathfrak{p}^n\right)\longrightarrow\lim H^i\left(M\otimes^L_R R/\mathfrak{p}^n\right)\longrightarrow 0.$$

By [1, Lemma 0A06], one has

$$H^i(M)\otimes_R\widehat{R}_\mathfrak{p}=H^i(R\lim_n M\otimes^L_R R/\mathfrak{p}^n).$$

Thus $H^i(M)\otimes_R\widehat{R}_\mathfrak{p}=0$ for all $i>b+1$ and all maximal ideal $\mathfrak{p}\subset R$. Consequently, $H^i(M)=0$ for all $i>b+1$ and we are done. □

Finally, we record the following result for a commutative Noetherian ring R.

Lemma 13. *For any finitely generated R-module M, the fiber rank function*

$$\beta(x):\ \mathrm{Spec}(R)\longrightarrow\mathbb{N};\quad x\longmapsto \dim_{k(x)} M\otimes_R k(x)$$

is upper-semicontinuous. If R is moreover reduced, then M is projective iff β is locally constant.

Proof. The first part follows from Proposition 6. For the second part, see [1, Lemma 0FWG]. □

3. Homological multiplicities

Let (G,H) be a spherical pair of reductive groups over p-adic field F. Let $I_H^G\mathbb{C}$ be the space

$$\{f:\ G(F)\longrightarrow\mathbb{C}\text{ smooth}\mid f(hg)=f(g)\ \forall\ h\in H(F),\ g\in G(F)\}$$

on which $G(F)$ acts by right translation and $i_H^G\mathbb{C}\subset I_H^G\mathbb{C}$ be the subspace consisting of functions which are compactly supported modulo $H(F)$. Since $H(F)$ is unimodular, $I_H^G\mathbb{C}$ and $i_H^G\mathbb{C}$ are just the normalized induction and normalized compact induction of the trivial representation $\mathbb{C}$ of $H(F)$ respectively.

By [16, Proposition 2.5],

$$m^i(\sigma)=\dim_{\mathbb{C}}\mathrm{Ext}^i_{G(F)}(\sigma,I_H^G\mathbb{C})=\dim_{\mathbb{C}}\mathrm{Ext}^i_{G(F)}(i_H^G\mathbb{C},\sigma^\vee),\quad\forall\ \sigma\in\mathrm{Rep}(G,\mathbb{C}).$$

For any compact open subgroup $K\subset G(F)$, let $\mathscr{H}(K,\mathbb{C})$ be the Hecke algebra of $\mathbb{C}$-valued bi-K-invariant Schwartz functions on $G(F)$. Then by Bernstein's decomposition theorem (see [3, Theorem 2.5(1)] etc.), there exists a neighborhood basis $\{K\}$ of $1\in G(F)$ consisting of *splitting* (see [2] for the notation) open compact subgroups such that $\mathscr{H}(K,\mathbb{C})$ is Noetherian, the subcategory $\mathscr{M}(G,K,\mathbb{C})$ of representations generated by their K-fixed vectors is a direct summand of $\mathrm{Rep}(G,\mathbb{C})$ and the functor $\sigma\mapsto\sigma^K$ induces an equivalence of categories $\mathscr{M}(G,K,\mathbb{C})\cong\mathrm{Mod}_{\mathscr{H}(K,\mathbb{C})}$. Thus for $\sigma^\vee\in M(G,K,\mathbb{C})$ with K splitting,

$$\mathrm{Ext}^i_{\mathscr{H}(K,\mathbb{C})}\left(\left(i_H^G\mathbb{C}\right)^K,(\sigma^\vee)^K\right)\cong\mathrm{Ext}^i_{G(F)}(i_H^G\mathbb{C},\sigma^\vee),\quad\forall\ i\in\mathbb{Z}.$$

Under our working hypothesis

the multiplicity $m(\sigma)$ is *finite* for all irreducible $\sigma\in\mathrm{Rep}(G,\mathbb{C})$,

$i_H^G\mathbb{C}$ is *locally finitely generated*, i.e. for any compact open subgroup $K\subset G(F)$, $(i_H^G\mathbb{C})^K$ is finitely generated over $\mathscr{H}(K,\mathbb{C})$, by [2, Theorem A]. Thus for K splitting, $(i_H^G\mathbb{C})^K$ admits a resolution by finite projective $\mathscr{H}(K,\mathbb{C})$-modules and consequently for $\sigma\in\mathrm{Rep}(G,\mathbb{C})$ such that $\sigma^\vee\in M(G,K,\mathbb{C})$,

$$m^i(\sigma)=\dim_{\mathbb{C}}\mathrm{Ext}^i_{\mathscr{H}(K,\mathbb{C})}\left(\left(i_H^G\mathbb{C}\right)^K,(\sigma^\vee)^K\right)<\infty,\quad\forall i\in\mathbb{N}.$$

In particular, for any $\sigma\in\mathrm{Rep}(G,\mathbb{C})$ of finite length, $m^i(\sigma)$ is finite for all i. Finally by [16, Corollary III.3.3], for any finite length $\sigma\in\mathrm{Rep}(G,\mathbb{C})$, $m^i(\sigma)=0$ for $i>d(G)$, the split rank of G.

Now we change the coefficient field $\mathbb{C}$ to a subfield $E\subset\mathbb{C}$. Let $\mathrm{Rep}(G,E)$ be category of smooth $G(F)$-representations over E. For any open compact subgroup $K\subset G(F)$, let $\mathscr{H}(K,E)$ be the Hecke algebra of E-valued bi-K-invariant Schwartz functions on $G(F)$, and $M(G,K,E)\subset\mathrm{Rep}(G,E)$ be the subcategory of representations generated by their K-fixed vectors. For any $\sigma\in\mathrm{Rep}(G,E)$, set

$$m^i(\sigma):=\dim_E\mathrm{Ext}^i_{H(F)}(\sigma,E),\ \forall\ i\in\mathbb{N},\quad \mathrm{EP}(\sigma):=\sum_i(-1)^i m^i(\sigma).$$

Let $i_H^G E\in\mathrm{Rep}(G,E)$ be the compact induction of the trivial $H(F)$-representation E. For any open compact subgroup $K\subset G(F)$,

Lemma 14. *For any splitting open compact subgroup $K\subset G(F)$, the Hecke algebra $\mathscr{H}(K,E)$ is Noetherian and $(i_H^G E)^K$ is finitely generated over $\mathscr{H}(K,E)$.*

Proof. Note that $(i_H^G E)^K \otimes_E \mathbb{C} = (i_H^G \mathbb{C})^K$. Take generators $\{y_i = \sum_j f_{i,j} \otimes a_{i,j}\}$ of $(i_H^G\mathbb{C})^K$ over $\mathscr{H}(K,\mathbb{C})$ with $f_{i,j} \in (i_H^G E)^K$. Let $V \subset i_H^G E$ be the $\mathscr{H}(K,E)$-submodule generated by $f_{i,j}$. Since y_i belongs to $V \otimes_E \mathbb{C}$ for each i, one has $V \otimes_E \mathbb{C} = (i_H^G E)^K \otimes_E \mathbb{C}$. Consequently, $V = i_H^G E$ and $i_H^G E$ is locally finitely generated.

Take any ascending chain of left ideals of $\mathscr{H}(K,E)$

$$I_0 \subset I_1 \subset \cdots \subset I_n \subset \cdots.$$

Then $I_i \otimes_E \mathbb{C}$ forms an ascending chain of left ideals of $\mathscr{H}(K,\mathbb{C}) \cong \mathscr{H}(K,E) \otimes_E \mathbb{C}$. Since $\mathscr{H}(K,\mathbb{C})$ is Noetherian, we have that for some n,

$$I_n \otimes_E \mathbb{C} = I_{n+1} \otimes_E \mathbb{C} = \cdots.$$

Consequently, $I_n = I_{n+1} = \cdots$ and $\mathscr{H}(K,E)$ is Noetherian. □

Proposition 15. *For any $\sigma \in \mathrm{Rep}(G,E)$ such that $\sigma^\vee \in M(G,K,E)$, the homological multiplicity*

$$m^i(\sigma) = \dim_E \mathrm{Ext}^i_{\mathscr{H}(K,E)}((i_H^G E)^K, (\sigma^\vee)^K) \quad \forall\ i \in \mathbb{N}.$$

If moreover σ has finite length, then $m^i(\sigma)$ is finite for each $i \geq 0$ and 0 for $i > d(G)$. In particular, $\mathrm{EP}(\sigma)$ *is actually a finite sum.*

Proof. For any $\sigma \in \mathrm{Rep}(G,E)$, set $\sigma_{\mathbb{C}} := \sigma \otimes_E \mathbb{C}$. Then for any $\sigma \in \mathrm{Rep}(G,E)$ and $\theta \in \mathrm{Rep}(G,\mathbb{C})$

$$\mathrm{Hom}_{G(F)}(\sigma,\theta) = \mathrm{Hom}_{G(F)}(\sigma_{\mathbb{C}},\theta).$$

Thus computing using any projective resolution of σ, one finds

$$\mathrm{Ext}^i_{G(F)}(\sigma, I_H^G E) \otimes_E \mathbb{C} \cong \mathrm{Ext}^i_{G(F)}(\sigma_{\mathbb{C}}, I_H^G \mathbb{C}) \quad \forall\ i \geq 0.$$

By Lemma 14, $(i_H^G E)^K \in D(\mathscr{H}(K,E))$ is pseudo-coherent for K splitting. Thus by Lemma 11,

$$\mathrm{Ext}^i_{\mathscr{H}(K,E)}((i_H^G E)^K, (\sigma^\vee)^K) \otimes_E \mathbb{C} \cong \mathrm{Ext}^i_{\mathscr{H}(K,\mathbb{C})}((i_H^G \mathbb{C})^K, (\sigma_{\mathbb{C}}^\vee)^K) \quad \forall\ i \geq 0.$$

From the corresponding results for $\sigma_{\mathbb{C}}$, one deduce that

- if $\sigma^\vee \in \mathrm{Rep}(G,K,E)$,
$$m^i(\sigma) = \dim_E \mathrm{Ext}^i_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, (\sigma^\vee)^K\right),\ \forall\ i \geq 0,$$
- if σ has finite length, $m^i(\sigma)$ is finite for all $i \geq 0$ and $m^i(\sigma) = 0$ if $i > d(G)$. □

Proposition 16. *For any splitting open compact subgroup $K \subset G(F)$, $(i_H^G E)^K \in D(\mathscr{H}(K,E))$ is perfect.*

Proof. Take $V \subset i_H^G \mathbb{C}$ be the sub-representation generated by $(i_H^G \mathbb{C})^K$. By [13, Appendix], V admits an explicit bounded above and below resolution by projective objects in $\mathrm{Rep}(G,\mathbb{C})$ (actually for certain K, the projective resolution can be made explicitly by the theory of Schneider-Stuhler, see [19, Theorem II.3.1] and [15, Theorem 1.2]). Thus there exists a positive integer N such that for any $W \in \mathscr{M}(G,K,\mathbb{C})$,

$$\mathrm{Ext}^i_G(V,W) = 0, \quad \forall\ i > N.$$

Note that $\sigma \mapsto \sigma^K$ induces an equivalence between $\mathscr{M}(G,K,\mathbb{C})$ and the category of $\mathscr{H}(K,\mathbb{C})$-modules (see [3, Theorem 2.5(1)]). Since $(i_H^G \mathbb{C})^K = V^K$, one finds

$$\mathrm{Ext}^i_{\mathscr{H}(K,\mathbb{C})}\left(\left(i_H^G \mathbb{C}\right)^K, M\right) = 0, \quad \forall\ i > N$$

for any $\mathscr{H}(K,\mathbb{C})$-module M. Thus by Lemma 11, for any $\mathscr{H}(K,E)$-module M,

$$\mathrm{Ext}^i_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, M\right) = 0, \quad \forall i > N.$$

Take any resolution $P^\bullet$ of $(i_H^G E)^K$

$$\cdots \longrightarrow P_{N+1} \longrightarrow P_N \longrightarrow \cdots P_1 \longrightarrow P_0 \longrightarrow 0 \cdots$$

by finite projective $\mathscr{H}(K,E)$-modules. Let $Q = \mathrm{coker}(P_{N+2} \to P_{N+1})$. Then Q admits a resolution

$$\cdots \longrightarrow P_{N+2} \longrightarrow P_{N+1} \longrightarrow 0 \cdots$$

Then $\mathrm{Ext}^1_{\mathscr{H}(K,E)}(Q,M) = 0$ for $M = \ker(P_{N+1} \to Q)$. Consequently, $P_{N+1} = Q \oplus M$ and Q is projective. Consequently, $(i_H^G E)^K$ is perfect as it is quasi-isomorphic to

$$\cdots 0 \longrightarrow Q \longrightarrow P_N \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0 \cdots . \qquad \square$$

Now we prove Theorem 3. Let R be a finitely generated reduced E-algebra and $\Sigma \subset \mathrm{Spec}(R)$ be a Zariski denseset subset of closed points. We restate Theorem 3 for the convenience of readers.

Theorem 17. *Let π be a torsion-free smooth admissible finitely generated $R[G(F)]$-module whose fiber rank is locally constant on Σ. Assume that there exists a finitely generated smooth admissible torsion-free $R[G(F)]$-module $\widetilde{\pi}$ such that $\widetilde{\pi}|_x \cong (\pi|_x)^\vee$ for any $x \in \Sigma$. Then $m^i(\pi|_x)$ is upper semi-continuous for each $i \in \mathbb{N}$ and $\mathrm{EP}(\pi|_x)$ is locally constant.*

Proof. Since splitting open subgroups form an neighborhood system of $1 \in G(F)$, one can take a splitting open compact subgroup $K \subset G(F)$ such that $\widetilde{\pi}^K$ generates $\widetilde{\pi}$ and $(i_H^G E)^K \in D(\mathscr{H}(K,E))$ is perfect by Proposition 16. Thus by Proposition 10,

$$\mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K\right) \otimes^L_R k(x) \cong \mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K \otimes^L_R k(x)\right)$$

By the duality between $\pi^K|_x$ and $\widetilde{\pi}^K|_x$ and Lemma 13, upon shrinking $\mathrm{Spec}(R)$ to an open subset containing Σ we can and will assume the fiber rank of $\widetilde{\pi}^U$ is locally constant on Σ and thus the R-module $\widetilde{\pi}^U$ is finite projective. Thus

$$\mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K \otimes^L_R k(x)\right) \cong \mathrm{RHom}_{\mathscr{H}(K,E)}\left(\left(i_H^G E\right)^K, \widetilde{\pi}^K|_x\right).$$

By Lemma 9 and Proposition 15, one has $m^i(\pi|_x) = \dim_{k(x)} \mathrm{Ext}^i_{\mathscr{H}(K,E)}\left((i_H^G E)^K, \widetilde{\pi}^K|_x\right)$. By Proposition 6, to finish the proof it suffices to show the complex $\mathrm{RHom}_{\mathscr{H}(K,E)}\left((i_H^G E)^K, \widetilde{\pi}^K\right)$ is perfect in $D(R)$. As $(i_H^G E)^K$ admits a bounded above and below resolution $P^\bullet$ by finite projective $\mathscr{H}(K,E)$-modules, the complex $\mathrm{RHom}_{\mathscr{H}(K,E)}\left((i_H^G E)^K, \widetilde{\pi}^K\right)$ is represented by $\mathrm{Hom}_{\mathscr{H}(K,E)}(P^\bullet, \widetilde{\pi}^K)$. Since $\widetilde{\pi}^K$ is finitely generated over R by the admissibility of $\widetilde{\pi}$, $\mathrm{Hom}_{\mathscr{H}(K,E)}(P^\bullet, \widetilde{\pi}^K)$ is a complex of finitely generated R-modules. Thus $H^i(\mathrm{RHom}_{\mathscr{H}(K,E)}\left((i_H^G E)^K, \widetilde{\pi}^K)\right)$ are finitely generated as R-modules for each $i \in \mathbb{Z}$. Now the desired perfectness follows from Lemma 12 and Proposition 15. $\square$

Finally, we remark that when $H(F)$ is compact, the upper semi-continuity of $m^i(\pi|_x)$ holds for all torsion-free finitely generated smooth admissible $R[G(F)]$-modules π (here we do not assume the existence of $\widetilde{\pi}$).

Proposition 18. *Assume $H(F)$ is compact. Then for any torsion-free finitely generated smooth admissible $R[G(F)]$-module π, the function $\mathrm{EP}(\pi|_x) = m^0(\pi|_x)$ is upper semi-continuous on $\mathrm{Spec}(R)$.*

Proof. By [2, Theorem 2.14], $m^i(\pi|_x) = 0$ for each $i \geq 1$ and $\mathrm{EP}(\pi|_x) = m^0(\pi|_x)$ for any $x \in \mathrm{Spec}(R)$. Let π_H (resp. π^H) be the $H(F)$-coinvariant (resp. $H(F)$-invariant) of π. Since $H(F)$ is compact, the natural map $\pi^H \to \pi_H$ is an isomorphism and for any $x \in \mathrm{Spec}(R)$, $(\pi^H)|_x \cong (\pi|_x)^H \cong (\pi|_x)_H$. In particular, $m^0(\pi|_x) = \dim_{k(x)}(\pi|_x)_H = \dim_{k(x)} \pi^H|_x$. By Lemma 13 to finish the proof, it suffices to show π^H is coherent.

Note that [3, Theorem 2.5(1)] actually works for any algebraically closed field of characteristic zero. Thus by Proposition 15, for each generic point η of $\mathrm{Spec}(R)$ and some splitting open subgroup K

$$\dim_{k(\eta)} \pi^H|_\eta = \dim_{k(\eta)} \mathrm{Hom}_{G(F)}\left(\left(i_H^G k(\eta)\right)^K, (\pi|_\eta)^{\vee,K}\right) < \infty.$$

Thus there exists an open compact subgroup $K' \subset G(F)$ such that $\pi^H|_\eta \subset (\pi|_\eta)^{K'}$ for all η. Then for any $v \in \pi^H$ and $k \in K'$, $k \cdot v = v$ in $\prod_\eta \pi|_\eta$. Since the diagonal map $\pi \hookrightarrow \prod_\eta \pi|_\eta$ is injective, one has $k \cdot v = v$ in π and consequently $\pi^H \subset \pi^{K'}$. As $\pi^{K'}$ is coherent by the admissibility of π, π^H is coherent and we are done. □

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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