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Asymptotic behaviour of Bohr's radii for polynomials on simply connected domains

Comportement asymptotique des rayons de Bohr pour les polynômes sur des domaines simplement connexes

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Abstract. Let $\mathscr{H}^\infty$ be the Banach space of bounded analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. The Bohr radius R for $\mathscr{H}^\infty$ is defined by

$$R = \sup_{0<r<1} \left\{ r : \sum_{n=0}^{\infty} |a_n(f)|\, r^n \le \|f\|_\infty \text{ for all } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}^\infty \right\}$$

and it is well-known that $R = 1/3$, the Bohr radius, is best possible. The class $\mathscr{P}_n$ of complex polynomials of degree at most n which are bounded by 1 and the Bohr radius for this class is defined by

$$R_n := \sup_{0<r<1} \left\{ r : \sum_{n=0}^{n} |a_n(p)|\, r^n \le \|p\|_\infty \text{ for all } p(z) = \sum_{n=0}^{n} a_n z^n \in \mathscr{P}_n \right\}$$

and the estimate

$$\frac{c_1}{3^{n/2}} < R_n - \frac{1}{3} < c_2 \frac{\log n}{n}$$

is valid for large values of n for absolute positive constants c_1 and c_2. In this paper, we prove the above estimate for simply connected domains.

Résumé. Soit $\mathscr{H}^\infty$ l'espace de Banach des fonctions analytiques bornées dans le disque unité $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, muni de la norme $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. Le rayon de Bohr R pour $\mathscr{H}^\infty$ est défini par

$$R = \sup_{0<r<1} \left\{ r : \sum_{n=0}^{\infty} |a_n(f)|\, r^n \le \|f\|_\infty \text{ pour tout } f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}^\infty \right\}$$

et il est bien connu que $R = 1/3$, le rayon de Bohr, est optimal. La classe $\mathscr{P}_n$ des polynômes complexes de degré au plus n, qui sont bornés par 1, et le rayon de Bohr pour cette classe sont définis par

$$R_n := \sup_{0<r<1} \left\{ r : \sum_{n=0}^{n} |a_n(p)|\, r^n \le \|p\|_\infty \text{ pour tout } p(z) = \sum_{n=0}^{n} a_n z^n \in \mathscr{P}_n \right\}$$

et l'estimation

$$\frac{c_1}{3^{n/2}} < R_n - \frac{1}{3} < c_2 \frac{\log n}{n}$$

est valable pour de grandes valeurs de n, pour des constantes absolues positives c_1 et c_2. Dans cet article, nous démontrons cette estimation pour les domaines simplement connexes.

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Keywords. Bohr inequality, Bohr radius, Bohr phenomenon, analytic functions, majorant series, Dirichlet series, harmonic mappings, holomorphic functions, Reinhardt domains, multidimensional Bohr radius.

Mots-clés. Inégalité de Bohr, rayon de Bohr, phénomène de Bohr, fonctions analytiques, série majorante, séries de Dirichlet, applications harmoniques, fonctions holomorphes, domaines de Reinhardt, rayon de Bohr multidimensionnel.

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1. Introduction

The theory of Diophantine approximation has important applications to the theory of Dirichlet's series. The solution of what is called "absolute convergence problem" for Dirichlet's series of the type $\sum_{n=0}^{\infty} a_n n^{-s}$ is based upon a study of the relation between the absolute value of a power series in an infinite number of variables on the one hand, and the sum of the absolute values of the individual terms of this series on the other hand. Let $\mathscr{B}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $|f(z)| < 1$ in $\mathbb{D}$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}$. In 1914, Harald Bohr [2] raised the following question in his investigations of the study of Dirichlet's series.

Question 1. Let x_1 be a positive number between 0 and 1. Is it possible to find a power series $\sum_{n=0}^{\infty} a_n x^n$ such that:

(i) $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is regular for $|x| < 1$ and continuous for $|x| \le 1$;

(ii) $|f(x)| < 1$ for $|x| \le 1$;

(iii) $\sum_{n=0}^{\infty} |a_n| x_1^n > 1$?

It is easy to see that the answer to Question 1 is affirmative when x_1 is sufficiently close to 1. In fact, the hypothesis (i) and (ii) are perfectly consistent with $\sum_{n=0}^{\infty} |a_n| x_1^n \to \infty$ as $x_1 \to 1$. In the study of Dirichlet series, Harald Bohr [2] discovered the following interesting phenomenon that if $f \in \mathscr{B}$, then its associated majorant series

$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \le 1 \quad \text{for } |z| = r \le \frac{1}{3}. \tag{1}$$

Originally the inequality (1) was actually obtained for $|z| \le 1/6$. The fact that 1/3 is the best possible constant was proved independently by M. Riesz, I. Schur and N. Wiener. This constant $r = 1/3$ is famously known as the Bohr radius for the class $\mathscr{B}$. Furthermore, for $\psi_a(z) = (a-z)/(1-az)$, $a \in [0,1)$, it follows easily that $M_{\psi_a}(r) > 1$ if, and only if, $r > 1/(1+2a)$ and hence the radius 1/3 is optimal as $a \to 1$. This simple result has attracted a great interest of mathematicians. Other proofs can also be found, for example, by Sidon [14], Tomić [15] and Paulsen et al. in [9] and [10,11]. Interest in this inequality was revived a few years ago when Dixon [4] used it to settle in the negative the conjecture that a non-unital Banach algebra that satisfies the von Neumann inequality must be isometrically isomorphic to a closed subalgebra of $B(H)$ for some Hilbert space H.

In 1997, Boas and Khavinson [1] generalized Bohr's theorem from one to several complex variables and found the bounds of Bohr radii in any complete Reinhard domain showing that the radii decrease to zero as dimension of the domain increases. This result stimulates the current wave of research on Bohr's radius and it becomes a source of inspiration for many subsequent papers, linking the asymptotic behavior of $K_{mh}(B_n)$ to various problems in functional analysis (for example, geometry of Banach spaces, unconditional basis constant of spaces of polynomials, etc.). We refer the to [3] for a survey of them. Hence, there was a huge interests in recent years in determining the behavior of $K_{mh}(B_n)$ for large values of n. In 2019, Popescu [13] studied the Bohr inequalities for free holomorphic functions on polyballs B_n, $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, which is a non-commutative analogue of the scalar polyball $(\mathbb{C}^{n_1}) \times \cdots \times (\mathbb{C}^{n_k})$ and studied Bohr radius $K_{mh}(B_n)$ (resp. $K_h(B_n)$) associated with the multi-homogeneous (resp. homogeneous) power series expansions of the free holomorphic functions. Furthermore, Popescu [13] has extended the result of Boas–Khavinson [1] for the scalar polydisk $\mathbb{D}^k$ to B_n and has shown that

$$\frac{1}{3\sqrt{k}} < K_{mh}(B_n) < \frac{2\sqrt{\log k}}{\sqrt{k}} \quad \text{for } k > 1.$$

In 2005, Guadarrama [8] studied the asymptotics of Bohr radii for the class of polynomials

$$\mathscr{P}_n := \left\{ p_n(z) = \sum_{k=0}^{n} a_k z^k : a_n \neq 0,\ \|p_n\|_\infty = 1 \right\}.$$

In this regard, we recall here the Bohr radius for the class $\mathscr{P}_n$.

Definition 2 ([8]). *R_n is called the Bohr radius for the class $\mathscr{P}_n$ if $\sum_{k=0}^{n} |a_k| r^k < 1$ for $r < R_n$ and all $p_n \in \mathscr{P}_n$, and there exists p_n such that $\sum_{k=0}^{n} |a_k| R_n^k = 1$, i.e., the inequality is sharp. In fact, it is easy to see that R_n can be defined by*

$$R_n := \sup_{0<r<1} \left\{ r : \sum_{n=0}^{n} \left| a_n(p) \right| r^n \le \|p\|_\infty \text{ for all } p(z) = \sum_{n=0}^{n} a_n z^n \in \mathscr{P}_n \right\}.$$

It is easy to see that R_n converges to $1/3$ as $n \to \infty$ which follows from Bohr's result. In [12, Section 1], Popescu showed that Bohr's inequality can be improved for analytic polynomials. More precisely, the author proved if $p(z) = \sum_{k=0}^{m-1} a_k z^k$ is a polynomial with $\|p\|_\infty < 1$, then

$$\sum_{k=0}^{m-1} |a_k| r^k \le 1 \quad \text{for } 0 \le r \le t_m,$$

where $t_m \in (0,1]$ is the solution of the equation

$$\sum_{k=0}^{m-1} t^k \cos\left(\frac{\pi}{\left[\frac{m-1}{k} \right] + 2} \right) = \frac{1}{2},$$

where $[x]$ is the integer part of x. Moreover, $\{t_m\}_{m=2}^{\infty}$ is a strictly decreasing sequence which converges to $1/3$.

In view of the Definition 2, Guadarrama [8] proved the following interesting result for polynomials in $\mathbb{C}$ of degree n which are bounded by 1 in the unit disk $\mathbb{D}$.

Theorem 3 ([8]). *If $p_n \in \mathscr{P}_n$ and R_n is Bohr's radius for $\mathscr{P}_n$, then there exist constants c_1 and c_2 such that*

$$\frac{c_1}{3^{n/2}} + \frac{1}{3} < R_n < \frac{1}{3} + c_2 \frac{\log n}{n} \quad \textit{for all } n \in \mathbb{N}.$$

In 2008, Fournier [6] obtained a characterization and conjecture asymptotics of the Bohr radius for the class of complex polynomials in one variable. The author proved that R_n is equal, for each $n \ge 1$, to the smallest root in $(0,1)$ of the equation

$$\det T_n\left(r, -r^2, r^3, \dots, (-1)^{n-1} r^n\right) = 0,$$

where T_n is the $(n+1) \times (n+1)$ Toeplitz matrix. Further, the equality

$$\sum_{k=0}^{n} \left| a_k(p) \right| r^k = \|p\|_\infty$$

holds only for the constant polynomials.

In this paper, we shall however concentrate on the asymptotic behavior of Bohr radii for the polynomials in one variable case. In fact, we shall quantify the rates at which the Bohr radii for the class of polynomials decays as the degree of the polynomial increases. More precisely, our aim is basically two-fold. One is to prove Theorem 3 for polynomials in the general setting, and the other is to find an improved bound of the Bohr radii R_n. Henceforth, let Ω be a simply connected domain containing $\mathbb{D}$ and $\mathscr{B}(\Omega)$ be the class of analytic functions in Ω such that $f(\Omega) \subseteq \overline{\mathbb{D}}$. We define the Bohr radius $B = B_\Omega$ for the class $\mathscr{B}(\Omega)$ by

$$B := \sup\left\{r \in (0,1) : \sum_{n=0}^{\infty} |a_n| r^n \leq 1 \text{ for all } f \in \mathscr{B}(\Omega) \text{ with } f(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in \mathbb{D}\right\}.$$

In particular, if $\Omega = \mathbb{D}$, then it is easy to see that $B_{\mathbb{D}} = 1/3$, which is the classical Bohr radius for the class $\mathscr{B}(\mathbb{D})$. Let $\mathbb{D}(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$, and it is clear that $\mathbb{D} := \mathbb{D}(0,1)$. Let $0 \leq \gamma < 1$. We consider the simply connected domain Ω_γ defined by

$$\Omega_\gamma := \left\{z \in \mathbb{C} : \left|z + \frac{\gamma}{1-\gamma}\right| < \frac{1}{1-\gamma}\right\}.$$

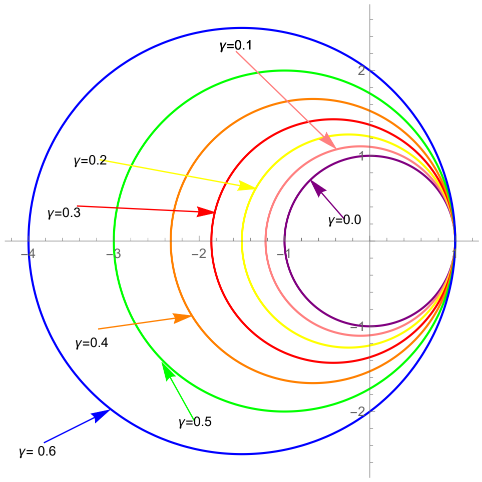

Figure 1. Different simply connected domains $|z+\gamma/(1-\gamma)| < 1/(1-\gamma)$ for different values of γ where $0 \leq \gamma < 1$.

It is easy to see that Ω_γ contains the unit disk $\mathbb{D}$. In 2010, the notion of classical Bohr inequality (1) has been generalized by Fournier and Ruscheweyh [7] to the class $\mathscr{B}(\Omega_\gamma)$. More precisely, the authors have obtained the following result.

Theorem 4 ([7]). *For $0 \leq \gamma < 1$, let $f \in \mathscr{B}(\Omega_\gamma)$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$. Then,*

$$\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \quad \text{for } r \leq \rho := \frac{1+\gamma}{3+\gamma}.$$

Moreover, $\sum_{n=0}^{\infty} |a_n| \rho^n = 1$ holds for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathscr{B}(\Omega_\gamma)$ if, and only if, $f(z) = c$ with $|c| = 1$.

Recently, an improved version of Theorem 4 for the class $\mathscr{B}(\Omega_\gamma)$ has been proved by Evdoridis et al. [5].

Theorem 5 ([5]). *For $0 \leq \gamma < 1$, let $f \in \mathscr{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$. Then*

$$\sum_{n=0}^{\infty} |a_n| r^n + \frac{8}{9}\left(\frac{S_{r(1-\gamma)}}{\pi}\right) \leq 1 \quad \text{for } r \leq \frac{1+\gamma}{3+\gamma},$$

where S_r denotes the area of the image of the disk $\mathbb{D}(0;r)$ under the mapping f. Moreover, the inequality is strict unless f is a constant function. The bound $8/9$ and the number $(1+\gamma)/(3+\gamma)$ cannot be replaced by a larger quantity.

To find the bounds of the coefficients of f with the series representation $f(z)=\sum_{n=0}^{\infty} a_n z^n$ in the unit disk $\mathbb{D}$, Evdoridis et al. [5] have proved the following lemma.

Lemma 6. *For $0\le\gamma<1$, let*

$$\Omega_\gamma := \left\{ z\in\mathbb{C} : \left| z+\frac{\gamma}{1-\gamma}\right| < \frac{1}{1-\gamma}\right\},$$

and f be an analytic function in Ω_γ, bounded by 1, with the series representation $f(z)=\sum_{n=0}^{\infty} a_n z^n$ in the unit disk $\mathbb{D}$. Then,

$$|a_n| \le \frac{1-|a_0|^2}{1+\gamma} \quad \text{for } n\ge 1. \tag{2}$$

In this paper, using the coefficient bounds in (2), we proved an improved version of Theorem 3 and also found the asymptotic Bohr's radius for the polynomials defined on a simply connected domain Ω_γ.

The present paper is organized as follows: in Section 2, we have stated the main result of this paper and the key lemmas also. In Section 3, we have discussed the proof of main result as well as the key lemmas.

2. Main results

We consider the function $f_{a,\gamma}$ which is defined by

$$f_{a,\gamma}(z) := \frac{1-\gamma-(1-\gamma)z}{1-a\gamma-a(1-\gamma)z} = A_0 - \sum_{n=1}^{\infty} A_n z^n, \tag{3}$$

where,

$$\begin{cases} A_0 = \dfrac{1-\gamma}{1-a\gamma}, \\ A_n = \dfrac{1-a}{a(1-a\gamma)}\left(\dfrac{a(1-\gamma)}{1-a\gamma}\right)^n \quad \text{for } n\ge 1. \end{cases}$$

It is easy to see that $f_{a,\gamma}\colon \mathbb{D}\to\Omega_\gamma$ and in particular $f_{a,0}=f_a$, where $f_a\colon\mathbb{D}\to\mathbb{D}$ is the Wiener's function given by

$$f_a(z) = \frac{1-z}{1-az} = 1+(a-1)z+(a^2-a)z^2+\ldots = \sum_{n=0}^{\infty} a_n z^n,$$

where $0<a<1$. We define the polynomial $P_{n,a,\gamma}(z)$ by

$$P_{n,a,\gamma}(z) := A_0 - A_1 z - A_2 z^2 - \cdots - A_n z^n = A_0 - \sum_{k=1}^{n} A_k z^k.$$

A simple computation shows that

$$\begin{aligned} P_{n,a,\gamma}(z) &= \frac{1-\gamma-(1-\gamma)z}{1-a\gamma-a(1-\gamma)z} - \sum_{k=n+1}^{\infty} A_k z^k \\ &= f_{a,\gamma}(z) + \frac{1-a}{a(1-a\gamma)}\left(\frac{a(1-\gamma)}{1-a\gamma}\right)^{n+1} z^{n+1}\left(\sum_{k=0}^{\infty}\left(\frac{a(1-\gamma)z}{1-a\gamma}\right)^k\right) \\ &= f_{a,\gamma}(z) + \frac{1-a}{a}\left(\frac{a(1-\gamma)}{1-a\gamma}\right)^{n+1} \frac{z^{n+1}}{(1-a\gamma)-a(1-\gamma)z}. \end{aligned} \tag{4}$$

We shall prove the following three lemmas which will play a key role to prove our main results of this paper.

Lemma 7. *For fixed a with $0 < a < \min\{\gamma, 1/(2+\gamma)\} < 1$, where $0 \le \gamma < 1$, if $\big((1-\gamma)/(1-a\gamma)\big)^{n+1} a^n < \pi^2/(n+1)^2$, then there exists $\widehat{n} \equiv \widehat{n}(a,\gamma) \in \mathbb{N}$ such that*

$$\|p_{n,a,\gamma}\|_\infty \le \frac{2}{1+a}$$

for all even $n \ge \widehat{n}$.

Lemma 8. *The equation*

$$x = \frac{1+\gamma}{1+2a+a\gamma} + \left(\frac{1-\gamma}{1-a\gamma}\right)^n \frac{(1+a)a^n}{1+2a+a\gamma} x^{n+1}$$

has the unique solution x_γ in the interval $\big((1+\gamma)/(3+\gamma), 1\big)$ for any fixed a with $0 < a < 1$ and any $n \in \mathbb{N}$.

Lemma 9. *The equation*

$$(3+\gamma)x - 2x^{n+1} - (1+\gamma) = 0$$

has a unique solution $r_{n,\gamma}$ in the interval $\big((1+\gamma)/(3+\gamma), 1\big)$.

By making use of the coefficient bounds of the polynomial which is different from that used by Guadarrama [8], in this paper, we obtain the following result which significantly improves Theorem 3.

Theorem 10. *Let $0 \le \gamma < 1$. If $p_n \in \mathscr{P}_n$ and R_n is Bohr's radius for $\mathscr{P}_n$, then there exist constants B_1, B_2 with $0 < B_1, B_2 < \infty$ such that*

$$\frac{B_1}{(3+\gamma)^{n+2}} + \frac{1+\gamma}{3+\gamma} < R_n < \frac{1+\gamma}{3+\gamma} + B_2 \frac{\log n}{n}. \tag{5}$$

The following is an obvious corollary of Theorem 10.

Corollary 11. *If $p_n \in \mathscr{P}_n$ and R_n is Bohr's radius for $\mathscr{P}_n$, then there exist constants B_1, B_2 with $0 < B_1, B_2 < \infty$ such that*

$$\frac{B_1}{3^{n+2}} + \frac{1}{3} < R_n < \frac{1}{3} + B_2 \frac{\log n}{n}. \tag{6}$$

We have the following remarks on Theorem 10 and Corollary 11.

Remark 12. The following observations are clear.

(i) In view of the sandwich theorem, it is easy to see from Theorem 10 and Corollary 11 that $\lim_{n\to\infty} R_n = (1+\gamma)/(3+\gamma)$.

(ii) In particular, if $\gamma = 0$, then Ω_γ reduces to the unit disk $\mathbb{D}$, and we have $\lim_{n\to\infty} R_n = 1/3$ which coincides exactly with the sharp Bohr's radius.

(iii) In particular, when $\gamma = 0$, it is worth pointing out that the left-hand inequality of (5) is $B_1/3^{n+2} + 1/3 < R_n$ which improves the bound of the left-hand inequality $c_1/3^{n/2} + 1/3 < R_n$ in Corollary 11.

3. Proof of the main results

Before proving the main result of this paper, we first prove the key lemmas.

Proof of Lemma 7. The polynomial $P_{n,a,\gamma}$ is defined by

$$\begin{aligned} P_{n,a,\gamma}(z) &= f_{a,\gamma}(z) + \frac{1-a}{a}\left(\frac{a(1-\gamma)}{1-a\gamma}\right)^{n+1} \frac{z^{n+1}}{(1-a\gamma) - a(1-\gamma)z} \\ &= f_{a,\gamma}(z) + \frac{(1-a)g_{a,\gamma}(z)}{(1-a\gamma) - a(1-\gamma)z}, \end{aligned}$$

where $g_{a,\gamma}(z) := a^n\big((1-\gamma)/(1-a\gamma)\big)^{n+1}z^{n+1}$. Then it is easy to see that $g_{a,\gamma}$ is $2\pi/(n+1)$-periodic. For each even $n\in\mathbb{N}$, first we note that

$$\begin{aligned}\left|P_{n,a,\gamma}\left(-\frac{1+\gamma}{1-\gamma}\right)\right| &= \left|f_{a,\gamma}\left(-\frac{1+\gamma}{1-\gamma}\right)+\frac{(-1)^{n+1}a^n(1-a)\left(\frac{1+\gamma}{1-a\gamma}\right)^{n+1}}{(1-a\gamma)+a(1-\gamma)\frac{1+\gamma}{1-\gamma}}\right|\\ &= \left|\frac{2}{1+a}-\left(\frac{1+\gamma}{1-a\gamma}\right)^n\frac{a^n(1-a)}{1+a}\right|\\ &\longrightarrow \frac{2}{1+a}\quad \text{as } n\to\infty,\end{aligned}$$

because $0<a<\min\{\gamma, 1/(1+2\gamma)\}<1$ shows that

$$0<\left(\frac{1+\gamma}{1-a\gamma}\right)<1 \text{ and hence, } \left(\frac{a(1+\gamma)}{1-a\gamma}\right)^n \longrightarrow 0 \text{ as } n\to\infty.$$

On the other hand, a simple computation shows that

$$\frac{2}{1+a}=\|f_{a,\gamma}\|_\infty = f_{a,\gamma}\left(-\frac{1+\gamma}{1-\gamma}\right). \tag{7}$$

By the periodicity of $g_{a,\gamma}$, for any $\tau\in[0,2\pi)$, there exists $\theta_0\in\big[\big((n-1)/(n+1)\big)\pi,\pi\big]$ such that

$$|P_{n,a,\gamma}(e^{i\tau})|\le|f_{a,\gamma}(e^{i\tau})|+|g_{a,\gamma}(e^{i\theta_0})|\le|f_{a,\gamma}(e^{i\theta_0})|+|g_{a,\gamma}(e^{i\theta_0})|.$$

Hence, if

$$|f_{a,\gamma}(e^{i\theta_0})|+|g_{a,\gamma}(e^{i\theta_0})|\le\frac{2}{1+a}$$

for all $\theta\in\big[\big((n-1)/(n+1)\big)\pi,\pi\big]$, then we have

$$\|P_{n,a,\gamma}\|_\infty\le\frac{2}{1+a}.$$

We define $\delta_n := (n-1)/(n+1)$ for $n>2$ and let $\psi := \arg\big(f_{a,\gamma}(e^{\delta_n\pi i})\big)$, then it is easy to see that

$$P_{n,a,\gamma}(e^{\delta_n\pi i}) = f_{a,\gamma}(e^{\delta_n\pi i})-\frac{a^n(1-a)\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}}{(1-a\gamma)-a(1-\gamma)e^{\delta_n\pi i}}$$

and

$$|f_{a,\gamma}(e^{\delta_n\pi i})|=\frac{2}{1+a}\cos\psi=\frac{2}{1+a}\frac{1-\cos(\delta_n\pi)}{\sqrt{2-2\cos(\delta_n\pi)}}=\frac{\sqrt{2-2\cos(\delta_n\pi)}}{1+a}.$$

Therefore, if we find $\widehat{n}=\widehat{n}(a,\gamma)\in\mathbb{N}$ such that

$$\left(\frac{1-\gamma}{1-a\gamma}\right)^{\widehat{n}+1}q^{\widehat{n}}<\frac{2}{1+a}-|f_{a,\gamma}(e^{e^{\delta_n\pi i}})|,$$

then a simple computation shows that

$$\begin{aligned}|P_{n,a,\gamma}(e^{\delta_n\pi i})| &= \left|f_{a,\gamma}(e^{\delta_n\pi i})-\frac{a^n(1-a)\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}}{1-a\gamma+a(1-\gamma)}\right|\\ &\le |f_{a,\gamma}(e^{\delta_n\pi i})|+a^n\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\left|\frac{1-a}{1+a-2a\gamma}\right|\\ &\le |f_{a,\gamma}(e^{\delta_n\pi i})|+a^n\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\quad(\text{since } 0<a,\ \gamma<1)\\ &< \frac{2}{1+a}\quad \text{for all } n\ge\widehat{n}.\end{aligned}$$

Next, our aim is to show that there exists $\widehat{n}=\widehat{n}(a,\gamma)\in\mathbb{N}$ such that, whenever $n\geq\widehat{n}$, then

$$a^n\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}<\frac{2}{1+a}-\frac{\sqrt{2-2\cos(\delta_n\pi)}}{1+a}$$

or, equivalently.

$$1+\cos(\delta_n\pi)>\frac{a^n(1+a)}{2}\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\left(4-a^n(1+a)\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\right).$$

In fact, since

$$-\cos(\delta_n\pi)=\cos\left(\pi-\frac{n-1}{n+1}\pi\right)=\cos\left(\frac{2\pi}{n+1}\right)=1-\frac{4\pi^2}{(n+1)^2}+O\left(\frac{1}{(n+1)^4}\right),$$

i.e.,

$$1+\cos(\delta_n\pi)=\frac{4\pi^2}{(n+1)^2}+O\left(\frac{1}{(n+1)^4}\right)$$

and

$$4a^n\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}>\frac{a^n(1+a)}{2}\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\left(4-a^n(1+a)\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\right),$$

it is easy to see that, if

$$1+\cos(\delta_n\pi)>4a^n\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1},$$

then

$$1+\cos(\delta_n\pi)>\frac{a^n(1+a)}{2}\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\left(4-a^n(1+a)\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\right).$$

We see that

$$a^n\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}<\left(\frac{\pi}{n+1}\right)^2+O\left(\frac{1}{(n+1)^4}\right)$$

is obviously the case for n large enough. We assume that $\widehat{n}$ is the smallest of such n and this completes the proof. □

Proof of Lemma 8. Let the function H be defined by

$$H(x):=\frac{1+\gamma}{1+2a+a\gamma}+\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^n}{1+2a+a\gamma}x^{n+1}-x.$$

For all $a\in(0,1)$, $\gamma\in[0,1)$ and all $n\in\mathbb{N}$, a routine and straightforward computation shows that

$$\begin{aligned}H\left(\frac{1+\gamma}{3+\gamma}\right)&=\frac{1+\gamma}{1+2a+a\gamma}+\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^n}{1+2a+a\gamma}\left(\frac{1+\gamma}{3+\gamma}\right)^{n+1}-\left(\frac{1+\gamma}{3+\gamma}\right)\\&=\frac{(1+\gamma)(1-a)(2+\gamma)}{(1+2a+a\gamma)(3+\gamma)}+\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^n}{1+2a+a\gamma}\left(\frac{1+\gamma}{3+\gamma}\right)^{n+1}\\&\geq\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^n}{1+2a+a\gamma}\left(\frac{1+\gamma}{3+\gamma}\right)^{n+1}\\&>0\end{aligned}$$

and

$$\begin{aligned}H(1)&=\frac{1+\gamma}{1+2a+a\gamma}+\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^n}{1+2a+a\gamma}-1\\&=\frac{\left(\frac{(1+\gamma)a}{1-a\gamma}\right)^n(1+a)+\gamma-a(2+\gamma)}{1+2a+a\gamma}\\&<0\end{aligned}$$

because

$$n > \log_{\frac{(1+\gamma)a}{1-a\gamma}}\Big(\big(a(2+\gamma)-\gamma\big)/(1+a)\Big) \quad \text{for all } n \in \mathbb{N}.$$

It is easy to see that

$$H''(x) = n(n+1)\left(\frac{1-\gamma}{1-a\gamma}\right)^n \frac{(1+a)a^n}{1+2a+a\gamma} x^{n-1} > 0 \quad \text{for all } x \in \left(\frac{1+\gamma}{3+\gamma}, 1\right),$$

hence, the function H is strictly convex.

On the other hand, it is easy to see that H is a polynomial, hence it is continuous and differentiable on the interval $\big((1+\gamma)/(3+\gamma), 1\big)$, therefore, by the Intermediate Value Theorem, there is a zero of H in that interval. On the other hand, it is easy to see that H is decreasing on the interval, hence the zero is unique. This completes the proof. □

Proof of Lemma 9. We fix $n > 1$ and suppose that the function $G(x)$ is defined by

$$G(x) := (3+\gamma)x - 2x^{n+1} - (1+\gamma).$$

It is easy to see that $G(1) = 0$ and hence, $x = 1$ is a root of G. Our aim is to show that G has the unique root in the interval $\big((1+\gamma)/(3+\gamma), 1\big)$. A simple computation shows that

$$G\left(\frac{1+\gamma}{3+\gamma}\right) = -2\left(\frac{1+\gamma}{3+\gamma}\right)^{n+1} < 0.$$

Since

$$\begin{cases} G'(x) = (3+\gamma) - 2(n+1)x^n, \\ G''(x) = -2n(n+1)x^{n-1}, \end{cases}$$

a simple computation shows that

$$G'(1) = (3+\gamma) - 2(n+1) = 2(1-n) < 0.$$

and

$$G'\left(\frac{1+\gamma}{3+\gamma}\right) = \frac{(3+\gamma)^{n+1} - 2(n+1)(1+\gamma)^n}{(1+\gamma)^n}. \tag{8}$$

Moreover, it is easy to see that $G''(x) < 0$ for all $n > 1$. Our aim is to show that $G'\big((1+\gamma)/(3+\gamma)\big) > 0$. By the binomial formula, it is easy to see that

$$\begin{aligned} (3+\gamma)^{n+1} &= \big(2+(1+\gamma)\big)^{n+1} \\ &= 2^{n+1} + {}^{n+1}C_1 2^n(1+\gamma) + {}^{n+1}C_2 2^{n-1}(1+\gamma)^2 + \cdots + {}^{n+1}C_n 2(1+\gamma)^n + (1+\gamma)^{n+1} \\ &= 2^{n+1} + (n+1)2^n(1+\gamma) + \frac{n(n+1)}{2}2^{n-1}(1+\gamma)^2 + \cdots + 2(n+1)(1+\gamma)^n + (1+\gamma)^{n+1} \\ &> 2(n+1)(1+\gamma)^n. \end{aligned} \tag{9}$$

In view of (9), it follows from (8) that $G'\big((1+\gamma)/(3+\gamma)\big) > 0$. Hence, there is a $\eta \in \big((1+\gamma)/(3+\gamma), 1\big)$, where $G'(\eta) = 0$ and such that $G(\eta) > 0$. Thus, G has the unique root in the interval $\big((1+\gamma)/(3+\gamma), 1\big)$, since the function is strictly increasing on that interval. We also see that there are no other roots in the interval $(\eta, 1)$ since the function G is decreasing in $(\eta, 1)$ and $G(1) = 0$. This completes the proof. □

Proof of Theorem 10. For simplicity, we assume n is even and writing $|z|=r$, a simple computation using the coefficient bounds in Lemma 6 shows that the majorant series for the polynomial $P_{n,a,\gamma}$ is given by

$$\begin{aligned} M_{P_{n,a,\gamma}}(r) &= \sum_{k=0}^{n} |a_k| r^k \\ &\leq |a_0| + \frac{1-|a_0|^2}{1+\gamma} \sum_{k=1}^{n} r^k \\ &= |a_0| + \frac{\big(1-|a_0|\big)\big(1+|a_0|\big)}{1+\gamma}\left(\frac{1-r^n}{1-r}\right) r \\ &\leq |a_0| + \frac{2\big(1-|a_0|\big)}{1+\gamma}\left(\frac{1-r^n}{1-r}\right) r. \end{aligned}$$

An involved computation shows that $M_{P_{n,a,\gamma}}(r)\leq 1$ if

$$(3+\gamma)r - 2r^{n+1} \leq 1+\gamma.$$

By Lemma 9, it is easy to see that the Bohr's radius $R_n \geq r_{n,\gamma}$ for all $n\in\mathbb{N}$. Therefore, we obtain

$$r_{n,\gamma} - \frac{1+\gamma}{3+\gamma} = \frac{2}{3+\gamma} r_{n,\gamma}^{n+1}$$

and since, $r_{n,\gamma} > (1+\gamma)/(3+\gamma)$, hence, it is easy to see that

$$R_n - \frac{1+\gamma}{3+\gamma} \geq r_{n,\gamma} - \frac{1+\gamma}{3+\gamma} = \frac{2}{3+\gamma} r_{n,\gamma}^{n+1} > \frac{2(1+\gamma)^{n+1}}{(3+\gamma)^{n+1}} \geq \frac{B_1}{(3+\gamma)^{n+2}} \tag{10}$$

for a constant B_1. This completes the proof of the first part of the inequality (5).

To prove the second part of the inequality (5), we define the majorant series for the function $f_{a,\gamma}$ by

$$M_{f_{a,\gamma}} = \sum_{n=0}^{\infty} |A_n| r^n,$$

where $f_{a,\gamma}$ is defined by (3). A simple computation shows that

$$M_{f_{a,\gamma}}(r) = A_0 + \sum_{n=1}^{\infty} |A_n| r^n = \frac{1-\gamma}{1-a\gamma} + \frac{(1-a)(1-\gamma)r}{(1-a\gamma)\big((1-a\gamma)-a(1-\gamma)r\big)}.$$

A simple computation shows that

$$\|f_{a,\gamma}\|_\infty = \frac{2}{1+a} = f_{a,\gamma}\left(-\frac{1+\gamma}{1-\gamma}\right).$$

Routine and straightforward calculations show that

$$M_{f_{a,\gamma}}(r) > \frac{2}{1+a} \quad \text{if } r > \frac{(1-a)(1+\gamma)}{(1-\gamma)(1+2a+a\gamma)} > \frac{1+\gamma}{1+2a+a\gamma},$$

since $a<\gamma$. Therefore,

$$\lim_{a\to 1} M_{f_{a,\gamma}}(r) = \lim_{a\to 1} \sum_{n=0}^{\infty} |A_n| r^n > 1, \quad \text{whenever } r > \frac{1+\gamma}{1+3\gamma}.$$

It is easy to see that the polynomial $P_{n,a,\gamma}(z)$ is

$$\begin{aligned} P_{n,a,\gamma}(z) &= A_0 - A_1 z - A_2 z^2 - \cdots - A_n z^n \\ &= f_{a,\gamma}(z) + \frac{(1-a)}{a}\left(\frac{a(1-\gamma)}{1-a\gamma}\right)^{n+1} \frac{z^{n+1}}{(1-a\gamma)-a(1-\gamma)z}. \end{aligned}$$

Since

$$0<a<\gamma<1 \text{ implies that } \frac{1-\gamma}{1-a\gamma} < 1,$$

hence, the sequence of polynomials $\{P_{n,a,\gamma}\}$ converges to the function $f_{a,\gamma}(z)$ uniformly on $\mathbb{D}$ because of

$$a^n(1-a)\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}\frac{z^{n+1}}{(1-a\gamma)-a(1-\gamma)z}\longrightarrow 0 \quad \text{as } n\to\infty.$$

The majorant function for the polynomial $P_{n,a,\gamma}(z)$ is defined by

$$\begin{aligned}M_{P_{n,a,\gamma}}(r) &:= \sum_{k=0}^{n}|A_k|r^k\\ &= \frac{1-\gamma}{1-a\gamma}+\frac{(1-a)(1-\gamma)r}{(1-a\gamma)\big(1-a\gamma-a(1-\gamma)r\big)}-a^n(1-a)\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{r^{n+1}}{1-a\gamma-a(1-\gamma)r}.\end{aligned}$$

Then by the Lemma 7, a simple computation shows that

$$M_{P_{n,a,\gamma}}(r) > \|P_{n,a,\gamma}\|_\infty$$

if

$$r > \frac{1+\gamma}{1+2a+a\gamma}+\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^n}{1+2a+a\gamma}r^{n+1}.$$

By Lemma 7 and Lemma 8, for a fixed a and any even $n\ge\hat{n}$, it is easy to that $r>R_{a,\gamma}$ implies that $M_{P_{n,a,\gamma}}(r)>\|P_{n,a,\gamma}\|_\infty$. Hence, the Bohr's radius R_n is less than $R_{a,\gamma}$ for all a and γ, where $0<a<\min\{\gamma,1/(2+\gamma)\}<1$. Therefore,

$$\begin{aligned}R_n &< \frac{1+\gamma}{1+2a+a\gamma}+\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^nR_{a,\gamma}^{n+1}}{1+2a+a\gamma}\\ &= \frac{1+\gamma}{3+\gamma}+\frac{1+\gamma}{3+\gamma}\frac{(1-a)(2+\gamma)}{1+2a+a\gamma}+\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^nR_{a,\gamma}^{n+1}}{1+2a+a\gamma}.\end{aligned}\tag{11}$$

Furthermore,

$$\left(\frac{1-\gamma}{1-a\gamma}\right)^{n+1}a^n<\left(\frac{\pi}{n+1}\right)^2$$

implies that

$$\|P_{n,a,\gamma}\|_\infty\le\frac{2}{1+a}\quad \text{if } 1-a>2\frac{\log n}{n}.$$

Hence, writing $1-a=A(\log n)/n$ with constant A, where it is easy to see that $A>2$, yields

$$\frac{1+\gamma}{3+\gamma}\frac{(1-a)(2+\gamma)}{1+2a+a\gamma}<A_1\frac{\log n}{n},$$

for some positive constant A_1. Hence, a simple computation shows that

$$\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^n}{1+2a+a\gamma}R_{a,\gamma}^{n+1}\frac{n}{\log n}\le\left(\frac{\pi}{n+1}\right)^2R_{a,\gamma}^{n+1}\frac{n}{\log n}\longrightarrow 0\quad \text{as } n\to\infty.$$

Thus, there exists a constant A_2 such that

$$\left(\frac{1-\gamma}{1-a\gamma}\right)^n\frac{(1+a)a^n}{1+2a+a\gamma}R_{a,\gamma}^{n+1}\frac{n}{\log n}<A_2\frac{\log n}{n}.$$

Hence, it follows from (11) that

$$R_n<\frac{1+\gamma}{3+\gamma}+B_2\frac{\log n}{n},\tag{12}$$

for all $n\ge\hat{n}$, where $B_2=A_1+A_2$. In fact, this result holds for all $n\ge\hat{n}$ since R_n decreases to $(1+\gamma)/(3+\gamma)$ as $n\to\infty$. Combining (10) and (12) yields the inequality (5). This completes the proof. □

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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