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The action of pseudo-differential operators on functions harmonic outside a smooth hyper-surface

L'action d'opérateurs pseudo-différentiels sur les fonctions harmoniques en dehors d'une hypersurface lisse

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Dedicated to the memory of Louis Boutet de Monvel.

Abstract. The goal of this note is to describe the action of pseudo-differential operators on the space of square integrable functions which are harmonic outside a smooth closed hyper-surface of a compact Riemannian manifold.

Résumé. Dans cette note, nous étudions un opérateur du type $B = \mathscr{P}^* A\mathscr{P}$ où A est un opérateur pseudo-différentiel et $\mathscr{P}$ l'opérateur de Poisson bilatéral associé à une hypersurface. Nous montrons que, sous certaines conditions, B est un opérateur pseudo-différentiel sur cette hypersurface dont nous calculons le symbole principal.

Keywords. Pseudo-differential operators, Dirichlet-to-Neumann map, Poisson operator.

Mots-clés. Opérateurs pseudos-différentiels, application Dirichlet-to-Neumann, Opérateurs de Poisson.

2020 Mathematics Subject Classification. 35S05, 35S15.

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The goal of this note is to describe the action of pseudo-differential operators on the space $\mathscr{H}$ of L^2 functions which are harmonic outside a smooth closed hyper-surface Z of a compact Riemannian manifold without boundary (X, g) and whose traces from both sides of Z coïncide. We will represent these L^2 harmonic functions as harmonic extensions of functions in the Sobolev space $H^{-1/2}(Z)$ by a Poisson operator $\mathscr{P}$. The main result says that, if A is a pseudo-differential operator of degree $d < 3$ on X, the operator

$$B = \mathscr{P}^\star \circ A \circ \mathscr{P}$$

is a pseudo-differential operator on Z of degree $d-1$ whose principal symbol of degree $d-1$ can be computed by integration of the principal symbol of A on the co-normal bundle of Z.

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These "bilateral" extensions are simpler (at least for the Laplace operator) than the "unilateral" ones whose study is the theory of pseudo-differential operators on manifolds with boundary (see [1–5, 7]).

1. Symbols

The following classes of symbols are defined in the books [5, Section 7.1] and in [6, Section 18.1]. A *symbol of degree d on $U_x \times \mathbb{R}^n_\xi$* where U is an open set in $\mathbb{R}^N$ is a smooth complex valued function $a(x,\xi)$ on $U \times \mathbb{R}^n$ which satisfies the following estimates: for any multi-indices (α,β), there exists a constant $C_{\alpha,\beta}$ so that

$$|D_x^\alpha D_\xi^\beta a(x;\xi)| \le C_{\alpha,\beta}(1+\|\xi\|)^{d-|\beta|} .$$

The symbol a is called *classical* if a admits an expansion $a \sim \sum_{l=0}^\infty a_{d-l}$ where a_j is homogeneous of degree j (j an integer) for $\xi \in \mathbb{R}^n$ large enough; more precisely, for any $J \in \mathbb{N}$, $a - \sum_{j=0}^J a_{d-j}$ is a symbol of degree $d-J-1$.

We will need the

Lemma 1. *If $a(x;\xi,\eta)$ is a symbol of degree $d < -1$ defined on $U_x \times \left(\mathbb{R}^n_\xi \times \mathbb{R}_\eta\right)$, $b(x;\xi) = \int_{\mathbb{R}} a(x;\xi,\eta)\,\mathrm{d}\eta$ is a symbol of degree $d+1$ defined on $U_x \times \mathbb{R}^n_\xi$. Moreover, if a is classical, b is also classical and the homogeneous components of b are given for $l \le d+1$, by $b_l(x;\xi) = \int_{\mathbb{R}} a_{l-1}(x;\xi,\eta)\mathrm{d}\eta$*

2. A general reduction Theorem for pseudo-differential operators

We choose local coordinates in some neighborhood of a point in Z denoted $x = (z,y) \in \mathbb{R}^{d-1} \times \mathbb{R}$, so that $Z = \{y = 0\}$. We denote by $(\Omega_j,\ j = 1,\dots,N)$ a finite cover of Z by such charts and denote by Ω_0 an open set disjoint from Z so that $X = \bigcup_{j=0}^N \Omega_j$. We choose the charts Ω_j so that the densities $|\mathrm{d}z|$ and $|\mathrm{d}x|$ are the Lebesgue measures.

If X is a smooth manifold, we denote by $\mathscr{D}'(X)$ the space of generalized functions on X of which the space of smooth functions on X is a dense subspace. We assume that X and Z are equipped with smooth densities $|\mathrm{d}x|$ and $|\mathrm{d}z|$. This allows to identify generalized functions with Schwartz distributions, i.e. linear functionals on test functions; this duality extending the L^2 product is denoted by $\langle\ |\ \rangle$. We introduce the extension operator $\mathscr{E} : \mathscr{D}'(Z) \to \mathscr{D}'(X)$ sending the distribution f to the distribution $f\delta(y=0)$ defined

$$\langle f\delta(y=0) \mid \phi(z,y)\rangle = \langle f | \phi(z,0)\rangle$$

and its adjoint, the trace $\mathscr{T} : C^\infty(X) \to C^\infty(Z)$ defined by $\phi \to \phi_{|Z}$. Let A be a pseudo-differential operator on X: let us call A_j the restriction of A to test functions compactly supported in Ω_j. We will work with one of the A_j's given by the following "quantization" rule

$$A_j u(z,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(\langle z-z'|\zeta\rangle + (y-y')\eta)}\, a_j(z,y;\zeta,\eta)\, u(z',y')\,\mathrm{d}z'\,\mathrm{d}y'\,\mathrm{d}\zeta\,\mathrm{d}\eta .$$

So we have formally, using the facts that the densities on X and Z are given by the Lebesgue measures in these local coordinates:

$$\mathscr{T}\circ A_j \circ \mathscr{E}\, v(z) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d-1}} \mathrm{e}^{\mathrm{i}\langle z-z'|\zeta\rangle}\, a_j(z,0;\zeta,\eta)\, v(z')\,\mathrm{d}z'\,\mathrm{d}\zeta\,\mathrm{d}\eta ,$$

which we can rewrite

$$\mathscr{T}\circ A_j \circ \mathscr{E}\, v(z) = \frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}\langle z-z'|\zeta\rangle}\, b_j(z;\zeta)\, v(z')\,\mathrm{d}z'\,\mathrm{d}\zeta ,$$

with

$$b_j(z;\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}} a_j(z,0;\zeta,\eta)\,\mathrm{d}\eta . \tag{1}$$

We have the

Theorem 2. *If A is a pseudo-differential operator on X of degree $m < -1$ whose full symbol in the chart Ω_j is a_j, then the operator $\mathscr{T} \circ A \circ \mathscr{E}$ is a pseudo-differential operator on Z of degree $m+1$ whose symbol is given in the charts $\Omega_j \cap Z$ by Equation (1).*

This is proved by looking at the actions on test functions compactly supported in the chart Ω_j, $j \geq 1$: then we use Lemma 1.

Remark 3. The principal symbol can be described in a more intrinsic way: let $z \in Z$ be given, from the smooth densities on $T_z X$ and on $T_z Z$ given by $|\mathrm{d}x|$ and $|\mathrm{d}z|$, we get, using the Liouville densities, densities on the dual bundles $T_z^\star Z$ and $T_z^\star X$. Let us denote by $\Omega^1(E)$ the 1-dimensional space of densities on the vector space E. From the exact sequence

$$0 \longrightarrow N_z^\star Z \longrightarrow T_z^\star X \longrightarrow T_z^\star Z \longrightarrow 0\,,$$

we deduce

$$\Omega^1(T^\star X) \equiv \Omega^1(N^\star Z) \otimes \Omega^1(T^\star Z)$$

and a canonical density $\mathrm{d}m(z)$ in $\Omega^1(N_z^\star Z)$. The principal symbol of $B = \mathscr{T} \circ A \circ \mathscr{E}$ is given in coordinates by $b(z,\zeta) = (1/2\pi)\int_{N_z^\star Z} a(z;\zeta,\eta)\mathrm{d}m(\eta)$.

3. The "bilateral" Dirichlet-to-Neumann operator

We will assume that the local coordinates $x = (z, y)$ along Z are chosen so that $g(z,0) = h(\mathrm{d}z)+\mathrm{d}y^2$ and the Riemannian volume along Z is $|\mathrm{d}x|_g = |\mathrm{d}z|_h\,|\mathrm{d}y|$. We will choose the associated densities on X and Z. We will denote by Δ_g the Laplace–Beltrami operator on (X, g) as defined by Riemannian geometers (i.e. with a minus sign in front of the second order derivatives).

If f is given on Z, let us denote by $\mathscr{D}\mathscr{N}(f)$ minus the sum of the interior normal derivatives on both sides of Z of the harmonic extension F of f; this always makes sense, even if the normal bundle of Z is not orientable. We have the

Lemma 4. *The distributional Laplacian of the harmonic extension F of a smooth function f on Z is $\Delta_g F = \mathscr{E}(\mathscr{D}\mathscr{N}(f))$.*

Proof. The proof is a simple application of the Green's formula: by definition of the action of the Laplacian on distributions, if ϕ is a test function on X, $\langle \Delta_g F \mid \phi\rangle := \langle F \mid \Delta_g \phi\rangle$. We can compute the righthandside integral as an integral on $X \setminus Z$ using Green's formula.

$$\int_{X\setminus Z} (F\Delta_g\phi - \phi\Delta_g F)|\mathrm{d}x|_g = \int_Z (F\delta\phi - \phi\delta F)|\mathrm{d}z|_h$$

where δ is the sum of the interior normal derivatives from both sides of Z. Using the fact that $\Delta_g F = 0$ in $X \setminus Z$ and $\delta\phi = 0$, we get the result. □

Denoting by Δ_g^{-1} the "quasi-inverse" of Δ_g defined by $\Delta_g^{-1}\phi_j = \lambda_j^{-1}\phi_j$ for the eigenfunctions ϕ_j of Δ_g with non-zero eigenvalue λ_j and $\Delta_g^{-1}1 = 0$, we have $f = \left(\mathscr{T} \circ \Delta_g^{-1} \circ \mathscr{E}\right) \circ \mathscr{D}\mathscr{N}(f)$ (mod constants). By Theorem 2, the operator $B = \mathscr{T} \circ \Delta_g^{-1} \circ \mathscr{E}$ is an elliptic self-adjoint pseudo-differential operator on Z. The operator $\mathscr{D}\mathscr{N}$ is a right inverse of B modulo smoothing operators and hence also a left inverse modulo smoothing operators. So that $\mathscr{D}\mathscr{N} = B^{-1}$ is an elliptic self-adjoint of principal symbol the inverse of

$$\frac{1}{2\pi}\int_{\mathbb{R}} (\|\zeta\|_h^2 + \eta^2)^{-1}\mathrm{d}\eta = \frac{1}{2\|\zeta\|_h}\,,$$

namely $2\|\zeta\|_h$. Hence

Theorem 5. *The bilateral Dirichlet-to-Neumann operator $\mathscr{D}\mathscr{N}$ is a self-adjoint elliptic pseudo-differential operator of degree 1 on $L^2(Z,|dz|)$ and of principal symbol $2\|\zeta\|_h$. The kernel of $\mathscr{D}\mathscr{N}$ is the space of constant functions.*

The full symbol of $\mathscr{D}\mathscr{N}$ can be computed in a similar way from the full symbol of the resolvent Δ_g^{-1} along Z.

4. The Poisson operator

Let A be an pseudo-differential operator on X of principal symbol a. We are interested to the restriction to the space $\mathscr{H}$ of the quadratic form $Q_A(F) = \langle AF|F\rangle$ associated to A. We will parametrize $\mathscr{H}$ as harmonic extensions of functions which are in $H^{-\frac{1}{2}}(Z)$ by the so-called Poisson operator denoted by $\mathscr{P}$; the pull-back R_A of Q_A on $L^2(Z)$ is defined by

$$R_A(f) = \langle A\mathscr{P} f \mid \mathscr{P} f\rangle = \langle \mathscr{P}^\star A\mathscr{P} f \mid f\rangle \ .$$

The goal of this section is to compute the operator $B = \mathscr{P}^\star A\mathscr{P}$ associated to the quadratic form R_A.

From Lemma 4, we have, modulo smoothing operators,

$$\mathscr{P} = \Delta_g^{-1} \circ \mathscr{E} \circ \mathscr{D}\mathscr{N} \ .$$

Hence

$$B = \mathscr{D}\mathscr{N} \circ \left[\mathscr{T} \circ \left(\Delta_g^{-1} \circ A \circ \Delta_g^{-1}\right) \circ \mathscr{E}\right] \circ \mathscr{D}\mathscr{N} \ .$$

The operator $\Delta_g^{-1} \circ A \circ \Delta_g^{-1}$ is a pseudo-differential operator of principal symbol $a/(\|\zeta\|_h^2 + \eta^2)^2$ near Z.

Applying Theorem 2 to the inner bracket and Theorem 5, we get the:

Theorem 6. *If A is a pseudo-differential operator of degree $d < 3$ on X and $\mathscr{P}$ the Poisson operator associated to Z, the operator $B = \mathscr{P}^\star A\mathscr{P}$ is a pseudo-differential operator of degree $d-1$ on Z of principal symbol*

$$b(z,\zeta) = \frac{2}{\pi}\|\zeta\|_h^2 \int_{\mathbb{R}} \frac{a(z,0;\zeta,\eta)}{(\|\zeta\|_h^2 + \eta^2)^2} \mathrm{d}\eta \ .$$

Remark 7. Note that if A is a pseudo-differential operator without the transmission property, the operator $A \circ \mathscr{P}$ may be ill-behaved and have disagreeable singularities along Z; however $\mathscr{P}^* A\mathscr{P}$ is always a good pseudo-differential operator on Z.

Note

This note was written with Louis in 2012. We had the project to publish it as an Appendix to a work still in progress with Gregory Berkolaiko. Finally, I decided to publish it independently and to dedicate it to the memory of Louis.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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