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Moduli of rank 2 vector bundles over a curve

Espace de modules des fibrés vectoriels de rang 2 sur une courbe

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Abstract. We show that among rank 2 vector bundles over a curve only semistable and simple objects admit a good moduli space.

Résumé. Nous montrons que parmi les fibrés vectoriels de rang 2 sur une courbe, seuls les objets semistables et simples admettent de bons espaces de modules.

Keywords. Vector bundles, Moduli space, Algebraic stacks.

Mots-clés. Fibrés vectoriels, Espaces de modules, Champs algébriques.

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1. Main result

It is a classical result that semistable vector bundles and simple vector bundles over a curve form a moduli space. The purpose of this note is to show that they are the only possibilities in rank 2 case. To be precise

Theorem 1 (Corollary 5 and Lemma 6). *Let C be a smooth projective connected curve of genus $g_C > 1$ over an algebraically closed field k of characteristic 0. Denote by $\mathscr{B}un_n^d$ the moduli stack of vector bundles of rank n and degree d over C.*

(1) *If $(2,d) = 1$, i.e., d is odd, then the open substack $\mathscr{B}un_2^{d,simple} \subseteq \mathscr{B}un_2^d$ of simple vector bundles is the unique maximal open substack that admits a good moduli space.*
(2) *If $(2,d) \neq 1$, i.e., d is even, then the open substack $\mathscr{B}un_2^{d,simple} \subseteq \mathscr{B}un_2^d$ of simple vector bundles and the open substack $\mathscr{B}un_2^{d,ss} \subseteq \mathscr{B}un_2^d$ of semstable vector bundles are the only maximal open substacks that admit a good moduli space.*

Similar results no longer hold in higher rank, as observed in [3, Theorem C] that there are other open substacks that admit (even separated) good moduli spaces.

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2. Preliminary

The key ingredient to prove Theorem 1 is the following existence criteria for algebraic stacks to admit good moduli spaces.

Theorem 2 ([1, Theorem 4.1]). *Let $\mathscr{X}$ be an algebraic stack, locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space. Then $\mathscr{X}$ admits a good moduli space if and only if*

(1) *every closed point of $\mathscr{X}$ has linearly reductive stabilizer;*
(2) *$\mathscr{X}$ is Θ-reductive; and*
(3) *$\mathscr{X}$ has unpunctured inertia.*

Let $\mathscr{U} \subseteq \mathscr{B}un_n^d$ be an open substack of vector bundles. Since automorphism groups of vector bundles (or in general, coherent sheaves) are connected, the open substack $\mathscr{U}$ has unpunctured inertia by [1, Proposition 3.55]. So the condition (3) is automatic for $\mathscr{U}$. Furthermore, [3, Proposition 3.6] gives a characterization for $\mathscr{U}$ to be Θ-reductive, which implies that in rank 2 case any Θ-reductive open substack cannot contain direct sum of line bundles of different degrees (see [3, Proposition 4.2]). Therefore, it remains to consider condition (1), i.e., we need to understand vector bundles with linearly reductive automorphism groups. In rank 2 case we have a complete classification of such vector bundles.

Proposition 3. *Let $\mathscr{E} \in \mathscr{B}un_2^d(k)$ be a point such that $\mathrm{Aut}(\mathscr{E})$ is linearly reductive. Then one of the following occurs:*

(1) *$\mathscr{E}$ is simple, in which case $\mathrm{Aut}(\mathscr{E}) = \mathbb{G}_m$.*
(2) *$\mathscr{E} \cong \mathscr{L}_1 \oplus \mathscr{L}_2$ for some line bundles $\mathscr{L}_1 \not\cong \mathscr{L}_2$ with $\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2) = 0 = \mathrm{Hom}(\mathscr{L}_2, \mathscr{L}_1)$, in which case $\mathrm{Aut}(\mathscr{E}) = \mathbb{G}_m^2$.*
(3) *$\mathscr{E} \cong \mathscr{L}^{\oplus 2}$ for some line bundle $\mathscr{L}$, in which case $\mathrm{Aut}(\mathscr{E}) = \mathrm{GL}_2$.*

Proof. In characteristic 0, linear reductivity is equivalent to reductivity.

Suppose $\mathscr{E}$ is indecomposable. By [2, Proposition 16] the set of nilpotent endomorphisms $\mathrm{End}^{\mathrm{nil}}(\mathscr{E}) \subseteq \mathrm{End}(\mathscr{E})$ forms a k-subalgebra and

$$\mathrm{End}(\mathscr{E}) = k \cdot \mathrm{id}_{\mathscr{E}} \oplus \mathrm{End}^{\mathrm{nil}}(\mathscr{E}).$$

In particular we see $\mathrm{id}_{\mathscr{E}} + \mathrm{End}^{\mathrm{nil}}(\mathscr{E})$ is a unipotent normal subgroup of $\mathrm{Aut}(\mathscr{E})$, so $\mathrm{End}^{\mathrm{nil}}(\mathscr{E}) = 0$ and then $\mathrm{End}(\mathscr{E}) = k$. This is case (1).

Suppose $\mathscr{E} = \mathscr{L}_1 \oplus \mathscr{L}_2$ for some line bundles $\mathscr{L}_1, \mathscr{L}_2$. If $\mathscr{L}_1 \cong \mathscr{L}_2$, then $\mathrm{Aut}(\mathscr{E}) = \mathrm{GL}_2$. This is case (3).

If $\mathscr{L}_1 \not\cong \mathscr{L}_2$, then any compositions $\mathscr{L}_1 \to \mathscr{L}_2 \to \mathscr{L}_1$ and $\mathscr{L}_2 \to \mathscr{L}_1 \to \mathscr{L}_2$ are zero. Using this it is straightforward to see that the following subset

$$\left\{ \begin{pmatrix} \mathrm{id}_{\mathscr{L}_1} & \phi_{12} \\ \phi_{21} & \mathrm{id}_{\mathscr{L}_2} \end{pmatrix} : \phi_{ij} \in \mathrm{Hom}(\mathscr{L}_i, \mathscr{L}_j) \right\} \subseteq \mathrm{Aut}(\mathscr{E})$$

is a unipotent normal subgroup of $\mathrm{Aut}(\mathscr{E})$, so $\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2) = 0 = \mathrm{Hom}(\mathscr{L}_2, \mathscr{L}_1)$ and then $\mathrm{Aut}(\mathscr{E}) = \mathbb{G}_m^2$. This is case (2). □

3. Proof

Now we are able to prove Theorem 1. Let $\mathscr{U} \subseteq \mathscr{B}un_2^d$ be an open substack, the road map is:

- If every closed point of $\mathscr{U}$ has linearly reductive stabilizer and $\mathscr{U}$ is Θ-reductive, then $\mathscr{U}$ is contained in $\mathscr{B}un_2^{d,ss} \cup \mathscr{B}un_2^{d,simple}$ (see Lemma 4).

- If $\mathscr{U} \nsubseteq \mathscr{B}un_2^{d,ss}$ and $\mathscr{U} \nsubseteq \mathscr{B}un_2^{d,simple}$, then $\mathscr{U}$ supports a degeneration family from a semistable vector bundle to an unstable vector bundle, which is not allowed by Θ-reductivity (see [3, Corollary 3.9]).

Thus, to obtain a good moduli space we must have $\mathscr{U} \subseteq \mathscr{B}un_2^{d,ss}$ or $\mathscr{U} \subseteq \mathscr{B}un_2^{d,simple}$.

Lemma 4. *If $\mathscr{U} \subseteq \mathscr{B}un_2^{d}$ is an open substack that admits a good moduli space, then $\mathscr{U} \subseteq \mathscr{B}un_2^{d,ss} \cup \mathscr{B}un_2^{d,simple}$.*

Proof. Any point $\mathscr{E} \in \mathscr{U}(\mathbb{K})$, for some field $\mathbb{K}/k$, specializes to a closed point $\mathscr{E}_0 \in \mathscr{U}(k)$ and we know $\mathrm{Aut}(\mathscr{E}_0)$ is linearly reductive by Theorem 2. Using Proposition 3

(1) either $\mathscr{E}_0$ is simple. Then $\mathscr{E}$ is also simple by the openness of simpleness.
(2) or $\mathscr{E}_0 \cong \mathscr{L}_1 \oplus \mathscr{L}_2$ for some line bundles $\mathscr{L}_1, \mathscr{L}_2$. By [3, Proposition 4.2] the Θ-reductivity of $\mathscr{U}$ forces that $\deg(\mathscr{L}_1) = \deg(\mathscr{L}_2)$. Then $\mathscr{E}_0$ is semistable and hence $\mathscr{E}$ is also semistable by the openness of semistability.

This shows that $\mathscr{E} \in \mathscr{B}un_2^{d,ss}(\mathbb{K}) \cup \mathscr{B}un_2^{d,simple}(\mathbb{K})$, as desired. □

Corollary 5. *If $(2,d) = 1$, i.e., d is odd, then $\mathscr{B}un_2^{d,simple} \subseteq \mathscr{B}un_2^{d}$ is the unique maximal open substack that admits a good moduli space.*

Proof. In this case $\mathscr{B}un_2^{d,ss} = \mathscr{B}un_2^{d,s} \subseteq \mathscr{B}un_2^{d,simple}$ and we are done by Lemma 4. □

Lemma 6. *If $(2,d) \neq 1$, i.e., d is even, then $\mathscr{B}un_2^{d,simple} \subseteq \mathscr{B}un_2^{d}$ and $\mathscr{B}un_2^{d,ss} \subseteq \mathscr{B}un_2^{d}$ are the only maximal open substacks that admit a good moduli space.*

Proof. If $\mathscr{U} \subseteq \mathscr{B}un_2^{d}$ is an open substack that admits a good moduli space, then $\mathscr{U} \subseteq \mathscr{B}un_2^{d,ss} \cup \mathscr{B}un_2^{d,simple}$ by Lemma 4. It suffices to show $\mathscr{U} \subseteq \mathscr{B}un_2^{d,ss}$ or $\mathscr{U} \subseteq \mathscr{B}un_2^{d,simple}$. Suppose otherwise, then

(1) $\mathscr{U} \nsubseteq \mathscr{B}un_2^{d,ss}$ implies that $\mathscr{U}$ contains an unstable vector bundle.
Let $\mathscr{E} \in \mathscr{U}(k)$ be an unstable vector bundle. Denote by $0 \to \mathscr{L}_1 \to \mathscr{E} \to \mathscr{L}_2 \to 0$ its Harder–Narasimhan filtration.
(2) $\mathscr{U} \nsubseteq \mathscr{B}un_2^{d,simple}$ implies that $\mathscr{U}$ contains a direct sum of line bundles.
Indeed, if $\mathscr{U}$ doesn't contain any direct sum of line bundles, then by Proposition 3 every point in $\mathscr{U}$ specializes to a simple vector bundle, thus $\mathscr{U} \subseteq \mathscr{B}un_2^{d,simple}$ by the openness of simpleness, a contradiction.
Let $\mathscr{V}_1 \oplus \mathscr{V}_2 \in \mathscr{U}(k)$ be a direct sum of line bundles, then $\mathscr{V}_1 \oplus \mathscr{V}_2$ is semistable by Lemma 4, i.e., $\deg(\mathscr{V}_1) = \deg(\mathscr{V}_2) = d/2$. Consider the direct sum morphism
$$\oplus : \mathrm{Pic}^{d/2} \times \mathrm{Pic}^{d/2} \to \mathscr{B}un_2^{d}$$
Then $\Omega := \oplus^{-1}(\mathscr{U}) \subseteq \mathrm{Pic}^{d/2} \times \mathrm{Pic}^{d/2}$ is open and dense since it contains $(\mathscr{V}_1, \mathscr{V}_2)$.

From these data, we can construct a degeneration family in $\mathscr{U}$, over some DVR R over k with fraction field K and residue field κ, from a semistable vector bundle $\mathscr{G}_K$ (as an extension of line bundles of the same degree, using the open subset Ω constructed in (2)) to an unstable vector bundle $\mathscr{G}_\kappa$ (using the unstable vector bundle $\mathscr{E}$ in (1)). Then any Jordan–Hölder filtration $\mathscr{G}_K^\bullet$ of $\mathscr{G}_K$ with the associated graded sheaf $\mathrm{gr}(\mathscr{G}_K^\bullet) \in \mathscr{U}(K)$ cannot even extend to a filtration of subbundles of $\mathscr{G}_\kappa$ (otherwise $\mathscr{G}_\kappa$ will be semistable as well), contradicting to the Θ-reductivity of $\mathscr{U}$. This concludes the proof.

Let us make this precise. By the same proof of [3, Lemma 4.7], the following subset
$$\Delta := \left\{\mathscr{L} \in \mathrm{Pic}^0 : \exists\, [e] \in \mathrm{Ext}^1(\mathscr{L} \otimes \mathscr{L}_2, \mathscr{L} \otimes \mathscr{L}_1) \text{ such that } \mathscr{E}([e]) \in \mathscr{U}\right\}$$
is open and dense in Pic^0, where $\mathscr{E}([e])$ is the extension vector bundle associated to $[e]$. To finish the proof, we claim that

Claim 7. *There exist a line bundle* $\mathscr{L} \in \Delta$ *and an effective divisor* $D \in \mathrm{Div}^a(C)$ *such that* $(\mathscr{L}(-D) \otimes \mathscr{L}_1) \oplus (\mathscr{L}(D) \otimes \mathscr{L}_2) \in \mathscr{U}(k)$, *where* $a := \deg(\mathscr{L}_1) - d/2 > 0$.

Indeed, for any line bundle $\mathscr{L} \in \Delta$ and effective divisor $D \in \mathrm{Div}^a(C)$ in Claim 7, by definition of Δ there exists a class $[e] \in \mathrm{Ext}^1(\mathscr{L} \otimes \mathscr{L}_2, \mathscr{L} \otimes \mathscr{L}_1)$ such that $\mathscr{E}([e]) \in \mathscr{U}(k)$. The extension

$$[e] : 0 \to \mathscr{L} \otimes \mathscr{L}_1 \to \mathscr{E}([e]) \to \mathscr{L} \otimes \mathscr{L}_2 \to 0$$

yields a twisted one

$$[e_D] : 0 \to \mathscr{L}(-D) \otimes \mathscr{L}_1 \to \mathscr{E}([e]) \to (\mathscr{L} \oplus \mathscr{O}_D) \otimes \mathscr{L}_2 \to 0.$$

Then by definition the quadruple

$$(\mathscr{L}(-D) \otimes \mathscr{L}_1, \mathscr{L}(D) \otimes \mathscr{L}_2, D, [e_D]) \\ \in \mathrm{Pic}^{d/2}(k) \times \mathrm{Pic}^{d/2}(k) \times \mathrm{Div}^a(C) \times \mathrm{Ext}^1((\mathscr{L} \oplus \mathscr{O}_D) \otimes \mathscr{L}_2, \mathscr{L}(-D) \otimes \mathscr{L}_1)$$

satisfies $(\mathscr{L}(-D) \otimes \mathscr{L}_1) \oplus (\mathscr{L}(D) \otimes \mathscr{L}_2) \in \mathscr{U}(k)$ and $\mathscr{E}([e_D]) \in \mathscr{U}(k)$. Then $\mathscr{U}$ is not Θ-reductive by [3, Corollary 3.9], a contradiction.

Proof of Claim 7. Consider the morphism

$$\lambda : \mathrm{Pic}^0 \times \mathrm{Div}^a(C) \to \mathrm{Pic}^{d/2} \times \mathrm{Pic}^{d/2} \text{ mapping } (\mathscr{L}, D) \mapsto (\mathscr{L}_1 \otimes \mathscr{L}(-D), \mathscr{L}_2 \otimes \mathscr{L}(D))$$

and it is surjective since for any $(\mathscr{N}_1, \mathscr{N}_2) \in \mathrm{Pic}^{d/2} \times \mathrm{Pic}^{d/2}$ we have

$$\left((\mathscr{N}_1 \otimes \mathscr{N}_2 \otimes \mathscr{L}_1^* \otimes \mathscr{L}_2^*)^{1/2}, (\mathscr{N}_1^* \otimes \mathscr{N}_2 \otimes \mathscr{L}_1 \otimes \mathscr{L}_2^*)^{1/2}\right) \mapsto (\mathscr{N}_1, \mathscr{N}_2).$$

Then $\lambda^{-1}(\Omega) \subseteq \mathrm{Pic}^0 \times \mathrm{Div}^a$ is an open dense subset of $\mathrm{Pic}^0 \times \mathrm{Div}^a$ and therefore

$$\lambda^{-1}(\Omega) \cap (\Delta \times \mathrm{Div}^a(C)) \neq \emptyset.$$

By definition any point $(\mathscr{L}, D)$ in this intersection satisfies the required condition. □

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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