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Some quasi-analytical solutions for propagative waves in free surface Euler equations

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Abstract. This note describes some quasi-analytical solutions for wave propagation in free surface Euler equations and linearized Euler equations. The obtained solutions vary from a sinusoidal form to a form with singularities. They allow a numerical validation of the free-surface Euler codes.

Résumé. Solutions quasi-analytiques d'ondes propagatives dans les équations d'Euler à surface libre. Cette note décrit des solutions quasi-analytiques correspondant à la propagation d'ondes dans les équations d'Euler et d'Euler linéarisées à surface libre. Les solutions obtenues varient d'une forme sinusoïdale à une forme présentant des singularités. Elles permettent de valider numériquement les codes de simulation des équations d'Euler à surface libre.

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1. Introduction

The water wave problem described by the Euler equations with a free surface has been widely studied in the literature, see e.g. [5, 8, 10–12]. This paper proposes some quasi-analytical solution of these equations that allow, for example, to validate the efficiency of the numerical tools. These analytical solutions exhibit singularities of the free surface when the wave amplitude increases. As far as the authors know, it is the first analytical solutions having such a behavior corresponding to an existence result given by W. Strauss, see [3, 13] and references therein. Some other explicit solutions have been presented in the literature, for example by Boulanger et al. [2] and Daboussy et al. [6] for the steady state. Following the methodology of Constantin and Strauss [3], Kalimeris [9] proposes an asymptotic expansion of the Euler system reducing the problem resolution to a cascade of ODEs. On the one hand, the result of Kalimeris is not reduced to flows without vorticity, on the other hand the proposed solutions – also exhibiting singularities of the free surface – are not analytical because obtained through an iterative numerical process.

Solutions presented in Section 2 are solutions of the Euler linearized system up to a negligible term. The proposed solutions are irrotationnal and are compared in Section 3 to the solutions of Airy and third order Stokes waves. In Section 4 this result is extended to the nonlinear Euler system through an additional pressure term on the free surface. Same type of quasi-analytical solutions are proposed in Section 5 for the stationnary waves.

We consider the Euler system and the linearized Euler system over a flat bottom for $x \in \mathbb{R}$ and $0 \le z \le h(t,x)$ given respectively by (1)-(3) and (4)-(6), where $u(t,x,z), w(t,x,z)$ are the two components of the velocity in the (x,z) domain, $h(t,x)$ is the water depth, $p(t,x,z)$ is the pressure and ρ_0 is the density assumed to be constant:

Euler system:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0, \tag{2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho_0}\frac{\partial p}{\partial z} = -g, \tag{3}$$

Linearized Euler system:

$$\frac{\partial \widetilde{u}}{\partial x} + \frac{\partial \widetilde{w}}{\partial z} = 0, \tag{4}$$

$$\frac{\partial \widetilde{u}}{\partial t} + u_0\frac{\partial \widetilde{u}}{\partial x} + \frac{1}{\rho_0}\frac{\partial \widetilde{p}}{\partial x} = 0, \tag{5}$$

$$\frac{\partial \widetilde{w}}{\partial t} + u_0\frac{\partial \widetilde{w}}{\partial x} + \frac{1}{\rho_0}\frac{\partial \widetilde{p}}{\partial z} = -g, \tag{6}$$

These systems are completed by initial conditions

$$\left(u(0,x,z) = u^0(x,z), w(0,x,z) = w^0(x,z), p(0,x,z) = p^0(x,z)\right),$$

a dynamic boundary condition at the free surface

$$p_s = p(t,x,h(t,x)) = p_a(t,x), \tag{7}$$

a kinematic boundary condition at the free surface and a non-penetration condition at the bottom

$$\frac{\partial h}{\partial t} + u_s\frac{\partial h}{\partial x} - w_s = 0, \quad w_b = 0. \tag{8}$$

where the subscript s (resp. b) denotes the considered quantity at the free surface (resp. at the bottom).

Remark 1. For the sake of simplicity we have used the same notations for the solution of the Euler and linearized Euler system but it is clear that they correspond to different solutions.

The linearized Euler system (4)-(6) is obtained by assuming that the velocity components u and w are such that $u = u_0 + \mathcal{O}(\varepsilon)$, $w = \mathcal{O}(\varepsilon)$ with $\varepsilon \ll 1$ and $u_0 = cst$. Around the solution $(u_0, 0, p_a + \rho_0 g(h-z))$, the solution $(\widetilde{u}, \widetilde{w}, \widetilde{p})$ of (4)-(6) yields a remainder term of order $\mathcal{O}(\varepsilon^2)$ in

(1)-(3). It is important to notice that in most cases the linearized Euler system does not admit any energy balance. However, simple computations show that when the quantity

$$e_{s,b} = \frac{|\mathbf{U}_s|^2}{2}\frac{\partial \eta}{\partial t} + \frac{|\mathbf{U}_s|^2}{2} u_0 \frac{\partial \eta}{\partial x},$$

where $\mathbf{U} = (u, w)$ and $|f|^2 = |(f_1, f_2)|^2 = f_1^2 + f_2^2$, can be written under the conservative form $e_{s,b} = \frac{\partial \alpha_{s,b}}{\partial x}$, with $\alpha_{s,b} = \alpha(h, u_0, \mathbf{U}_s)$, the linearized Euler system (4)-(6) completed with (7) and (8) admits an energy balance of the form

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta}\left(E + p_a\right) dz + \frac{\partial}{\partial x}\left[\int_{z_b}^{\eta}\left(u_0 E + u(p+gz)\right) dz + \alpha_{s,b}\right] = h\frac{\partial p_a}{\partial t}, \tag{9}$$

with E defined by $E = \frac{u^2+\tilde{w}^2}{2} + gz$.

The water depth $h(t,x)$ does not appear directly in systems (1)-(3) and (4)-(6), it can be obtained by integrating equation (1) from $z = 0$ to $z = h(t,x)$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^{h(t,x)} u(t,x,\xi)\mathrm{d}\xi\right) = 0. \tag{10}$$

2. Propagating waves for the linearized Euler system

We consider the linearized Euler system (4)-(6) completed by the boundary conditions (7) and (8). A large part of the results are based on the properties of the *LambertW* functions (LW). The *LambertW* functions is the main branch of a set of functions corresponding to the inverse relation of the function $f(z) = ze^z$ where z is any complex number [4]. $\mathrm{LW}(x)$ is the unique real solution of

$$\mathrm{LW}(x)e^{\mathrm{LW}(x)} = x. \tag{11}$$

For $x \geq -1/e$ and for $x \neq 0$ and $x \neq -1/e$ we have $\mathrm{LW}'(x) = \frac{\mathrm{LW}(x)}{x(1+\mathrm{LW}(x))}$. Moreover, $LW(x) \underset{x\to 0}{\sim} x$.

Then the following Proposition 2 holds.

Proposition 2. *Let* $(b, u_0, h_0, k, a, c, \omega) \in \mathbb{R}^2 \times \mathbb{R}_+^5$ *such that* $|a| < 1/e$ *and* $h_0 k/c > \mathrm{LW}(-|a|)$ *and*

$$f : x \mapsto -\frac{c}{k}\mathrm{LW}\left(a\cos\left(\frac{x}{c} + b\right)\right), \tag{12}$$

where LW represents the LambertW function. Let $p^a(t)$ *be any given function.*

The functions h, u, w and p defined by

$$h(t,x) = h_0 + f\left(kx - (\omega + ku_0)\, t\right), \tag{13}$$

$$u(t,x,z) = u_0 - \frac{\omega a}{k} e^{\frac{k}{c}(z-h_0)} \cos\left(\frac{kx - (\omega + ku_0)\, t}{c} + b\right), \tag{14}$$

$$w(t,x,z) = -\frac{c}{k}\frac{\partial u}{\partial x}, \tag{15}$$

$$p(t,x,z) = p_a(t) + \rho_0 g\,(h_0 - z) - \frac{\rho_0 g c a}{k} e^{\frac{k}{c}(z-h_0)} \cos\left(\frac{kx - (\omega + ku_0)\, t}{c} + b\right), \tag{16}$$

are quasi-analytical solutions of the linearized Euler system (4)-(6) *completed by the boundary conditions* (7) *and* (8) *if and only if the following relation holds*

$$\frac{\omega}{k} = \sqrt{\frac{gc}{k}}. \tag{17}$$

More precisely, equations (4)-(6)*, dynamic pressure condition* (7) *and the kinematic free surface condition in equation* (8) *are verified exactly while. For equations* (8) *and* (10)*, we get:*

$$\widetilde{w}_b = \mathcal{O}\left(e^{-\frac{h_0 k}{c}}\right) \quad \textit{and} \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^{h(t,x)} \widetilde{u}(t,x,\xi)\mathrm{d}\xi\right) = \mathcal{O}\left(e^{-\frac{h_0 k}{c}}\right). \tag{18}$$

Sketch of proof. To make the proof more readable, we set $u_0 = 0$, $b = 0$ and we take

$$Y = a\cos\left((kx-\omega t)/c\right).$$

From (4) and using the definition of u given in (14) we obtain easily the expression of w. From (14)-(16), by simple derivation of u and p, equation (5) gives

$$\left(-\frac{\omega^2 a}{k} + ga\right) e^{\frac{k}{c}(z-h_0)} \sin(Y) = 0.$$

This equation is verified for all Y only if the relation (17) holds. This relation is similar to the dispersion relation in the Airy theory when $kh_0 \gg 1$. Expression of $\widetilde{p}$ is obtained by integrating (6) from z to h. It is easy to verify a posteriori that (6) is verified by taking the derivative of $\widetilde{w}$ and $\widetilde{p}$ appearing in this equation. From (16), we observe that

$$\begin{aligned}\widetilde{p}(t,x,h(t,x)) &= p^a(t) - gf(kx-\omega t)) - \frac{gc}{k} e^{\frac{k}{c} f(kx-\omega t))} Y,\\ &= p^a(t) + \frac{gc}{k}\mathrm{LW}(Y) - \frac{gc}{k} e^{-\mathrm{LW}(Y)} Y = p^a(t).\end{aligned}$$

The main difficulty is to verify the surface evolution equation (10). From (13) we obtain

$$\frac{\partial h}{\partial t} = \left(\frac{\omega a}{k}\right)\sin\left(\frac{kx-\omega t}{c}\right)\frac{\mathrm{LW}(Y)}{Y(1+\mathrm{LW}(Y))}. \tag{19}$$

With the expression of $\widetilde{u}$ given in (14) we deduce using (11) that

$$\begin{aligned}\int_0^{h(t,x)} \widetilde{u}(t,x,\xi)\mathrm{d}\xi &= -\frac{\omega}{k} Y \int_0^{h(t,x)} e^{\frac{k}{c}(\xi-h_0)}\mathrm{d}\xi,\\ &= -\frac{\omega c}{k^2} Y \left(e^{-\frac{k}{c}(h(t,x)-h_0)} - e^{\frac{k}{c}h_0}\right),\\ &= \frac{\omega c}{k^2}\left(Ye^{-\frac{k}{c}h_0} - \mathrm{LW}(Y)\right).\end{aligned}$$

Then, using the same expression of the derivative of the LW function, we have

$$\frac{\partial}{\partial x}\left(\int_0^{h(t,x)} \widetilde{u}(t,x,\xi)\mathrm{d}\xi\right) = -\frac{\omega a}{k}\sin\left(\frac{kx-\omega t}{c}+b\right)\left(e^{-\frac{k}{c}h_0} + \frac{\mathrm{LW}(Y)}{Y(1+\mathrm{LW}(Y))}\right). \tag{20}$$

We deduce that equation (10) is verified up to a term

$$\frac{\omega a}{k}\sin\left(\frac{kx-\omega t}{c}+b\right)e^{-\frac{k}{c}h_0}.$$

In the same way, vertical velocity at the bottom is given by $\widetilde{w}_b = -t\frac{\omega a}{c}e^{-\frac{k}{c}h_0}$. The solutions proposed in Proposition 2 for the linearized Euler system are not exactly analytical solutions in the sense that additional terms in

$$\mathcal{O}\left(e^{-\frac{h_0 k}{c}}\right)$$

appear. But when $h_0 k \gg 1$, considering e.g. $h_0 = 100$ m, $k = 0.2$ m^{-1} and $c = 1$ gives $e^{-\frac{h_0 k}{c}} \approx 10^{-9} \ll 1$. These solutions satisfy the energy balance (9) with

$$\alpha_{s,b} = -\frac{g\omega}{4k}\frac{c^2\,\mathrm{LW}\left(a\cos\left(\frac{kx-(\omega+ku_0)t}{c}+b\right)\right)^2}{k^2\cos\left(\frac{kx-(\omega+ku_0)t}{c}+b\right)^2}.$$

□

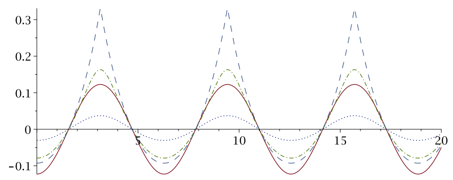

Figure 1. Free surface given by the function $f : x \mapsto -\mathrm{LW}(a\cos(x))$ for three values of the parameter a : 0.10 (dot line), 0.30 (dash-dot line), 1/e (dash line), and comparison with function $(-1/e)\cos(x)$ (solid line).

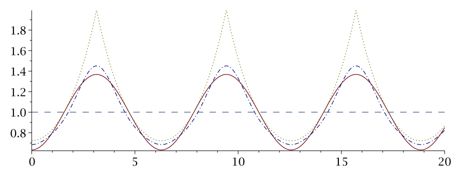

Figure 2. Free surface given by Airy theory (solid line), third order Stokes theory (dash-dot line) and equation (13) (dot line) for the given set of parameters : $(a = 1/e,\ k = 1,\ c = 1,\ t = 0,\ b = 0)$.

3. Comparison with the Airy and Stokes wave theories

The form of the free surface of our quasi-analytical solution depends of the parameter a. When a is small, surface elevation is close to a sinusoidal function since we have seen that $LW(x) \underset{x\to 0}{\sim} x$. When $|a|$ is near to $1/e$, the top of the wave is narrower than its bottom. This property is presented in Figure 1 with parameters ($c = 1,\ k = 1,\ t = 0$ and $b = 0$). For $|a| = 1/e$, the function f is not differentiable in $(2m+1)\pi,\ \forall\ m \in \mathbb{Z}$. The non-differentiable character of the solution was studied from a theoretical point of view by Strauss [3]. Here we give an explicit expression of this solution.

In the literature, some analytical solutions of free surface simplified models have been proposed [5]. The most known are the Airy wave and the third-order Stokes wave, and we propose here some numerical comparisons.

By setting $\theta = (k_0 x - \omega_0 t + b_0)$, in the Airy wave theory, surface elevation is given by

$$h(t,x) = h_0 + a_0\cos(\theta), \tag{21}$$

and in the third-order Stokes wave on deep water, surface elevation is given by [7]

$$h(t,x) = h_0 + a_0\left(\left(1 - \frac{1}{16}(k_0 a_0)^2\right)\cos(\theta) + \frac{1}{2}k_0 a_0\cos(2\theta) + \frac{3}{8}(k_0 a_0)^2\cos(3\theta)\right). \tag{22}$$

To make a comparison with water depth given by equation (13), we have to set $a = a_0 k_0$ (using again $LW(x) \underset{x\to 0}{\sim} x$), $k = ck_0$, $b = b_0$ and $\omega = c\omega_0$. We have plotted in Figure 2 a comparison between the free surface obtained by Airy theory, third order Stokes theory and equation (13) for the given set of parameters $(a = 1/e,\ k = 1,\ c = 1,\ t = 0,\ b = 0)$. With this value of a we observe numerically the maximum of differences between the three waves. Of course, for small value of a, solution of (13) can be very close to the two other waves.

4. Propagating waves for the Euler system

We consider the Euler system (1)-(3) completed by the boundary conditions (7) and (8). The previous solution can be extended by considering a small pressure term at the free surface and the following Proposition 3 holds.

Proposition 3. *Under the same conditions of the Proposition 2 the functions h, u, w and p defined by*

$$h(t,x) = h_0 + f(kx - (\omega + ku_0)t),$$
$$u(t,x,z) = u_0 - \frac{\omega a}{c} e^{\frac{k}{c}(z-h_0)} \cos\left(\frac{kx - (\omega + ku_0)\,t}{c} + b\right),$$
$$w(t,x,z) = -\frac{c}{k}\frac{\partial u}{\partial x},$$
$$p(t,x,z) = g\,(h_0 - z) + \frac{ga^2c}{2k}\left(1 - e^{\frac{2k}{c}(z-h_0)}\right) - \frac{gca}{k} e^{\frac{k}{c}(z-h_0)} \cos\left(\frac{kx - (\omega + ku_0)t}{c} + b\right),$$

are quasi-analytical solutions of the Euler system (1)-(3) *completed by the boundary conditions* (7) *and* (8) *iff the relation* (17) *holds. At the free surface, the pressure p is such that* $p_s = p^a(t,x)$ *with*

$$p^a(t,x) = \frac{ga^2c}{2k}\left(1 - e^{\frac{2k}{c}(h(t,x)-h_0)}\right). \tag{23}$$

More precisely, equations (4)-(6)*, dynamic pressure condition* (7) *and the kinematic free surface condition in equation* (8) *are verified exactly. For equations* (8) *and* (10)*, we get*

$$\widetilde{w}_b = \mathscr{O}\left(e^{-\frac{h_0 k}{c}}\right) \quad \textit{and} \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^{h(t,x)} \widetilde{u}(t,x,\xi)\mathrm{d}\xi\right) = \mathscr{O}\left(e^{-\frac{h_0 k}{c}}\right). \tag{24}$$

Pressure at the surface is equal to zero when $h(t,x) = h_0$. If $h(t,x) - h_0$ is small, this pressure is of the order a^3. The proof is similar to the one given in Proposition 2 but is more tedious due to the nonlinearity.

Remark 4. The solutions proposed in Propositions 2 and 3 are irrotational. In [1], C. J. Amick proved that for any irrotational wave, the angle of inclination of the free surface with respect to the horizontal must be less than 31.15°. In our analytical solutions, the angle of inclination is less or equal to 45°, but we have an additional source term (23) that can justify this inclination.

5. Standing waves

Now we consider the situation of standing waves that occur when two progressive waves of same amplitude travel in opposite direction. The results depicted in this paragraph are based on the following remark: for small values of the parameter a, one has the Taylor expansion

$$\mathrm{LW}\left(a\cos\left(\frac{kx-\omega t}{c}\right)\right) = a\cos\left(\frac{kx-\omega t}{c}\right) - a^2\cos^2\left(\frac{kx-\omega t}{c}\right) + \mathscr{O}\left(a^3\right).$$

Proposition 5. *Under the same conditions as in Proposition 2 the functions h, u, w and p defined by*

$$h(t,x) = h_0 + f(kx - \omega t) + f(kx + \omega t), \tag{25}$$
$$u(t,x,z) = \frac{\omega}{c} e^{\frac{k}{c}(z-h(t,x))}\left(f(kx-\omega t) - f(kx+\omega t)\right), \tag{26}$$
$$w(t,x,z) = -\frac{c}{k}\frac{\partial u}{\partial x}, \tag{27}$$
$$p(t,x,z) = p^a(t) + g(h-z) + \int_z^{h(t,x)} \frac{\partial w}{\partial t} dz, \tag{28}$$

are quasi-analytical solutions of the linearized Euler system (4)-(6) *completed by the boundary conditions* (7) *and* (8) *iff the relation* (17) *holds. At the free surface, the pressure p is such that* $p_s = p^a(t,x)$ *with*

$$p^a(t,x) = \frac{ga^2c}{2k}\left(1 - e^{\frac{2k}{c}(h(t,x)-h_0)}\right). \tag{29}$$

More precisely, equations (4) *and* (6)*, dynamic pressure condition* (7) *and the kinematic free surface condition in equation* (8) *are verified exactly. For equations* (5)*,* (8) *and* (10)*, we get*

$$\frac{\partial \widetilde{u}}{\partial t} + u_0 \frac{\partial \widetilde{u}}{\partial x} + \frac{1}{\rho_0} \frac{\partial \widetilde{p}}{\partial x} = \mathscr{O}\left(a^3\right), \tag{30}$$

$$\widetilde{w}_b = \mathscr{O}\left(a e^{-\frac{h_0 k}{c}}\right) \quad and \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^{h(t,x)} \widetilde{u}(t,x,\xi)\mathrm{d}\xi\right) = \mathscr{O}\left(a e^{-\frac{h_0 k}{c}}\right). \tag{31}$$

Since a Taylor expansion of Equation (25) *gives*

$$h(t,x) = h_0 - \frac{2ac}{k}\cos\left(\frac{kx}{c}\right)\cos\left(\frac{\omega t}{c}\right) + \mathscr{O}\left(\frac{a^2}{k}\right),$$

the proposed solution corresponds, up to terms in $\mathscr{O}(\frac{a^2}{k})$*, to a standing wave.*

The proofs rely on simple but tedious computations similar to those performed in the proof of Proposition 2.

Proposition 6. *Under the same conditions as in Proposition 3 the functions h, u, w and p defined by*

$$h(t,x) = h_0 + f(kx - \omega t) + f(kx + \omega t), \tag{32}$$

$$u(t,x,z) = \frac{\omega}{c} e^{\frac{k}{c}(z - h(t,x))}\left(f(kx - \omega t) - f(kx + \omega t)\right), \tag{33}$$

$$w(t,x,z) = -\frac{c}{k}\frac{\partial u}{\partial x}, \tag{34}$$

$$p(t,x,z) = g(h-z) + \frac{g a^2 c}{2k}\left(1 - e^{\frac{2k}{c}(z-h_0)}\right) + g(h-z) + \int_z^{h(t,x)} \frac{\partial w}{\partial t} dz, \tag{35}$$

are quasi-analytical solutions of the Euler system (1)*-*(3) *completed by the boundary conditions* (7) *and* (8) *iff the relation* (17) *holds. At the free surface, the pressure p is such that $p_s = p^a(t,x)$ with*

$$p^a(t,x) = \frac{g a^2 c}{k}\cos\left(\frac{2kx}{n}\right). \tag{36}$$

More precisely, equations (1) *and* (3)*, dynamic pressure condition* (7) *and the kinematic free surface condition in equation* (8) *are verified exactly. For equations* (1)*,* (8) *and* (10)*, we get*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = \mathscr{O}\left(a^3\right), \tag{37}$$

$$\widetilde{w}_b = \mathscr{O}\left(a e^{-\frac{h_0 k}{c}}\right) \quad and \quad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^{h(t,x)} \widetilde{u}(t,x,\xi)\mathrm{d}\xi\right) = \mathscr{O}\left(a^2 e^{-\frac{h_0 k}{c}}\right). \tag{38}$$

Since a Taylor expansion of equation (25) *gives*

$$h(t,x) = h_0 - \frac{2ac}{k}\cos\left(\frac{kx}{c}\right)\cos\left(\frac{\omega t}{c}\right) + \mathscr{O}\left(\frac{a^2}{k}\right),$$

the proposed solution corresponds, up to terms in $\mathscr{O}(\frac{a}{k})$*, to a standing wave.*

The proofs rely on simple but tedious computations similar to those performed in the proof of Proposition 2.

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References

[1] C. J. Amick, "Bounds for water waves", *Arch. Ration. Mech. Anal.* **99** (1987), no. 2, p. 91-114.

[2] A.-C. Boulanger, M.-O. Bristeau, J. Sainte-Marie, "Analytical solutions for the free surface hydrostatic Euler equations", *Commun. Math. Sci.* **11** (2013), no. 4, p. 993-1010.

[3] A. Constantin, W. A. Strauss, "Exact steady periodic water waves with vorticity", *Commun. Pure Appl. Math.* **57** (2004), no. 4, p. 481-527.

[4] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, D. E. Knuth, "On the Lambert *w* function", *Adv. Comput. Math.* **5** (1996), no. 4, p. 329-359.

[5] A. D. D. Craik, "The origins of water wave theory", in *Annual review of fluid mechanics*, Annual Review of Fluid Mechanics, vol. 36, Annual Reviews, 2004, p. 1-28.

[6] D. Daboussy, F. Dias, J.-M. Vanden-Broeck, "On explicite solutions of the free-surface Euler equations in the presence of gravity", *Phys. Fluids* **9** (1997), no. 10, p. 2828-2834.

[7] M. W. Dingemans, "Water wave propagation over uneven bottoms. Part 1: Linear wave propagation. Part 2: Nonlinear wave propagation. (2 vol.).", Advanced Series on Ocean Engineering, vol. 13, World Scientific, 1997, NASA Sti/Recon Technical Report N, pages 171-184, §2.8.

[8] D. D. Houghton, A. Kasahara, "Nonlinear shallow fluid flow over an isolated ridge", *Commun. Pure Appl. Math.* **21** (1968), p. 1-23.

[9] K. Kalimeris, "Analytical approximation and numerical simulations for periodic travelling water waves", *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **376** (2017), no. 21111, article no. 20170093 (19 pages).

[10] D. Lannes, *The water waves problem. Mathematical analysis and asymptotics.*, Mathematical Surveys and Monographs, vol. 188, American Mathematical Society, 2013.

[11] M. S. Longuet-Higgins, "A Theory of the Origin of Microseisms", *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **243** (1950), no. 857, p. 1-35.

[12] J. J. Stoker, "The formation of breakers and bores the theory of nonlinear wave propagation in shallow water and open channels", *Commun. Pure Appl. Math.* **1** (1948), p. 1-87.

[13] W. A. Strauss, M. H. Wheeler, "Bound on the Slope of Steady Water Waves with Favorable Vorticity", *Arch. Ration. Mech. Anal.* **222** (2016), no. 3, p. 1555-1580.