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A note on free divergence-free vector fields

Note sur les champs de vecteurs libres à divergence nulle

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Abstract. We exhibit an orthonormal basis of cyclic gradients and a (non-orthogonal) basis of the homogeneous free divergence-free vector field on the full Fock space and determine the dimension of Voiculescu's free divergence-free vector field of degree k or less. Moreover, we also give a concrete formula for the orthogonal projection onto the space of cyclic gradients as well as the free Leray projection.

Résumé. Nous présentons une base orthonormée de gradients cycliques et une base (non orthogonale) du champ de vecteurs libre homogène à divergence nulle sur l'espace de Fock plein et déterminons la dimension du champ de vecteurs libre au sens de Voiculescu à divergence nulle de degré k ou moins.

En outre, nous donnons une formule concrète pour la projection orthogonale sur l'espace des gradients cycliques ainsi que pour la version libre de la projection de Leray.

Keywords. Free probability, Free semi-circular system, Free divergence-free vector fields, Cyclic gradients.

Mots-clés. Probabilité libre, système semi-circulaire libre, champ de vecteurs libre à divergence nulle, gradients cycliques.

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1. Introduction

In the 1980s, Voiculescu introduced free probability theory to address the free group factor isomorphism problem (see [5]). Within this theoretical framework, the concept of the *free semi-circular system* emerges, defined as a tuple of freely independent *semi-circular distributions* given by $\frac{1}{2\pi}\sqrt{4-t^2}\,1_{[-2,2]}\,\mathrm{d}t$ (with the Lebesgue measure $\mathrm{d}t$). The free semi-circular system plays a role analogous to independent Gaussian distributions, as demonstrated in the free analogues of the central limit theorem, Wick's theorem, and the Stein equation.

In the 1990s, Voiculescu also introduced free probabilistic analogues of entropy and Fisher's information measure, naming them free entropy and free Fisher's information measure, respectively (see a survey article [9]). In particular, Voiculescu [6] introduced the so-called *non-microstates* free entropy. In this approach, Voiculescu introduced a certain non-commutative

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differential operator, which is called the *free difference quotient* and plays the role of a non-commutative counterpart of Hilbert transform, in order to define the free Fisher's information measure.

Then, the study of the *cyclic derivative* associated with the free difference quotient naturally emerged in relation to free entropy (see [7, 8]). In the work [7], Voiculescu determined the range of the cyclic gradient associated with the free difference quotient and established a certain exact sequence, which Mai and Speicher [4] and the first-named author [2] revisited in more general contexts. In the work [8], Voiculescu studied more geometric aspects of cyclic gradients associated with the free difference quotient. In particular, he introduced the notion of the free divergence-free vector field (originally called τ-preserving non-commutative vector fields) which is a free probabilistic analogue of the divergence-free vector field, and he showed that, for a vector in the free divergence-free vector field, the associated derivative exponentiates a one-parameter automorphism of a free group factor.

This work was motivated by Voiculescu's work [10]. The paper [10] gave a free probabilistic analogue of the Euler equation (called free Euler equation) of ideal incompressible fluids based on the techniques developed in [7, 8] with replacing Euclidean space $\mathbb{R}^n$ with a free semi-circular system $s_1, s_2, \dots, s_n$ on the full Fock space, following the method of [1]. Recently, Jekel-Li-Shlyakhtenko [3] extended Voiculescu's framework to tracial non-commutative smooth functions, and they connect a solution of the free Euler equation with a geodesic in the free Wasserstein manifold. In our quest for examples of (non-stationary) solutions of the free Euler equation, we realized that it is difficult to analyze the free probabilistic analogue of Leray projection (called *free Leray projection*), which is an ingredient of free Euler equation and the orthogonal projection from the non-commutative L^2-space generated by a free semi-circular system onto the free divergence-free vector field. Hence, we tried to understand the structure of the free divergence-free vector field. In [8], Voiculescu gave linearly independent sets of vectors in the free divergence-free vector field represented on the full Fock space. However, it was not clear whether they span the whole free divergence-free vector field or not.

The purpose of this note is to clarify the dimensions of the homogeneous parts of the free divergence-free vector field and deduce an exact formula for the free Leray projection. To show this, we focus on the space of cyclic gradients whose dimension can be computed by a group action of cyclic groups on words of finite length. We hope that our results will be used to find concrete solutions to the free Euler equation in future work.

2. Preliminaries

In this section, we recall some basic notations and some facts from [8, 10]. The full Fock space $\mathscr{F}(\mathbb{C}^n)$ over $\mathbb{C}^n$ is the Hilbert space defined as follows:

$$\mathscr{F}(\mathbb{C}^n) = \mathbb{C}1 \oplus \bigoplus_{k\geq 1} (\mathbb{C}^n)^{\otimes k},$$

where 1 is the vacuum vector. Throughout this note, we fix an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$, and $\{f_1, \dots, f_n\}$ denotes the standard basis of $\mathbb{C}^n$, i.e., the i-th component of f_i is 1 and other components are 0.

For any $n \in \mathbb{N}$, we set $[n] = \{1, 2, \dots, n\}$. We denote by $[n]^*$ the free monoid with the identity ϵ and n generators $1, \dots, n$, that is, $[n]^* = \{\epsilon\} \cup \{i_1 i_2 \cdots i_k \mid k \in \mathbb{N}, i_j \in [n], 1 \leq j \leq k\}$. For any word $w = i_1 \cdots i_k \in [n]^*$, we define the length of w by k (the length of ϵ is defined by 0), and $[n]^k$ denotes the subset of $[n]^*$ which consists of all elements whose lengths are k. For any $w \in [n]^*$ and $k \in \mathbb{N}$, w^k denotes the k-product of w, that is, $w \cdots w$. In addition, for the identity ϵ and $i_1 \cdots i_k \in [n]^*$, let e_ϵ and $e_{i_1 \cdots i_k}$ denote the vacuum vector 1 and $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$, respectively.

Let l_j and r_j denote the left and right creation operator with respect to e_j, respectively, for each $j = 1,\dots,n$. Namely, for each $w \in [n]^*$, $l_j e_w$ and $r_j e_w$ are given by

$$l_j e_w = e_{jw},$$
$$r_j e_w = e_{wj}.$$

Set $s_j = l_j + l_j^*$ for each $j = 1,\dots,n$. Then, $\{s_1,\dots,s_n\}$ becomes a free semi-circular system with respect to the vacuum state $\tau(\cdot) := \langle \cdot 1, 1\rangle$. Let $\mathbb{C}\langle s_1,\dots,s_n\rangle = \mathbb{C}^s_{\langle n\rangle}$ denote the unital (algebraic) $*$-subalgebra of $B(\mathscr{F}(\mathbb{C}^n))$ generated by $\{1\}\cup\{s_1,\dots,s_n\}$ and by M the von Neumann subalgebra of $B(\mathscr{F}(\mathbb{C}^n))$ generated by $\mathbb{C}^s_{\langle n\rangle}$. For each $k \in \mathbb{Z}_{\geq 0}$, we define $(\mathbb{C}^s_{\langle n\rangle})_k$ as the subspace of homogeneous polynomials of degree k in $\mathbb{C}^s_{\langle n\rangle}$.

We then work in the non-commutative L^2-space $L^2(M,\tau)$. We have the unitary isomorphism $U : L^2(M,\tau) \to \mathscr{F}(\mathbb{C}^n)$ which sends an element $U_{k_1}(s_{i_1})\cdots U_{k_p}(s_{i_p})$ to $e_{i_1^{k_1} i_2^{k_2}\cdots i_p^{k_p}}$ where $i_1,\dots,i_p \in [n]$ with $i_j \neq i_{j+1}$, $k_1,\dots,k_p \in \mathbb{N}$, and $\{U_k(t)\}_{k=0}^{\infty}$ are the Chebyshev polynomials (of degree k) of the second kind, which are orthogonal to each other with respect to the semi-circular distribution ([8, section 1.4]).

In a similar way, let $\mathbb{C}\langle l_1,\dots,l_n\rangle = \mathbb{C}^l_{\langle n\rangle}$ denote the unital subalgebra of $B(\mathscr{F}(\mathbb{C}^n))$ generated by $\{1\}\cup\{l_1,\dots,l_n\}$ and $(\mathbb{C}^l_{\langle n\rangle})_k$ denotes the subspace of homogeneous polynomials of degree k in $\mathbb{C}^l_{\langle n\rangle}$. Now, we have two cyclic gradients $\delta^s = (\delta^s_j)_{j=1}^n$ and $\delta^l = (\delta^l_j)_{j=1}^n$ with respect to $s_1,\dots,s_n$ and with respect to $l_1,\dots,l_n$ defined by

$$\delta^s(s_{i_1}s_{i_2}\cdots s_{i_p}) = \sum_{j=1}^{p} s_{i_{j+1}}\cdots s_{i_p}s_{i_1}s_{i_2}\cdots s_{i_{j-1}} \otimes f_{i_j} \in (\mathbb{C}^s_{\langle n\rangle})^n,$$
$$\delta^l(l_{i_1}l_{i_2}\cdots l_{i_p}) = \sum_{j=1}^{p} l_{i_{j+1}}\cdots l_{i_p}l_{i_1}l_{i_2}\cdots l_{i_{j-1}} \otimes f_{i_j} \in (\mathbb{C}^l_{\langle n\rangle})^n,$$

where $i_1,\dots,i_p \in [n]$ and we identify $(\mathbb{C}^s_{\langle n\rangle})^n \simeq \mathbb{C}^s_{\langle n\rangle}\otimes\mathbb{C}^n$ and $(\mathbb{C}^l_{\langle n\rangle})^n \simeq \mathbb{C}^l_{\langle n\rangle}\otimes\mathbb{C}^n$. In general, δ^l is different from δ^s as an operator, but we have the following fact:

Theorem 1 ([8, Theorem 7.4]). *We have* $(\delta^s\mathbb{C}^s_{\langle n\rangle})[1\oplus\cdots\oplus 1] = (\delta^l\mathbb{C}^l_{\langle n\rangle})[1\oplus\cdots\oplus 1]$ *in* $\mathscr{F}(\mathbb{C}^n)^n$.

Following [8, 10], we write $\mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle}) = (\mathbb{C}^s_{\langle n\rangle})^n$. The next object is the main target of this note.

Definition 2 ([8, Section 3.5], [10, Section 2]). *The free divergence-free vector field (with respect to a free semi-circular system) is defined as follows.*

$$\mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle}|\tau) = \left\{(p_1,\dots,p_n)\in\mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle}) \;\middle|\; \sum_{1\leq j\leq n}\tau(p_j\delta^s_j[r]) = 0 \textit{ for all } r\in\mathbb{C}^s_{\langle n\rangle}\right\}.$$

By definition, it is clear that $\mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle}|\tau) = \mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle})\ominus\delta^s\mathbb{C}^s_{\langle n\rangle}$. Moreover, we have the next fact:

Theorem 3 ([8, Theorem 7.5]). *We have*

$$L^2(M,\tau)^n\ominus\overline{\delta^s\mathbb{C}_{\langle\rangle}} \simeq \mathscr{F}(\mathbb{C}^n)^n\ominus\overline{\delta^s\mathbb{C}_{\langle\rangle}[\mathbb{1}\oplus\cdots\oplus\mathbb{1}]} = \left\{\left((l_j^*-r_j^*)\xi\right)_{j=1}^n \;\middle|\; \xi\in\mathscr{F}(\mathbb{C}^n)\right\}.$$

In particular, we also have the orthogonal decomposition:

$$\mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle}|\tau)[1\oplus\cdots\oplus 1] = \bigoplus_{k\geq 0}\mathscr{X}_k^{(n)},$$

where $\mathscr{X}_k^{(n)} = [(\mathbb{C}^n)^{\otimes k}]^n\ominus\left(\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1\oplus\cdots\oplus 1]\right) = \left\{\left((l_j^*-r_j^*)\xi\right)_{j=1}^n \;\middle|\; \xi\in(\mathbb{C}^n)^{\otimes k+1}\right\}$.

We call $\mathscr{X}_k^{(n)}$ the *homogeneous free divergence-free vector field of degree k*. Here is a consequence of these available facts:

Lemma 4 ([8, Lemma 7.7]). *We have* $\ker\big((\theta^l)^*|_{(\mathbb{C}^n)^{\otimes k}}\big) = \ker\big((I-R)|_{(\mathbb{C}^n)^{\otimes k}}\big)$ *for* $k \geq 1$*, where* θ^l *is the linear map from* $\mathscr{F}(\mathbb{C}^n)^n$ *to* $\mathscr{F}(\mathbb{C}^n)$ *such that* $\theta^l[(\xi_1,\dots,\xi_n)] = \sum_{j=1}^n (l_j - r_j)\xi_j$ *for any* $\xi_1,\dots,\xi_n \in \mathscr{F}(\mathbb{C}^n)$ *and R is the cyclic permutation, that is,* $R(e_{i_1 i_2 \cdots i_p}) = e_{i_p i_1 \cdots i_{p-1}}$.

We use the same notation R for the cyclic permutation on $[n]^* \setminus \{\epsilon\}$, that is, $R(i_1 \cdots i_{k-1} i_k) = i_k i_1 \cdots i_{k-1}$ for all $i_1 \cdots i_k \in [n]^*$.

At the end of this section, we exhibit an interesting example of vectors in the free divergence-free vector field, which is inspired by the classical case when the stream function is radially symmetric.

Proposition 5. *For any* $m \in \mathbb{N}$*, we have*

$$\begin{pmatrix} \delta_2^s(s_1^2+s_2^2)^m \\ -\delta_1^s(s_1^2+s_2^2)^m \end{pmatrix} \in \mathrm{Vect}(\mathbb{C}^s_{\langle 2\rangle}|\tau).$$

Proof. Throughout the proof, we suppress s in the notation δ_i^s and simply write δ_i for each $i \in [2]$. We use the fact that the free semi-circular system (s_1, s_2) satisfies the analogue of the Stein equation (cf. [6]):

$$\tau[s_i P(s_1,s_2)] = \tau\otimes\tau[\partial_i P(s_1,s_2)]$$

for any non-commutative polynomial $P(s_1,s_2)$ where $\partial_i : \mathbb{C}^s_{\langle 2\rangle} \to \mathbb{C}^s_{\langle 2\rangle}\otimes\mathbb{C}^s_{\langle 2\rangle}$ is the free (partial) difference quotient which is a linear map defined for each monomial P by

$$\partial_i P = \sum_{P = A s_i B} A\otimes B.$$

We also use the following relation between ∂_i and δ_j for any $i, j \in [2]$ (see [4, Lemma 3.4]):

$$\partial_i \circ \delta_j = \sigma\circ\partial_j\circ\delta_i$$

where $\sigma : \mathbb{C}^s_{\langle 2\rangle}\otimes\mathbb{C}^s_{\langle 2\rangle} \to \mathbb{C}^s_{\langle 2\rangle}\otimes\mathbb{C}^s_{\langle 2\rangle}$ is the flip defined by linear extension of $\sigma(r_1\otimes r_2) = r_2\otimes r_1$.

Now, we show the proposition by induction on m. When $m=1$, we have $(\delta_2(s_1^2+s_2^2), -\delta_1(s_1^2+s_2^2)) = 2(s_2,-s_1)$. Then, for a given $r\in\mathbb{C}^s_{\langle 2\rangle}$, we obtain by the analog of the Stein equation,

$$\tau[s_2\delta_1 r] - \tau[s_1\delta_2 r] = \tau\otimes\tau[\partial_2(\delta_1 r)] - \tau\otimes\tau[\partial_1(\delta_2 r)],$$

which is equal to zero by the formula $\partial_i\circ\delta_j = \sigma\circ\partial_j\circ\delta_i$. This implies $(\delta_2(s_1^2+s_2^2), -\delta_1(s_1^2+s_2^2)) \in \mathrm{Vect}(\mathbb{C}^s_{\langle 2\rangle}|\tau)$ (one can also show this by using Theorem 3 with $\xi = e_{12}$). Suppose that we have $(\delta_2 f, -\delta_1 f) \in \mathrm{Vect}(\mathbb{C}^s_{\langle 2\rangle}|\tau)$ for $f = (s_1^2+s_2^2)^k$ $(1\le k\le m)$. From the Leibniz rule of free difference quotients (note that $\partial_i(s_1^2+s_2^2) = s_i\otimes 1 + 1\otimes s_i$ for $i = 1,2$), we have for $f = (s_1^2+s_2^2)^{m+1}$

$$(\delta_2 f, -\delta_1 f) = (m+1)[(s_1^2+s_2^2)^m v + v(s_1^2+s_2^2)^m]$$

where $v = (s_2,-s_1)$. Therefore, for a given $r\in\mathbb{C}^s_{\langle 2\rangle}$, we want to show

$$\tau[s_2(\delta_1 r)(s_1^2+s_2^2)^m] + \tau[s_2(s_1^2+s_2^2)^m(\delta_1 r)] - \tau[s_1(\delta_2 r)(s_1^2+s_2^2)^m] - \tau[s_1(s_1^2+s_2^2)^m(\delta_2 r)] = 0. \quad (1)$$

By using the formula $\tau[s_i P(s_1,s_2)] = \tau\otimes\tau[\partial_i P(s_1,s_2)]$, we have from the Leibniz rule,

$$\begin{aligned}
\tau[s_2(\delta_1 r)(s_1^2+s_2^2)^m] &= \tau\otimes\tau[(\partial_2(\delta_1 r))\cdot 1\otimes(s_1^2+s_2^2)^m] \\
&\quad + \sum_{k=1}^m \tau\otimes\tau[(\delta_1 r)(s_1^2+s_2^2)^{k-1}(s_2\otimes 1+1\otimes s_2)(s_1^2+s_2^2)^{m-k}], \\
\tau[s_2(s_1^2+s_2^2)^m(\delta_1 r)] &= \tau\otimes\tau[(s_1^2+s_2^2)^m\otimes 1\cdot(\partial_2(\delta_1 r))] \\
&\quad + \sum_{k=1}^m \tau\otimes\tau[(s_1^2+s_2^2)^{m-k}(s_2\otimes 1+1\otimes s_2)(s_1^2+s_2^2)^{k-1}(\delta_1 r)], \\
\tau[s_1(\delta_2 r)(s_1^2+s_2^2)^m] &= \tau\otimes\tau[(\partial_1(\delta_2 r))\cdot 1\otimes(s_1^2+s_2^2)^m] \\
&\quad + \sum_{k=1}^m \tau\otimes\tau[(\delta_2 r)(s_1^2+s_2^2)^{k-1}(s_1\otimes 1+1\otimes s_1)(s_1^2+s_2^2)^{m-k}], \\
\tau[s_1(s_1^2+s_2^2)^m(\delta_2 r)] &= \tau\otimes\tau[(s_1^2+s_2^2)^m\otimes 1\cdot(\partial_1(\delta_2 r))] \\
&\quad + \sum_{k=1}^m \tau\otimes\tau[(s_1^2+s_2^2)^{m-k}(s_1\otimes 1+1\otimes s_1)(s_1^2+s_2^2)^{k-1}(\delta_2 r)].
\end{aligned}$$

Since odd moments of the free semi-circular system are zero, we have $\tau[s_i(s_1^2+s_2^2)^{m-k}] = 0$ for $i = 1,2$, and thus we have

$$\begin{aligned}
\tau[s_2(\delta_1 r)(s_1^2+s_2^2)^m] &= \tau\otimes\tau[(\partial_2(\delta_1 r))\cdot 1\otimes(s_1^2+s_2^2)^m] \\
&\quad + \sum_{k=1}^m \tau[(\delta_1 r)(s_1^2+s_2^2)^{k-1}s_2]\tau[(s_1^2+s_2^2)^{m-k}], \\
\tau[s_2(s_1^2+s_2^2)^m(\delta_1 r)] &= \tau\otimes\tau[(s_1^2+s_2^2)^m\otimes 1\cdot(\partial_2(\delta_1 r))] \\
&\quad + \sum_{k=1}^m \tau[(s_1^2+s_2^2)^{m-k}]\tau[s_2(s_1^2+s_2^2)^{k-1}(\delta_1 r)], \\
\tau[s_1(\delta_2 r)(s_1^2+s_2^2)^m] &= \tau\otimes\tau[(\partial_1(\delta_2 r))\cdot 1\otimes(s_1^2+s_2^2)^m] \\
&\quad + \sum_{k=1}^m \tau[(\delta_2 r)(s_1^2+s_2^2)^{k-1}s_1]\tau[(s_1^2+s_2^2)^{m-k}], \\
\tau[s_1(s_1^2+s_2^2)^m(\delta_2 r)] &= \tau\otimes\tau[(s_1^2+s_2^2)^m\otimes 1\cdot(\partial_1(\delta_2 r))] \\
&\quad + \sum_{k=1}^m \tau[(s_1^2+s_2^2)^{m-k}]\tau[s_1(s_1^2+s_2^2)^{k-1}(\delta_2 r)].
\end{aligned}$$

Then, the left-hand side of (1) is equal to

$$\begin{aligned}
&\tau\otimes\tau[(\partial_2(\delta_1 r))\cdot 1\otimes(s_1^2+s_2^2)^m] + \tau\otimes\tau[(s_1^2+s_2^2)^m\otimes 1\cdot(\partial_2(\delta_1 r))] \\
&\quad - \tau\otimes\tau[(\partial_1(\delta_2 r))\cdot 1\otimes(s_1^2+s_2^2)^m] - \tau\otimes\tau[(s_1^2+s_2^2)^m\otimes 1\cdot(\partial_1(\delta_2 r))] \\
&\quad + \sum_{k=1}^m \tau[(s_1^2+s_2^2)^{m-k}]\Big(\tau[(\delta_1 r)(s_1^2+s_2^2)^{k-1}s_2] + \tau[s_2(s_1^2+s_2^2)^{k-1}(\delta_1 r)] \\
&\qquad - \tau[(\delta_2 r)(s_1^2+s_2^2)^{k-1}s_1] - \tau[s_1(s_1^2+s_2^2)^{k-1}(\delta_2 r)]\Big).
\end{aligned}$$

By using the trace property of τ and the assumption of induction $f = (s_1^2+s_2^2)^k$, the sum in the third and fourth lines is equal to 0. Moreover, by using the trace property of τ and the formula $\partial_i\circ\delta_j = \sigma\circ\partial_j\circ\delta_i$ again, we have

$$\tau\otimes\tau[(\partial_i(\delta_j r))\cdot 1\otimes X] - \tau\otimes\tau[X\otimes 1\cdot(\partial_i(\delta_j r))] = 0$$

for any $X \in M$ and $i, j \in [2]$. Therefore, we can see the identity (1) for any $r \in \mathbb{C}^s_{\langle 2\rangle}$, and $f = (s_1^2 + s_2^2)^{m+1}$ satisfies $(\delta_2 f, -\delta_1 f) \in \mathrm{Vect}(\mathbb{C}^s_{\langle 2\rangle}|\tau)$, which completes the induction. □

3. The dimension of homogeneous free divergence-free vector field

In this section, we exhibit an orthonormal basis of $\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1} \simeq \delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1 \oplus \cdots \oplus 1]$ and compute the dimension of the homogeneous free divergence-free vector field $\mathscr{X}_k^{(n)}$ of degree k for each $k \in \mathbb{N}$ and each $n \in \mathbb{N}$. The key point of our argument is that the cyclic gradient δ^l is invariant under the cyclic permutation, i.e., $\delta^l(l_{Ru}) = \delta^l(l_u)$ for any $u \in [n]^*$ where we write $l_u = l_{i_1} l_{i_2} \cdots l_{i_p}$ for $u = i_1 i_2 \cdots i_p$. Note that the cyclic permutation R on $[n]^k$ induces a group action of $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ on $[n]^k$. Thus, we can decompose $[n]^k$ into the orbits of this action, and we have $\delta^l(l_u) = \delta^l(l_{u'})$ if u' is in the orbit $[u] = \mathbb{Z}_k u = \{gu \in [n]^k \mid g \in \mathbb{Z}_k\}$.

In the following theorem, we see that $\delta^l(l_u)[1 \oplus \cdots \oplus 1]$ is orthogonal to $\delta^l(l_{u'})[1 \oplus \cdots \oplus 1]$ if u and u' are not in the same orbit. Moreover, we can normalize $\delta^l(l_u)[1 \oplus \cdots \oplus 1]$ by using the order of the stabilizer subgroup $(\mathbb{Z}_k)_u = \{g \in \mathbb{Z}_k \mid gu = u\}$ of $u \in [n]^k$. Note that, if $u' \in [u]$, $(\mathbb{Z}_k)_u$ and $(\mathbb{Z}_k)_{u'}$ are conjugate, and therefore $|(\mathbb{Z}_k)_u|$ only depends on the orbit $[u]$ but not on the concrete representative u.

Theorem 6. *For each $k \in \mathbb{Z}_{\geq 0}$, the subset of vectors in $\left[(\mathbb{C}^n)^{\otimes k}\right]^n$*

$$S_k = \left\{ F([u]) := \frac{\delta^l(l_u)[1 \oplus \cdots \oplus 1]}{|(\mathbb{Z}_{k+1})_u| \sqrt{|[u]|}} \,\middle|\, [u] \in {[n]^{k+1}}\big/{\mathbb{Z}_{k+1}} \right\}$$

is an orthonormal basis of $\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1 \oplus \cdots \oplus 1]$.

Proof. First, we see that $\delta^l(l_u)[1 \oplus \cdots \oplus 1]$ is orthogonal to $\delta^l(l_{u'})[1 \oplus \cdots \oplus 1]$ if $u' \notin [u]$. Under the identification $\left[(\mathbb{C}^n)^{\otimes k+1}\right]^n \simeq (\mathbb{C}^n)^{\otimes k+1} \otimes \mathbb{C}^n$, we write the orthonormal basis of $\left[(\mathbb{C}^n)^{\otimes k+1}\right]^n$ by $\{e_w \otimes f_i\}_{w \in [n]^{k+1}, i \in [n]}$ (recall that $\{f_i\}_{i=1}^n$ denotes the standard basis). Then, for each $u = i_1 i_2 \cdots i_{k+1} \in [n]^{k+1}$, the cyclic derivative $\delta^l(l_u)[1 \oplus \cdots \oplus 1]$ is written by

$$\sum_{j=1}^{k+1} e_{i_{j+1} \cdots i_{k+1} i_1 \cdots i_{j-1}} \otimes f_{i_j}.$$

If $\delta^l(l_u)[1 \oplus \cdots \oplus 1]$ is not orthogonal to $\delta^l(l_{u'})[1 \oplus \cdots \oplus 1]$ with $u = i_1 i_2 \cdots i_{k+1}$ and $u' = i'_1 i'_2 \cdots i'_{k+1}$, there exists $j, j' \in [k+1]$ such that $i_j = i'_{j'}$ and

$$i_{j+1} \cdots i_{k+1} i_1 \cdots i_{j-1} = i'_{j'+1} \cdots i'_{k+1} i'_1 \cdots i'_{j'-1},$$

implying that u and u' are in the same orbit. Therefore, $\delta^l(l_u)[1 \oplus \cdots \oplus 1]$ is orthogonal to $\delta^l(l_{u'})[1 \oplus \cdots \oplus 1]$ if $u' \notin [u]$.

Note that, if p is the minimal number (generator) in the stabilizer subgroup $(\mathbb{Z}_{k+1})_u$ (which is also a cyclic group), then we have $u = v^m$ with $v = i_1 i_2 \cdots i_p$ and $m = |(\mathbb{Z}_{k+1})_u|$ and $p = \frac{k+1}{m} = |[u]|$. Thus, we obtain

$$\delta^l(l_u)[1 \oplus \cdots \oplus 1] = m \sum_{j=1}^{p} e_{i_{j+1} \cdots i_p v^{m-1} i_1 \cdots i_{j-1}} \otimes f_{i_j}.$$

The minimality of p implies that all vectors in the sum are orthonormal, and hence we have

$$\|\delta^l(l_u)[1 \oplus \cdots \oplus 1]\|^2 = m^2 p = |(\mathbb{Z}_{k+1})_u|^2 \cdot |[u]|.$$

Since $\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1 \oplus \cdots \oplus 1]$ is spanned by $\{\delta^l(l_u)[1 \oplus \cdots \oplus 1] \mid u \in [n]^{k+1}\}$ and $F([u])$ does not depend on the choice of words in the same orbit $[u]$, we can conclude that $S_k = \{F([u]) \mid [u] \in {[n]^{k+1}}/{\mathbb{Z}_{k+1}}\}$ is an orthonormal basis of $\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1 \oplus \cdots \oplus 1]$. □

Corollary 7. *We have* $\dim\left(\delta^s(\mathbb{C}^s_{\langle n\rangle})_{k+1}\right) = \dim\left(\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}\right) = \left|[n]^{k+1}/\mathbb{Z}_{k+1}\right|$*, and hence*

$$\dim\left(\mathscr{X}^{(n)}_k\right) = n^{k+1} - \left|[n]^{k+1}\middle/\mathbb{Z}_{k+1}\right|$$

for any $n \in \mathbb{N}$ *and* $k \in \mathbb{Z}_{\geq 0}$*. Thus, we obtain that*

$$\dim\left(\mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle}|\tau)_{\leq k}\right) = \frac{n(n^{k+1}-1)}{n-1} - \sum_{j=0}^{k}\left|[n]^{j+1}\middle/\mathbb{Z}_{j+1}\right|,$$

where $\mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle}|\tau)_{\leq k}$ *is the subspace of* $\mathrm{Vect}(\mathbb{C}^s_{\langle n\rangle}|\tau)$ *of all elements whose degrees as polynomials with respect to* $\{s_i\}_{i=1}^n$ *are k or less.*

Proof. It is a direct consequence from Theorem 6 with the facts that $\dim\left[(\mathbb{C}^n)^{\otimes k}\right]^n = n^{k+1}$ and that $[(\mathbb{C}^n)^{\otimes k}]^n = \mathscr{X}^{(n)}_k \oplus \left(\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1\oplus\cdots\oplus 1]\right)$ by Theorem 1. □

Remark 8. The number $\left|[n]^k/\mathbb{Z}_k\right|$ is equal to the number of necklaces of length k such that each bead is chosen from n colors. From Burnside's lemma, we have

$$\left|[n]^k\middle/\mathbb{Z}_k\right| = \frac{1}{k}\sum_{g\in\mathbb{Z}_k}|([n]^k)^g|,$$

where $([n]^k)^g$ is the set of elements in $[n]^k$ which are fixed by g. Moreover, we have

$$|([n]^k)^g| = n^{\gcd(g,k)},$$

where $\gcd(g,k)$ is the greatest common divisor of g and k.

Thanks to the orthonormal basis in Theorem 6, we can compute the orthogonal projection onto the subspace of cyclic gradients. Therefore, we can obtain a concrete formula for the free Leray projection.

Corollary 9. *For any* $\begin{bmatrix} e_{u_1} \\ \vdots \\ e_{u_n}\end{bmatrix} \in \mathscr{F}(\mathbb{C}^n)^n$ *with* $u_j \in [n]^{k_j}$*, we have*

$$P_{\overline{\delta^l(\mathbb{C}_{\langle\rangle})[\mathbb{1}\oplus\cdots\oplus\mathbb{1}]}}\left(\begin{bmatrix} e_{u_1} \\ \vdots \\ e_{u_n}\end{bmatrix}\right) = \sum_{1\leq j\leq n}\frac{\delta^l(l_{ju_j})[1\oplus\cdots\oplus 1]}{k_j+1}.$$

where $P_{\overline{\delta^l(\mathbb{C}_{\langle\rangle})[\mathbb{1}\oplus\cdots\oplus\mathbb{1}]}}$ *denotes the orthogonal projection onto* $\overline{\delta^l(\mathbb{C}_{\langle\rangle})[\mathbb{1}\oplus\cdots\oplus\mathbb{1}]}$.

Proof. By linearlity of $P_{\overline{\delta^l(\mathbb{C}_{\langle\rangle})[\mathbb{1}\oplus\cdots\oplus\mathbb{1}]}}$, it suffices to confirm the desired identity for $e_{u_i}\otimes f_i \in (\mathbb{C}^n)^{\otimes k}\otimes\mathbb{C}^n$ with $u_i \in [n]^k$ and $i \in [n]$ (recall that $\{f_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{C}^n$). Note that we have the orthogonal decomposition $\delta^l(\mathbb{C}^l_{\langle n\rangle})[1\oplus\cdots\oplus 1] = \bigoplus_{k\geq 0}\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1\oplus\cdots\oplus 1]$ and $\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1\oplus\cdots\oplus 1] \subset (\mathbb{C}^n)^{\otimes k}\otimes\mathbb{C}^n$. This implies

$$P_{\overline{\delta^l(\mathbb{C}_{\langle\rangle})[\mathbb{1}\oplus\cdots\oplus\mathbb{1}]}}\left(e_{u_i}\otimes f_i\right) = P_{\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1\oplus\cdots\oplus 1]}\left(e_{u_i}\otimes f_i\right).$$

Since $S_k = \{F([w]) \mid w \in [n]^{k+1}/\mathbb{Z}_{k+1}\}$, as we have seen in Theorem 6, is an orthonormal basis of $\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1\oplus\cdots\oplus 1]$, $\langle e_{u_i}\otimes f_i, F([w])\rangle \neq 0$ implies $[w]=[iu_i]$. We also have $\langle e_{u_i}\otimes f_i, F([iu_i])\rangle = \frac{1}{\sqrt{|[iu_i]|}}$ by the arguments in a similar fashion to the proof of Theorem 6. Hence we observe that

$$\begin{aligned}
P_{\delta^l(\mathbb{C}^l_{\langle n\rangle})_{k+1}[1\oplus\cdots\oplus 1]}(e_{u_i}\otimes f_i) &= \sum_{[w]\in[n]^{k+1}/\mathbb{Z}_{k+1}} \langle e_{u_i}\otimes f_i, F([w])\rangle F([w]) \\
&= \langle e_{u_i}\otimes f_i, F([iu_i])\rangle F([iu_i]) \\
&= \frac{1}{\sqrt{|[iu_i]|}} \cdot \frac{\delta^l(l_{iu_i})[1\oplus\cdots\oplus 1]}{|(\mathbb{Z}_{k+1})_{iu_i}|\sqrt{|[iu_i]|}} \\
&= \frac{\delta^l(l_{iu_i})[1\oplus\cdots\oplus 1]}{k+1},
\end{aligned}$$

where we use the well-known identity of the group action $|[iu_i]|\cdot|(\mathbb{Z}_{k+1})_{iu_i}| = |\mathbb{Z}_{k+1}| = k+1$. □

Let Π be the free Leray projection, which is the orthogonal projection onto the norm-closure of the free divergence-free vector field in the full Fock space (see [10, section 3]). By definition, we have $\Pi = I - P_{\overline{\delta^l(\mathbb{C}_{\langle\rangle})[\mathbb{1}\oplus\cdots\oplus\mathbb{1}]}}$, and therefore we obtain the following corollary.

Corollary 10. *For any* $\begin{bmatrix} e_{u_1} \\ \vdots \\ e_{u_n}\end{bmatrix} \in \mathscr{F}(\mathbb{C}^n)^n$ *with* $u_j \in [n]^{k_j}$, *we have*

$$\Pi\left(\begin{bmatrix} e_{u_1} \\ \vdots \\ e_{u_n}\end{bmatrix}\right) = \sum_{1\le j\le n}\left(e_{u_j}\otimes f_j - \frac{\delta^l(l_{ju_j})[1\oplus\cdots\oplus 1]}{k_j+1}\right).$$

Remark 11. We can also describe a (non-orthogonal) basis of the homogeneous free divergence-free vector field $\mathscr{X}_k^{(n)}$ on the full Fock space. Indeed, Lemma 4 tells us

$$(\theta^l)^* : \operatorname{ran}\left((I-R)|_{(\mathbb{C}^n)^{\otimes k+1}}\right) \longrightarrow \operatorname{ran}\left((\theta^l)^*|_{(\mathbb{C}^n)^{\otimes k+1}}\right)$$

is a linear isomorphism. Here, note that $\operatorname{ran}\left((\theta^l)^*|_{(\mathbb{C}^n)^{\otimes k+1}}\right) = \mathscr{X}_k^{(n)}$, and hence

$$\dim\left(\operatorname{ran}\left((I-R)|_{(\mathbb{C}^n)^{\otimes k+1}}\right)\right) = \dim\left(\mathscr{X}_k^{(n)}\right) = n^{k+1} - \left|[n]^{k+1}/\mathbb{Z}_{k+1}\right|.$$

Thus, in order to obtain a basis of $\mathscr{X}_k^{(n)}$, it suffices to find a basis of $\operatorname{ran}\left((I-R)|_{(\mathbb{C}^n)^{\otimes k+1}}\right)$. In [8], Voiculescu introduced a linearly independent subset in $\mathscr{X}_k^{(n)}$ by $\{(\theta^l)^*(I-R)e_w \mid w\in\Omega_{k+1}\}$ where $\Omega_{k+1} = \{w \in [n]^{k+1} \mid w \prec Rw, w \neq Rw\}$ and $\prec$ is the lexicographic order. However, since we have the decomposition $[n]^{k+1} = \{Rw = w\} \sqcup \Omega_{k+1} \sqcup \{w \succ Rw, w\neq Rw\}$ with $|\{Rw = w\}| = n$ and $|\Omega_{k+1}| = |\{w\succ Rw, w\neq Rw\}|$ due to the bijective map $w_1w_2\ldots w_{k+1} \mapsto (n+1-w_1)(n+1-w_2)\cdots(n+1-w_{k+1})$, the cardinality of Ω_{k+1} is $\frac{1}{2}(n^{k+1}-n)$, which is smaller than our dimension $n^{k+1} - \left|[n]^{k+1}/\mathbb{Z}_{k+1}\right|$. In fact, we can modify the set Ω_{k+1} and take a basis of $\operatorname{ran}\left((I-R)|_{(\mathbb{C}^n)^{\otimes k+1}}\right)$ by considering the action of $\mathbb{Z}_{k+1}$.

Proposition 12. *The set* $B = \bigsqcup_{[u]\in[n]^{k+1}/\mathbb{Z}_{k+1}} B_{[u]}$, *where*

$$B_{[u]} = \{(I-R)e_v = e_v - e_{Rv} \mid v \in [u]\setminus\{u\}\},$$

is a basis of $\operatorname{ran}\left((I-R)|_{(\mathbb{C}^n)^{\otimes k+1}}\right)$, *and thus, the set* $\widetilde{B} = \bigsqcup_{[u]\in[n]^{k+1}/\mathbb{Z}_{k+1}} \widetilde{B}_{[u]}$, *where*

$$\widetilde{B}_{[u]} = \left\{\left(\delta_{j,i_1}e_{i_2\cdots i_{k+1}} - 2\delta_{j,i_{k+1}}e_{i_1\cdots i_k} + \delta_{j,i_k}e_{i_{k+1}i_1\cdots i_{k-1}}\right)_{1\le j\le n} \,\middle|\, i_1i_2\cdots i_{k+1} \in [u]\setminus\{u\}\right\},$$

is a basis of $\mathscr{X}_k^{(n)}$.

Proof. Since $|B| = n^{k+1} - \left|[n]^{k+1}/\mathbb{Z}_{k+1}\right| = \dim\left(\mathrm{ran}\left((I-R)|_{(\mathbb{C}^n)^{\otimes k+1}}\right)\right)$, we have to confirm only that all elements of B are linearly independent. Remark that if $[u] \neq [v]$ in $[n]^{k+1}/\mathbb{Z}_{k+1}$, then $B_{[u]}$ and $B_{[v]}$ are orthogonal to each other. Hence, it suffices to show that all the elements of $B_{[u]}$ are linearly independent of each other for each $[u] \in [n]^{k+1}/\mathbb{Z}_{k+1}$.

Choose an arbitrary $[u] \in [n]^{k+1}/\mathbb{Z}_{k+1}$. Assume that $\sum_{v\in[u]\setminus\{u\}} \alpha(v)\cdot(I-R)e_v = 0$ in $(\mathbb{C}^n)^{\otimes k+1}$ with $\alpha(v) \in \mathbb{C}$. Let us write

$$[u]\setminus\{u\} = \{v, Rv, R^2v, \ldots, R^pv\} \subset (\mathbb{C}^n)^{\otimes k+1} \quad (p = [u]-2 \text{ and } R^{p+1}v = u).$$

Remark that $R^iv \neq R^jv$ for any $0 \le i \neq j \le p$. Then, we observe that

$$\alpha(v)e_v + \sum_{1\le j\le p} (\alpha(R^jv) - \alpha(R^{j-1}v))\cdot e_{R^jv} - \alpha(R^pv)u = 0.$$

By the linear independence of $\{e_u\}_{u\in[n]^*}$, we have $\alpha(R^jv) = 0$ for all $0 \le j \le p$. Applying $(\theta^l)^* = (l_j^* - r_j^*)_{1\le j\le n}$ to the basis B, we obtain $\widetilde{B}$ as a basis of $\mathscr{X}_k^{(n)}$. □

Remark 13. In [8], Voiculescu introduced another linearly independent subset in $\mathscr{X}_k^{(n)}$. For $u = i_0\cdots i_k \in [n]^{k+1}$, let $\mathrm{per}(u)$ be the period of u, i.e. the least $m \in \{1,\ldots,k+1\}$ such that $i_s = i_t$ whenever $s \equiv t \pmod m$. Let $\rho(\mathrm{per}(u))$ be the set of the non-unital roots of $\zeta^{\mathrm{per}(u)} = 1$ (we set $\rho(1) = \emptyset$). Then, Voiculescu introduced the following set for the basis of $\mathscr{X}_k^{(n)}$

$$\left\{\sum_{j=1}^{\mathrm{per}(u)-1} \zeta^j F_{i_j i_{j+1}\cdots i_k i_0\cdots i_{j-1}} \mid u = i_0\cdots i_k \in \omega(k+1), \zeta \in \rho(\mathrm{per}(u))\right\},$$

where $F_w = (\theta^l)^*(I-R)e_w$ for $w \in [n]^*$ and $\omega(k+1)$ is defined by

$$\left\{u = i_0\cdots i_k \in [n]^{k+1} \mid i_0\cdots i_k \prec i_j i_{j+1}\cdots i_k i_0\cdots i_{j-1},\ j = 1,\ldots,k\right\}.$$

Note that the element u in $\omega(k+1)$ is the minimal element in the orbit of u with respect to the lexicographic order, and thus there is a bijection between $\omega(k+1)$ and $[n]^{k+1}/\mathbb{Z}_{k+1}$. Since $|\rho(\mathrm{per}(u))| = \mathrm{per}(u) - 1 = |[u]| - 1$, the number of (u,ζ) such that $u \in \omega(k+1)$ and $\zeta \in \rho(\mathrm{per}(u))$ is given by

$$\sum_{u\in\omega(k+1)} |\rho(\mathrm{per}(u))| = \sum_{u\in\omega(k+1)} (|[u]|-1) = \left(\sum_{u\in\omega(k+1)} |[u]|\right) - |\omega(k+1)| = n^{k+1} - \left|[n]^{k+1}/\mathbb{Z}_{k+1}\right|,$$

which coincides with our dimension.

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Declaration of interests

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