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Mathématique

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Volume 362 (2024), p. 1253-1260

Online since: 5 November 2024

https://doi.org/10.5802/crmath.638

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www.centre-mersenne.org — e-ISSN : 1778-3569

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https://doi.org/10.5802/crmath.638

Research article / *Article de recherche*
Harmonic analysis / *Analyse harmonique*

On the sharpness of some quantitative Muckenhoupt–Wheeden inequalities

Optimalité des inégalités quantitatives de Muckenhoupt–Wheeden

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Abstract. In the recent work [Cruz-Uribe et al. (2021)] it was obtained that

$$|\{x\in\mathbb{R}^d : w(x)|G(fw^{-1})(x)|>\alpha\}|\lesssim \frac{[w]_{A_1}^2}{\alpha}\int_{\mathbb{R}^d}|f|\,\mathrm{d}x$$

both in the matrix and scalar settings, where G is either the Hardy–Littlewood maximal function or any Calderón–Zygmund operator. In this note we show that the quadratic dependence on $[w]_{A_1}$ is sharp. This is done by constructing a sequence of scalar-valued weights with blowing up characteristics so that the corresponding bounds for the Hilbert transform and maximal function are exactly quadratic.

Résumé. Dans le récent travail [Cruz-Uribe et al. (2021)], il a été démontré

$$|\{x\in\mathbb{R}^d : w(x)|G(fw^{-1})(x)|>\alpha\}|\lesssim \frac{[w]_{A_1}^2}{\alpha}\int_{\mathbb{R}^d}|f|\,\mathrm{d}x$$

à la fois dans les contextes matriciel et scalaire, où G est soit la fonction maximale de Hardy-Littlewood ou tout opérateur de Calderón-Zygmund. Dans cette note, nous démontrons que la dépendance quadratique par rapport à $[w]_{A_1}$ est optimale. Cela est réalisé en construisant une séquence de poids à valeurs scalaires avec des caractéristiques d'éclatements, de sorte que les bornes correspondantes à la transformation de Hilbert et la fonction maximale soient exactement quadratiques.

Keywords. Matrix weights, quantitative bounds, endpoint estimates.

Mots-clés. Poids matriciel, borne quantitative, estimations de type faible $(1,1)$.

2020 Mathematics Subject Classification. 42B20, 42B25.

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ISSN (electronic): 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

Funding. The first author was supported by ISF grant no. 1035/21. The second author was supported by the National Natural Science Foundation of China through project numbers 12222114 and 12001400. The fourth author was supported by Spanish Ministerio de Ciencia e Innovación grant PID2022-136619NB-I00 and by Junta de Andalucía grant FQM-354.

Manuscript received 12 March 2024, accepted 14 April 2024.

1. Introduction

Recall that a non-negative locally integrable function w satisfies the A_1 condition if there exists a constant $C > 0$ such that for every cube $Q \subset \mathbb{R}^d$,

$$\frac{1}{|Q|}\int_Q w \le C \operatorname*{ess\,inf}_Q w.$$

The smallest constant C for which this property holds is denoted by $[w]_{A_1}$.

In the 70s, Muckenhoupt and Wheeden [10] established weighted weak type $(1,1)$ bounds of the form

$$|\{x \in \mathbb{R} : w(x)|T(fw^{-1})(x)| > \alpha\}| \lesssim \frac{C_w}{\alpha}\int_{\mathbb{R}} |f|\,\mathrm{d}x, \tag{1}$$

where T is either the Hilbert transform H or the Hardy–Littlewood maximal operator M. They showed as well that $w \in A_1$ is a sufficient condition for those inequalities to hold, even though it is not necessary. Muckenhoupt and Wheeden observed that (1) could be regarded as a first step to settle inequalities of the form

$$u^{1-r}\left(\left\{u^r|Tf| > \alpha\right\}\right) \lesssim \frac{1}{\alpha}\int_{\mathbb{R}} |f|\,w\,\mathrm{d}x, \tag{2}$$

where u, w are non-negative functions and $r \in [0,1]$. Note that for $u = w$, for $r = 0$ this inequality is the standard weak type inequality and in the case $r = 1$ it reduces to (1). Their idea was to combine (2) with interpolation with change of measures in order to obtain two weighted estimates.

Pushing forward that idea, Sawyer [13] showed that

$$uv\left(\left\{\frac{M(fv)}{v} > \alpha\right\}\right) \lesssim C_{u,v}\frac{1}{\alpha}\int_{\mathbb{R}} |f|\,uv\,\mathrm{d}x, \tag{3}$$

where $u, v \in A_1$. This estimate combined with interpolation with change of measures allowed him to reprove Muckenhoupt's maximal theorem.

Since the aforementioned papers a number of works have been devoted to estimates related to the ones above, that are known in the literature as mixed weak type estimates. Some worth mentioning are [3] where (3) is extended to higher dimensions and further operators such as Calderón–Zygmund operators via extrapolation, or [9] where it is shown that $u \in A_1$ and $v \in A_\infty$ is sufficient for (3) to hold.

In terms of quantitative estimates for $C_{u,v}$ in (3), and up to very recently for C_w in (1), as we will mention soon, not very much is known. Some results are provided in the aforementioned work [9] or in some other papers such as [12] or [1] for $C_{u,v}$ in (3). The purpose of this note is to provide some insight on C_w in (1). However, before presenting our main result we would like to connect this problem with the matrix weighted setting. We devote the following lines to that purpose.

In the last years quantitative matrix weighted estimates have attracted the attention of a number of authors. Up until now only few sharp quantitative results in the matrix weight setting are known. Among them the sharp $L^p(W)$ bounds for the maximal operator [7], the sharp $L^2(W)$ bound for the square function [6], and also the sharp $L^p(W)$ bounds in terms of the $[W]_{A_q}$ constants with $1 \le q < p$ obtained in [8] for the maximal operator, Calderón–Zygmund operators and commutators. Very recently Domelevo, Petermichl, Treil and Volberg [4] showed the sharpness of the $L^2(W)$ bound by $[W]_{A_2}^{3/2}$ for Calderón–Zygmund operators obtained previously in [11].

Making sense of endpoint matrix weighted estimates is a tricky problem. Quite recently, Cruz-Uribe et al. [2] managed to obtain the first quantitative endpoint estimates in that setting. In order to state this result, we first give several definitions.

Assume that W is a matrix weight, that is, W is an $n\times n$ self-adjoint matrix function with locally integrable entries such that $W(x)$ is positive definite for a.e. $x \in \mathbb{R}^d$. Define the operator norm of W by

$$\|W(x)\| := \sup_{e\in\mathbb{C}^n:|e|=1} |W(x)e|.$$

We say that $W \in A_1$ if

$$[W]_{A_1} := \sup_Q \operatorname*{ess\,sup}_{y\in Q} \frac{1}{|Q|}\int_Q \|W(x)W(y)^{-1}\|\,\mathrm{d}x < \infty.$$

It is easy to see that the matrix A_1 constant $[W]_{A_1}$ coincides with $[w]_{A_1}$ when $n = 1$.

Given a matrix weight W, a vector-valued function $\vec{f} : \mathbb{R}^d \to \mathbb{C}^n$ and a Calderón–Zygmund operator T, define

$$T_W\vec{f}(x) := W(x)T(W^{-1}\vec{f})(x).$$

Next, define the maximal operator by

$$M_W\vec{f}(x) := \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |W(x)W^{-1}(y)\vec{f}(y)|\,\mathrm{d}y.$$

The operators above have the obvious interpretation in the scalar setting.

Theorem A ([2]). *We have*

$$\left|\left\{x\in\mathbb{R}^d : |T_W\vec{f}(x)| > \alpha\right\}\right| \lesssim \frac{[W]_{A_1}^2}{\alpha}\int_{\mathbb{R}^d} |\vec{f}|\,\mathrm{d}x, \tag{4}$$

and the same holds for M_W.

At this point we are in the position to state the main result of this note.

Theorem 1. *In the scalar-valued setting the quadratic dependence on $[w]_{A_1}$ in (4) is sharp for T_w and for M_w.*

As a direct consequence of Lemma 2 below, this result shows the sharpness of $[W]_{A_1}^2$ in Theorem A in the matrix setting as well.

An interesting phenomenon here is the contrast between the strong $L^2(W)$ and the weak $L^1(W)$ bounds for Calderón–Zygmund operators. As we mentioned above, the recent work [4] establishes the sharpness of $[W]_{A_2}^{3/2}$ in the matrix setting. Comparing this with the linear A_2 bound in the scalar case [5], we see that the sharp weighted L^2 bounds for Calderón–Zygmund operators are different in the matrix and scalar settings. However, Theorem 1 shows that the sharp weighted weak L^1 bounds are the same in both settings.

2. Proof of Theorem 1

2.1. *Connection between the scalar and the matrix weighted estimates*

Lemma 2. *Assume that G_W stands either for T_W or for M_W. Then if*

$$\left|\left\{x\in\mathbb{R}^d : |G_W\vec{f}(x)| > \alpha\right\}\right| \le c\varphi\left([W]_{A_1}\right)\frac{1}{\alpha}\int_{\mathbb{R}^d} |\vec{f}|\,\mathrm{d}x,$$

we have that for every $w \in A_1$

$$\left|\left\{x \in \mathbb{R}^d \,:\, |J_w f(x)| > \alpha\right\}\right| \le c\varphi\left([w]_{A_1}\right) \frac{1}{\alpha} \int_{\mathbb{R}^d} |f| \,\mathrm{d}x,$$

where J_w stands, respectively, for $wT(fw^{-1})$ or for $wM(fw^{-1})$.

Proof. Let $w \in A_1$. It is clear that $W = wI_n$ is a matrix A_1 weight. Furthermore,

$$[W]_{A_1} = [w]_{A_1}.$$

Now given a scalar function f, we build $\vec{f} = (f, 0, \ldots, 0)^t$. Note that for these choices of $\vec{f}$ and W, clearly

$$\left|G_W \vec{f}(x)\right| = |J_w f(x)|.$$

This ends the proof. □

2.2. *Construction of a family of weights providing the lower bound*

We prove Theorem 1 via the following result.

Theorem 3. *For any integer $N > 20$, there exists a scalar weight $w \in A_1(\mathbb{R})$ satisfying the following properties:*

(1) $\int_0^1 w = 1$;
(2) $[w]_{A_1} \simeq N$;
(3) $|\{x \in (1, \infty) : w(x) > x\}| \gtrsim N^2$.

Observe that Theorem 3 implies Theorem 1 immediately because if we take $f = \chi_{[0,1]}$, then for $x > 1$,

$$Mf(x) = \frac{1}{x}, \qquad Hf(x) = \int \frac{f(y)}{x - y} \,\mathrm{d}y > \frac{1}{x}.$$

Hence, if T is either M or H, then

$$\begin{aligned} \|wTf\|_{L^{1,\infty}} &\ge \left|\left\{x \in (1, \infty) : w(x)|Tf(x)| > 1\right\}\right| \\ &\ge \left|\left\{x \in (1, \infty) : w(x) \cdot \frac{1}{x} > 1\right\}\right| \\ &\gtrsim N^2 \simeq [w]_{A_1}^2 \|f\|_{L^1(w)}. \end{aligned}$$

The rest of this section will be devoted to proving Theorem 3.

Proof of Theorem 3. For $k = 2, 3, \ldots, N$ we denote $J_k = [2^k, 2^{k+1})$. We will split J_k into small intervals. Set $I_k = [2^k, 2^k + k)$ and $L_k = J_k \setminus I_k = [2^k + k, 2^{k+1})$. Let L_k^- and L_k^+ be the left and right halves of L_k, respectively. Next we define $(L_k^-)^1$ to be the right half of L_k^- and $(L_k^+)^1$ the left half of L_k^+. Then

- when $\left(L_k^-\right)^j = \left[a_k^j, b_k^j\right)$ is defined, let $\left(L_k^-\right)^{j+1} = \left[a_k^{j+1}, b_k^{j+1}\right)$ satisfy that

$$b_k^{j+1} = a_k^j, \qquad \left|(L_k^-)^{j+1}\right| = \frac{1}{2}\left|(L_k^-)^j\right|;$$

- when $\left(L_k^+\right)^j = \left[c_k^j, d_k^j\right)$ is defined, let $\left(L_k^+\right)^{j+1} = \left[c_k^{j+1}, d_k^{j+1}\right)$ satisfy that

$$c_k^{j+1} = d_k^j, \qquad \left|(L_k^+)^{j+1}\right| = \frac{1}{2}\left|(L_k^+)^j\right|.$$

The process is stopped when we have $(L_k^-)^{k-1}$ and $(L_k^+)^{k-1}$ defined, and we simply define

$$(L_k^-)^k = \left[2^k + k, 2^k + k + \frac{|L_k^-|}{2^{k-1}}\right), \ (L_k^+)^k = \left[2^{k+1} - \frac{|L_k^+|}{2^{k-1}}, 2^{k+1}\right).$$

Now we have split J_k into disjoint intervals, namely,

$$J_k = I_k \cup \bigcup_{j=1}^{k} (L_k^-)^j \cup \bigcup_{j=1}^{k} (L_k^+)^j.$$

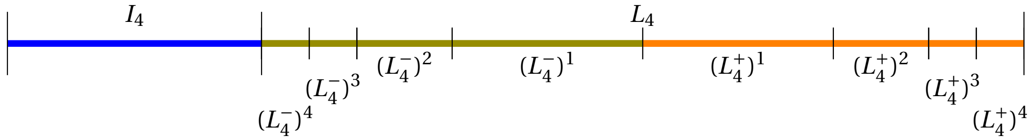

Figure 1. Component intervals of J_4.

Define

$$w_k = 2^{k+1}\chi_{I_k} + \sum_{j=1}^{k} 2^j \chi_{(L_k^-)^j \cup (L_k^+)^j}$$

and our weight on $[0, 2^{N+2}]$ is

$$w(x) = \begin{cases} \chi_{[0,4)}(x) + \sum_{k=2}^{N} w_k(x), & x \in [0, 2^{N+1}), \\ 2^N, & x = 2^{N+1}, \\ w(2^{N+2} - x), & x \in [2^{N+1}, 2^{N+2}]. \end{cases}$$

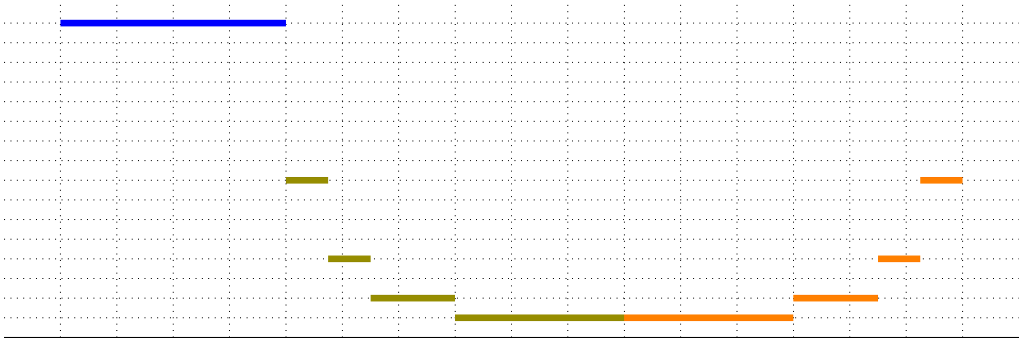

Figure 2. Graph of w_4

Finally we extend $w(x)$ from $[0, 2^{N+2}]$ to $\mathbb{R}$ periodically with period 2^{N+2}. Such a weight trivially satisfies $\int_0^1 w = 1$. Moreover, since $w(x) > x$ on I_k, we have

$$\left|\{x \in (1, \infty) : w(x) > x\}\right| \geq \sum_{k=2}^{N} |I_k| = \sum_{k=2}^{N} k \simeq N^2.$$

Hence it remains to check that $[w]_{A_1} \simeq N$. Since w is periodic on $\mathbb{R}$ and symmetrical on $[0, 2^{N+2}]$, it suffices to prove that

$$\sup_{I \subset [0, 2^{N+1}]} \frac{w(I)}{|I| \operatorname*{ess\,inf}_{x \in I} w(x)} \simeq N$$

(we use the standard notation $w(E) = \int_E w$).

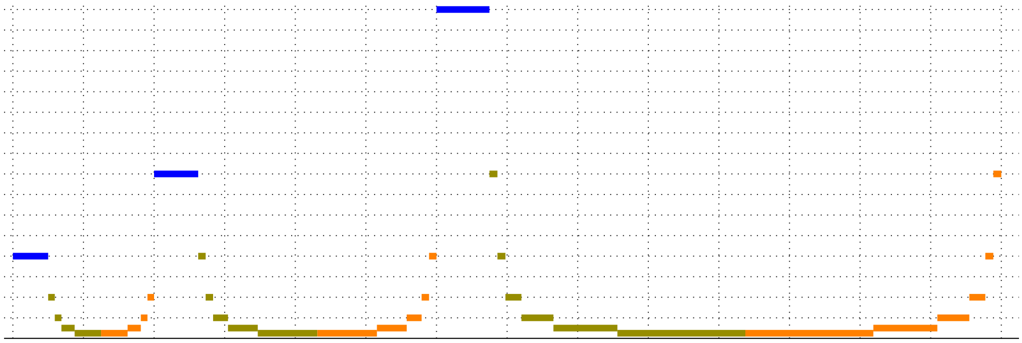

Figure 3. Joint graph of w_4, w_5 and w_6.

Observe that $|L_k^-| = |L_k^+| = \frac{1}{2}(2^k - k)$. Further,

$$\left|(L_k^-)^j\right| = \left|(L_k^+)^j\right| = \frac{1}{2^{j+1}}(2^k - k), \quad j = 1, \dots, k-1,$$

and

$$\left|(L_k^-)^k\right| = \left|(L_k^+)^k\right| = \frac{1}{2^k}(2^k - k).$$

Hence,

$$\begin{aligned} w_k(J_k) &= 2^{k+1}|I_k| + \sum_{j=1}^{k} 2^j \left(\left|(L_k^-)^j\right| + \left|(L_k^+)^j\right|\right) \\ &= 2^{k+1}k + (k-1)(2^k - k) + 2(2^k - k) \simeq k2^k. \end{aligned}$$

From this, when $I = [0, 2^{N+1}]$ we have

$$\begin{aligned} \frac{w(I)}{|I|} = 2^{-(N+1)}\left(4 + \sum_{k=2}^{N} w_k(J_k)\right) &\simeq 2^{-(N+1)}\left(4 + \sum_{k=2}^{N} k2^k\right) \\ &\simeq N = N \operatorname*{ess\,inf}_{x \in I} w(x). \end{aligned}$$

Therefore, we are left to prove that for any $I \subset [0, 2^{N+1}]$,

$$\frac{w(I)}{|I|} \lesssim N \operatorname*{ess\,inf}_{x \in I} w(x). \tag{5}$$

At this point we will make the following elementary observation. Our weight w is a step function, and for each two adjacent intervals from its definition, the “jump" of w is at most 2. Since the “jumps" are multiplicative we have the following.

Claim A. *If I intersects at most m intervals from the definition of w, then*

$$\frac{w(I)}{|I|} \le \max_{x \in I} w(x) \le 2^m \min_{x \in I} w(x) = 2^m \operatorname*{ess\,inf}_{x \in I} w(x).$$

In what follows we will prove (5) according to the size of I.

Case 1. $|I| \le 4$. In this case, note that in each J_k $(k \ge 2)$, $(L_k^-)^{k-1}, (L_k^-)^k$ and $(L_k^+)^{k-1}, (L_k^+)^k$ are the smallest intervals, and

$$\left|(L_k^-)^{k-1}\right| = \left|(L_k^-)^k\right| = \left|(L_k^+)^{k-1}\right| = \left|(L_k^+)^k\right| = 1 - \frac{k}{2^k} \ge \frac{1}{2}.$$

Hence I intersects at most 9 intervals from the definition of w, and we are in position to apply Claim A with $m = 9$.

Case 2. $|I| > 4$. In this case, we may assume $|I| \in (2^{k_0}, 2^{k_0+1}]$ with some $k_0 \geq 2$. We may further assume $k_0 < N - 10$ as otherwise

$$\frac{w(I)}{|I|} \lesssim 2^{-N} w([0, 2^{N+1}]) \simeq N \operatorname*{ess\,inf}_{x \in I} w(x).$$

Case 2a. $I \subset [0, 2^{k_0+10}]$. Then similarly to above,

$$\frac{w(I)}{|I|} < 2^{-k_0} w([0, 2^{k_0+10}]) \simeq k_0 \operatorname*{ess\,inf}_{x \in I} w(x).$$

Case 2b. $I \not\subset [0, 2^{k_0+10}]$. Then $I \subset [2^{k_0+9}, 2^{N+1}]$. Denote by c_k the center of L_k.

Case 2b-a. I contains some c_k with $k \geq k_0 + 9$. Then the estimate is trivial since $I \subset (L_k^-)^1 \cup (L_k^+)^1$ and we apply Claim A with $m = 2$.

Case 2b-b. I does not contain any c_k. In this case we may assume $I \subset (c_\ell, c_{\ell+1})$ for some $k_0 + 8 \leq \ell \leq N$.

Suppose that $I = [a, b]$ and $a \in (L_\ell^+)^j$ for some j. If $j \leq \ell - k_0 - 4$, then

$$\left|(L_\ell^+)^{j+1}\right| = |L_\ell^+| 2^{-(j+1)} = \frac{2^\ell - \ell}{2^{j+2}} > 2^{k_0+1},$$

so that I will intersect at most $(L_\ell^+)^j$ and $(L_\ell^+)^{j+1}$ and we again apply Claim A with $m = 2$.

If $j \geq \ell - k_0 - 3$, note that then

$$I \subset \bigcup_{j=\ell-k_0-3}^{\ell} (L_\ell^+)^j \cup \bigcup_{i=\ell-k_0-2}^{\ell+1} (L_{\ell+1}^-)^i \cup I_{\ell+1}.$$

Here $i \geq \ell - k_0 - 2$ since

$$\left|(L_{\ell+1}^-)^{\ell-k_0-2}\right| = \frac{2^{\ell+1} - (\ell+1)}{2^{\ell-k_0-1}} > \frac{2^\ell}{2^{\ell-k_0-1}} = 2^{k_0+1} \geq \ell(I).$$

Hence we have

$$\begin{aligned}
\frac{w(I)}{|I| \operatorname*{ess\,inf}_{x \in I} w(x)} &\leq \frac{\sum_{j=\ell-k_0-3}^{\ell} w\big((L_\ell^+)^j\big) + \sum_{i=\ell-k_0-2}^{\ell+1} w\big((L_{\ell+1}^-)^i\big) + w(I_{\ell+1})}{2^{k_0} 2^{\ell-k_0-3}} \\
&\lesssim \frac{\sum_{j=\ell-k_0-3}^{\ell} 2^j \cdot 2^{\ell-j} + \sum_{i=\ell-k_0-2}^{\ell+1} 2^i \cdot 2^{\ell+1-i} + (\ell+1) 2^{\ell+2}}{2^\ell} \lesssim \ell.
\end{aligned}$$

It remains to consider the case $a \in I_{\ell+1} \cup L_{\ell+1}^-$. However, in this case we just need to discuss whether $b \in (L_{\ell+1}^-)^j$ with some $j \leq \ell - k_0 - 3$ or not, which is completely similar. This completes the proof. □

Acknowledgement

We would like to thank the anonymous referee for his/her careful reading that helped improving the presentation of the paper.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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