ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Junpeng Wang, Zhongkui Liu and Renyu Zhao

When are the classes of Gorenstein modules (co)tilting?

Volume 362 (2024), p. 1301-1325

Online since: 14 November 2024

https://doi.org/10.5802/crmath.639

This article is licensed under the
Creative Commons Attribution 4.0 International License.
http://creativecommons.org/licenses/by/4.0/

CENTRE MERSENNE

The Comptes Rendus. Mathématique *are a member of the Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2024, Vol. 362, p. 1301-1325
https://doi.org/10.5802/crmath.639

Research article / *Article de recherche*
Algebra / *Algèbre*

When are the classes of Gorenstein modules (co)tilting?

Quand les classes de modules de Gorenstein sont-elles (co)basculantes ?

Junpeng Wang [a], **Zhongkui Liu** [a] **and Renyu Zhao** [*, a]

[a] Department of Mathematics, Northwest Normal University, Lanzhou 730070, People's Republic of China

E-mails: wangjunpeng1218@163.com, liuzk@nwnu.edu.cn, zhaory@nwnu.edu.cn

Abstract. For the class of Gorenstein projective (resp. injective and flat) modules, we investigate and settle the questions when the middle class is tilting and the other ones are cotilting. The applications have in three directions. The first is to obtain the coincidence between the 1-tilting and silting property, as well as the 1-cotilting and cosilting property of such classes respectively. The second is to characterize Gorenstein modules via finitely generated modules, which provides a proof of that left Noetherian rings with finite left Gorenstein global dimension satisfy First Finitistic Dimension Conjecture and a result related to a question posed by Bazzoni in [J. Algebra 320 (2008) 4281-4299]. The last is to give some new characterizations of Dedekind and Prüfer domains and commutative Gorenstein Artin algebras as well as general (possibly not commutative) Gorenstein rings and Ding–Chen rings.

Résumé. Pour la classe des modules Gorenstein-projectifs (respectivement G-injectifs et G-plats), nous étudions et réglons les questions de savoir quand la seconde est basculante et les autres cobasculantes. Les applications vont dans trois directions. La première est d'obtenir la coïncidence entre les propriétés de ces classes d'être, respectivement, 1-basculante et bousculante, ainsi que la propriété d'être 1-cobasculante et cobousculante. La deuxième consiste à caractériser les modules de Gorenstein via des modules finiment engendrés, ce qui prouve que les anneaux noethériens à gauche de dimension globale de Gorenstein gauche finie satisfont la première conjecture de la dimension finitiste et un résultat lié à une question posée par Bazzoni dans [J. Algebra 320 (2008) 4281-4299]. Le dernier objectif est de donner de nouvelles caractérisations des domaines de Dedekind et de Prüfer et des algèbres d'Artin de Gorenstein commutatives, ainsi que des anneaux de Gorenstein généraux (éventuellement non commutatifs) et des anneaux de Ding–Chen.

Keywords. Gorenstein projective (resp. injective and flat) module, (co)tilting class, finitistic dimension conjecture, (strongly) finite type.

Mots-clés. Module Gorenstein-projectif (respectivement G-injectif et G-plat), classe de (co)basculement, conjecture de la dimension finitiste, type (fortement) fini.

2020 Mathematics Subject Classification. 18G25, 18E45, 16E60, 16E65.

Funding. This work is partially supported by the National Natural Science Foundation of China (Grant no. 12361008, 12061061, 11861055, 11761060), the Natural Science Foundation of Gansu Province of China (Grant no. 22JR5RA153) and the Foundation for Innovative Fundamental Research Group Project of Gansu Province (Grant no. 23JRRA684).

Manuscript received 22 April 2023, revised 7 September 2023, accepted 15 April 2024.

[*] Corresponding author

1. Introduction

Enochs, Jenda and Torrecillas [25, 28] introduced Gorenstein projective, injective and flat modules for any ring and then established Gorenstein homological algebra. Such a relative homological algebra has been developed rapidly during the past several decades and now becomes a rich theory. Tilting theory has its origin as Morita equivalence for derived categories, and becomes an important tool to deal with many famous conjecture in homological algebra and algebra representation theory, such as "Telescope Conjecture for Modules categories" (see [6]) and "Finitistic Dimension Conjecture" (see [7]) and so on. The goal of this manuscript is to investigate some connections between Gorenstein homological algebra theory and tilting theory. Such a work can go back to the construction of Bass (co)tilting modules over Gorenstein rings, and was extensively studied by Angeleri Hügel, Herbera, Trlifaj, Wang, Li, Hu, Di, Wei, Zhang, Chen, Moradifar, Saroch and Yassemi in [3, 20, 37, 38, 42, 46].

Given a ring R, we denote by $\mathcal{GP}(R)$ (resp. $\mathcal{GI}(R)$, $\mathcal{GF}(R)$) the class of all Gorenstein projective (resp. injective and flat) left R-modules. For the corresponding classes of right R-modules, we use the notation $\mathcal{GP}(R^{\mathrm{op}})$ and so on, where R^{op} is the oppositive ring of R. Recall that a class $\mathcal{X}$ of modules is *tilting* (resp. *cotilting*) if $\mathcal{X}$ is an n-tilting (resp. n-cotilting) class for some nonnegative integer n, that is, there is an n-tilting (resp. n-cotilting) module T such that $\mathcal{X} = \{T\}^{\perp}$ (resp. $\mathcal{X} = {}^{\perp}\{T\}$). Such properties for a general class $\mathcal{X}$ were classified by Angeleri Hügel, Trlifaj, Göbel and Bazzoni in [3, 9, 32]. In particular, the properties for the classes of Gorenstein modules were classified in [3]. Let R be a two-sided Noetherian ring. Then [3, Theorem 3.4] proved that the classes $\mathcal{GI}(R)$ and $\mathcal{GI}(R^{\mathrm{op}})$ are tilting if and only if the class $\mathcal{GI}(R)$ is tilting and the class $\mathcal{GF}(R)$ is cotilting, if and only if R is Gorenstein. If R is an Artin algebra, then [3, Corollary 3.8] showed that the class $\mathcal{GP}(R)$ is cotilting if and only if R is Gorenstein. Motivated by these results, the following questions are posed naturally:

Question 1. *When does the class $\mathcal{GP}(R)$ (resp. $\mathcal{GF}(R)$) form a cotilting class ?*

Question 2. *When does the class $\mathcal{GI}(R)$ form a tilting class ?*

Our main results are as follows, which give a thorough-paced answer to Questions 1 and 2.

Theorem 3 (see Theorems 33 and 34). *The following are equivalent for any ring R:*

(1) *The class $\mathcal{GP}(R)$ (resp. $\mathcal{GF}(R)$) is cotilting.*
(2) *R is a right coherent and left perfect (resp. right coherent) ring admitting finite left Gorenstein global dimension (resp. finite Gorenstein weak global dimension).*

Theorem 4 (see Theorem 35). *The following are equivalent for any ring R:*

(1) *The class $\mathcal{GI}(R)$ is tilting.*
(2) *R is a left Noetherian ring admitting finite left Gorenstein global dimension.*

It is well-known that, injective and flat modules can be characterized by finitely generated modules over a ring. As the first application of the preceding theorems, we characterize Gorenstein projective (resp. injective and flat) modules by finitely generated modules having finite projective dimension over left Noetherian rings with finite left Gorenstein global dimension (see Theorem 45 and Lemma 47). On one hand, the characterizations provides us a proof that any left Noetherian ring with finite left Gorenstein global dimension satisfies "First Finitistic Dimension Conjecture" (see Corollary 46). It covers the same result for Gorenstein rings [3, Theorem 3.2] (see Christensen, Estrada, and Thompson [17, Remark 3.11] for the fact of that Gorenstein rings can be described as two-sided Noetherian rings with finite left Gorenstein global dimension and see Example 39 for the existence of a left Noetherian ring with finite left Gorenstein global dimension which is not Gorenstein). On the other hand, the characterizations provides us a result related to a question posed by Bazzoni in [9, Question 1(1)].

Silting modules was introduced by Angeleri Hügel, Marks and Vitória [5], which provide a common generalization of 1-tilting modules and support τ-tilting modules, and correspond bijectively to the two-terms silting complexes. Cosilting modules, as the dual notion, was introduced by Breaz and Pop [14]. Recall that a class $\mathscr{X}$ of modules is *silting* (resp. *cosilting*) if there is a silting (resp. cosilting) module T such that $\mathscr{X} = \operatorname{Gen} T$ (resp. $\mathscr{X} = \operatorname{Cogen} T$). Note from [32, Lemma 6.1.2] (resp. [32, Lemma 8.2.2]) that a class $\mathscr{X}$ is 1-tilting (resp. 1-cotilting) if and only if there is a 1-tilting (resp. 1-cotilting) module T such that $\mathscr{X} = \operatorname{Gen} T$ (resp. $\mathscr{X} = \operatorname{Cogen} T$). Since the inclusions {1-tilting modules} ⊆ {silting modules} and {1-cotilting classes} ⊆ {cosilting classes} are strict, it is a routine to check that the another one {1-tilting classes} ⊆ {silting classes} and {1-cotilting classes} ⊆ {cosilting classes} are strict as well. As the second application of the preceding theorems, we completely settle the questions when the classes $\mathscr{GP}(R)$ and $\mathscr{GF}(R)$ are cosilting and when the class $\mathscr{GI}(R)$ is silting (see Propositions 54, 55 and 56). These results shows that the silting (resp. cosilting) and 1-tilting (resp. 1-cotilting) property for the classes $\mathscr{GP}(R)$ and $\mathscr{GF}(R)$ (resp. the class $\mathscr{GI}(R)$) coincide.

As an example of Gorenstein rings, Gorenstein Artin algebras play an important role in representation theory of Artin algebras. On the other hand, many authors, such as Ding, Chen, Mao, Li and Gillespie [22, 23, 29, 31], pointed out that Ding–Chen rings are natural generalizations of Gorenstein rings. The third application of the theorems is to give some characterizations of Gorenstein rings (including Gorenstein Artin algebras) and Ding–Chen rings. Note that Corollary 5 below is a slight improvement of [3, Theorem 3.4].

Corollary 5 (see Theorem 37 and Corollary 38). *A ring R is Gorenstein if and only if both the classes $\mathscr{GI}(R)$ and $\mathscr{GI}(R^{\mathrm{op}})$ are tilting. In particular, a commutative (or two-sided Noetherian) ring R is Gorenstein if and only if the class $\mathscr{GI}(R)$ is tilting.*

Corollary 6 (see Theorem 40 and Corollary 41). *A ring R is Ding–Chen if and only if both the classes $\mathscr{GF}(R)$ and $\mathscr{GF}(R^{\mathrm{op}})$ are cotilting. In particular, a commutative (or two-sided coherent) ring R is Ding–Chen if and only if the class $\mathscr{GF}(R)$ is cotilting.*

Corollary 7 (see Theorem 42). *A commutative ring R is a Gorenstein Artin algebra if and only if the class $\mathscr{GP}(R)$ is cotilting.*

As shown in [32], both the (co)tilting modules and classes over a Dedekind (resp. Prüfer) domain have a nice description. The last application of the theorems is to characterize Dedekind and Prüfer domains using some special (co)tilting classes.

Corollary 8 (see Theorems 60 and 61). *Let R be a domain. Then the following hold:*

(1) *R is Dedekind if and only if the class $\mathscr{GI}(R)$ is 1-tilting and $\mathscr{GI}(R) = \mathscr{I}(R)$.*
(2) *R is Prüfer if and only if the class $\mathscr{GF}(R)$ is 1-cotilting and $\mathscr{GF}(R) = \mathscr{F}(R)$.*

Here $\mathscr{I}(R)$ (resp. $\mathscr{F}(R)$) denotes the class of all injective (resp. flat) left R-modules.

We conclude this section by summarizing the contents of this paper. Section 2 contains some notations, definitions and lemmas for use throughout this paper. Section 3 is devoted to proving Theorems 3 and 4. Section 4 gives some applications of Theorems 3 and 4.

2. Preliminaries

Throughout this article, all rings R are assumed to be associative rings with identity and all modules are unitary. By an "R-module" we always mean a left R-module, for a right R-module, we view it as an R^{op}-module, where R^{op} is the oppositive ring of R.

In this section we mainly recall some necessary notions and facts, which will be used in the paper. Let R be a ring. As usual, denote by R-Mod the class of all R-modules; by $\mathscr{P}(R)$ (resp.

$\mathscr{I}(R)$ and $\mathscr{F}(R)$) its subclass of all projective (resp. injective and flat) R-modules; by $\mathrm{pd}_R(M)$ (resp. $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$) the projective (resp. injective and flat) dimension of an R-module M; by $\mathrm{gldim}(R)$ (resp. $\mathrm{wgldim}(R)$) the left global (resp. weak global) dimension of R. In addition, we write $M^+ = \mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$.

For an R-module M, denote by $\mathrm{Add}\,M$ (resp. $\mathrm{Prod}\,M$) the class of all R-modules that are isomorphic to a direct summand of a copy coproduct (resp. product) of M; by $\mathrm{Gen}\,M$ (resp. $\mathrm{Cogen}\,M$) is the class formed by all R-modules that are isomorphic to the epimorphic images of some R-module in $\mathrm{Add}\,M$ (resp. to the submodules of some R-module in $\mathrm{Prod}\,M$).

2.1. *Cotorsion pairs and induced dimensions*

Let R be a ring and $\mathscr{X},\mathscr{Y}$ classes of R-modules. A pair $(\mathscr{X},\mathscr{Y})$ is called a *cotorsion pair* if $\mathscr{X}^\perp = \mathscr{Y}$ and ${}^\perp\mathscr{Y} = \mathscr{X}$. Here $\mathscr{X}^\perp = \{M \in R\text{-Mod} \mid \mathrm{Ext}^1_R(X,M) = 0, \forall X \in \mathscr{X}\}$, and ${}^\perp\mathscr{Y}$ is defined dually. A cotorsion pair $(\mathscr{X},\mathscr{Y})$ is said to be *hereditary* if $\mathrm{Ext}^n_R(X,Y) = 0$ for all $X \in \mathscr{X}$, $Y \in \mathscr{Y}$ and $n \geq 1$. A cotorsion pair $(\mathscr{X},\mathscr{Y})$ is called *complete* if for any $M \in R\text{-Mod}$, there are exact sequences of R-modules $0 \to Y \to X \to M \to 0$ and $0 \to M \to Y' \to X' \to 0$ with $X, X' \in \mathscr{X}$ and $Y, Y' \in \mathscr{Y}$. A cotorsion pair $(\mathscr{X},\mathscr{Y})$ is said to be *cogenerated* (resp. *generated*) *by a set* if there is a set $\mathscr{S}$ of R-modules in $\mathscr{X}$ (resp. $\mathscr{Y}$) such that $\mathscr{S}^\perp = \mathscr{Y}$ (resp. ${}^\perp\mathscr{S} = \mathscr{X}$).

Given an R-module M and a class $\mathscr{X}$ of R-modules, a *special* $\mathscr{X}$*-preenvelope* of M is defined as a monic homomorphism $\alpha : M \to X$ with $X \in \mathscr{X}$ and $\mathrm{Coker}\,\alpha \in {}^\perp\mathscr{X}$. A class $\mathscr{X}$ is said to be *special preenveloping* if every R-module has a special $\mathscr{X}$-preenvelope. A class $\mathscr{X}$ is *injectively coresolving* if it is closed under extensions and cokernels of monic morphisms, and $I(R) \in \mathscr{X}$. Dually, we have the definitions of *special* $\mathscr{X}$*-precover* and that $\mathscr{X}$ is *special precovering* (resp. *projectively resolving*).

For an R-module M and a class $\mathscr{X}$ of R-modules, the $\mathscr{X}$-projective dimension of M, denoted by $\mathscr{X}\text{-pd}_R(M)$, is defined as follows:

$$\mathscr{X}\text{-pd}_R(M) = \inf\{n \in \mathbb{N} \mid \text{there is an exact sequence of } R\text{-modules}$$
$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0, \text{ where each } X_i \in \mathscr{X}\}.$$

If no such an exact sequence exists, then we set $\mathscr{X}\text{-pd}_R(M) = \infty$. Dually, we have the definition of $\mathscr{X}$*-injective dimension* of M, $\mathscr{X}\text{-id}_R(M)$.

Next two lemmas give some characterizations of relative projective dimensions of modules, which are applied for the proof of Lemma 26.

Lemma 9. *Let $\mathscr{X}$ be a class of R-modules such that $(\mathscr{X},\mathscr{X}^\perp)$ is a complete and hereditary cotorsion pair. Then the following are equivalent for any R-module M and any integer $n \geq 0$:*

(1) $\mathscr{X}\text{-pd}_R(M) \leq n$.
(2) *There is an exact sequence of R-modules* $0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$ *with each $X_i \in \mathscr{X}$.*
(3) *For any exact sequence of R-modules* $0 \to K \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$*, if each X_i is in $\mathscr{X}$, then so is K.*
(4) $\mathrm{Ext}^{n+1}_R(M,Y) = 0$ *for all $Y \in \mathscr{X}^\perp$.*

Proof.

(1) ⇔ (2). It is clear.

(2) ⇒ (3). Note that $\mathscr{X}$ is projectively resolving and closed under arbitrary direct sums and summands, as $(\mathscr{X},\mathscr{X}^\perp)$ forms a complete and hereditary cotorsion pair. So the result follows from [8, Lemma 3.12].

(3) ⇒ (4). Consider the exact sequence $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ of R-modules with each P_i projective. Then by (3) K is in $\mathscr{X}$ since so is each P_i. Now, for any $Y \in \mathscr{X}^{\perp}$, by dimension shifting one has $0 = \mathrm{Ext}^1_R(K, Y) \cong \mathrm{Ext}^{n+1}_R(M, Y)$.

(4) ⇒ (2). Consider an exact sequence $0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$ of R-modules with $X_i \in \mathscr{X}$ for all $0 \le i \le n-1$ (one can choose that all such X_i projective). Then, for any $Y \in \mathscr{X}^{\perp}$, by (4) and dimension shifting one has $\mathrm{Ext}^1_R(X_n, Y) \cong \mathrm{Ext}^{n+1}_R(M, Y) = 0$. Thus, $X_n \in \mathscr{X}$ since $(\mathscr{X}, \mathscr{X}^{\perp})$ is a cotorsion pair. □

Lemma 10. *Let $\mathscr{X}$ be a class of R-modules such that $(\mathscr{X}, \mathscr{X}^{\perp})$ is a complete and hereditary cotorsion pair. Then the following are equivalent:*

(1) $\sup\{\mathscr{X}\text{-pd}_R(M) \mid M \in R\text{-Mod}\} < \infty$.
(2) *Every R-module in $\mathscr{X}^{\perp}$ has finite injective dimension.*
(3) $\mathscr{X}\text{-pd}_R(M) < \infty$ *for any R-module M.*

Proof.

(1) ⇒ (3). It is obvious.

(2) ⇔ (3). It follows from Lemma 9.

(3) ⇒ (1). We first claim that

$$\mathscr{X}\text{-pd}_R\left(\coprod_{j\in J} M_j\right) = \sup\{\mathscr{X}\text{-pd}_R(M_j) \mid j \in J\}$$

for any family $\{M_j \mid j \in J\}$ of R-modules. Indeed, let $\sup\{\mathscr{X}\text{-pd}_R(M_j) \mid j \in J\} = m < \infty$. Then $\mathscr{X}\text{-pd}_R(M_j) \le m$ for each $j \in J$. By Lemma 9, there exists an exact sequence of R-modules $0 \to X_{m,j} \to \cdots \to X_{1,j} \to X_{0,j} \to M_j \to 0$ with each $X_{i,j} \in \mathscr{X}$. This induces another exact sequence of R-modules $0 \to \coprod_{j\in J} X_{m,j} \to \cdots \to \coprod_{j\in J} X_{1,j} \to \coprod_{j\in J} X_{0,j} \to \coprod_{j\in J} M_j \to 0$ with each $\coprod_{j\in J} X_{i,j} \in \mathscr{X}$ as $\mathscr{X}$ is closed under arbitrary direct sums. Hence, $\mathscr{X}\text{-pd}_R(\coprod_{j\in J} M_j) \le m$ by Lemma 9. Conversely, let $\mathscr{X}\text{-pd}_R(\coprod_{j\in J} M_j) = m < \infty$. Then Lemma 9 yields that $\mathrm{Ext}^{m+1}_R(\coprod_{j\in J} M_j, Y) = 0$ for all $Y \in \mathscr{X}^{\perp}$. So, using the isomorphism $\mathrm{Ext}^{m+1}_R(\coprod_{j\in J} M_j, Y) \cong \prod_{j\in J} \mathrm{Ext}^{m+1}_R(M_j, Y)$ one has $\mathrm{Ext}^{m+1}_R(M_j, Y) = 0$ for all $j \in J$. Thus, $\sup\{\mathscr{X}\text{-pd}_R(M_j) \mid j \in J\} \le m$ by Lemma 9. This shows the claim.

Now assume that $\mathscr{X}\text{-pd}_R(M) < \infty$ for any R-module M. If $\sup\{\mathscr{X}\text{-pd}_R(M) \mid M \in R\text{-Mod}\} = \infty$, then there is an R-module M_n such that $\mathscr{X}\text{-pd}_R(M_n) \ge n$ for any integer $n \ge 0$. This leads to a contradiction:

$$\infty = \sup\{\mathscr{X}\text{-pd}_R(M_n) \mid n \in \mathbb{N}\} \overset{\text{the above claim}}{=\!=\!=\!=\!=} \mathscr{X}\text{-pd}_R(\oplus_{n\in\mathbb{N}} M_n) < \infty.$$

Consequently, $\sup\{\mathscr{X}\text{-pd}_R(M) \mid M \in R\text{-Mod}\} < \infty$. □

Lemmas 11 and 12 below are the dual of Lemmas 9 and 10 respectively, which are applied for the proof of Lemma 27.

Lemma 11. *Let $\mathscr{X}$ be a class of R-modules such that $({}^{\perp}\mathscr{X}, \mathscr{X})$ is a complete and hereditary cotorsion pair. Then the following are equivalent for any R-module M and any integer $n \ge 0$:*

(1) $\mathscr{X}\text{-id}_R(M) \le n$.
(2) *There is an exact sequence of R-modules* $0 \to M \to X^0 \to X^1 \to \cdots \to X^n \to 0$ *with each* $X^i \in \mathscr{X}$.
(3) *For any exact sequence of R-modules* $0 \to M \to X^0 \to X^1 \to \cdots \to X^{n-1} \to C \to 0$*, if each X^i is in $\mathscr{X}$, then so is C.*
(4) $\mathrm{Ext}^{n+1}_R(Y, M) = 0$ *for all* $Y \in {}^{\perp}\mathscr{X}$.

Lemma 12. *Let $\mathscr{X}$ be a class of R-modules such that $({}^{\perp}\mathscr{X}, \mathscr{X})$ is a complete and hereditary cotorsion pair. Then the following are equivalent:*

(1) $\sup\{\mathscr{X}\text{-id}_R(M) \mid M \in R\text{-Mod}\} < \infty$.
(2) *Every R-module in ${}^{\perp}\mathscr{X}$ has finite projective dimension.*
(3) $\mathscr{X}\text{-id}_R(M) < \infty$ *for any R-module M.*

2.2. *Gorenstein homological modules and dimensions*

An R-module M is said to be *Gorenstein projective* [25] if there exists a $\mathrm{Hom}_R(-, \mathscr{P}(R))$-exact exact sequence of projective R-modules $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ such that $M \cong \mathrm{Im}(P_0 \to P^0)$. Dually we have the definition of *Gorenstein injective* [25] R-modules. An R-module M is said to be *Gorenstein flat* [28] if there exists an exact sequence of flat R-modules $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and that remains exact whenever the functor $I \otimes_R -$ is applied for any injective R^{op}-module I.

We denote by $\mathscr{GP}(R)$ (resp. $\mathscr{GI}(R)$ and $\mathscr{GF}(R)$) the class of all Gorenstein projective (resp. injective and flat) R-modules; by $\mathrm{Gpd}_R(M)$ (resp. $\mathrm{Gid}_R(M)$ and $\mathrm{Gfd}_R(M)$) the Gorenstein projective (resp. injective and flat) dimension of an R-module M, that is, $\mathscr{GP}(R)$-projective (resp. $\mathscr{GI}(R)$-injective and $\mathscr{GF}(R)$-projective) dimension of M.

As a refinement of the usual global (resp. weak global) dimension of rings, Gorenstein global (resp. Gorenstein weak global) dimension of rings is defined as follows:

Definition 13.

(1) *For any ring R, its left (resp. right) Gorestein global dimension, denoted by Ggldim(R) (resp. Ggldim(R^{op})), is defined via the following formula*

$$\sup\{\mathrm{Gpd}_R(M) \mid M \in R\text{-Mod}\} = \text{G-gldim}(R) = \sup\{\mathrm{Gid}_R(M) \mid M \in R\text{-Mod}\}$$

$$(\textit{resp.}\ \sup\{\mathrm{Gpd}_R(M) \mid M \in R^{\mathrm{op}}\text{-Mod}\} = \text{G-gldim}(R^{\mathrm{op}}) = \sup\{\mathrm{Gid}_R(M) \mid M \in R^{\mathrm{op}}\text{-Mod}\}).$$

(2) *We say that R admits* finite left Gorenstein global dimension *(resp.* finite right Gorenstein global dimension*) if* G-gldim$(R) < \infty$ *(resp.* G-gldim$(R^{\mathrm{op}}) < \infty$*).*

Definition 14.

(1) *For any ring R, its Gorestein weak global dimension, denoted by Gwgldim(R), is defined via the following formula*

$$\sup\{\mathrm{Gfd}_R(M) \mid M \in R\text{-}Mod\} = \text{G-wgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \in R^{\mathrm{op}}\text{-}Mod\}.$$

(2) *We say that R admits* finite Gorenstein weak global dimension *if* G-wgldim$(R) < \infty$.

Remarks 15.

(1) For any ring R, the equality

$$\sup\{\mathrm{Gpd}_R(M) \mid M \in R\text{-Mod}\} = \sup\{\mathrm{Gid}_R(M) \mid M \in R\text{-Mod}\}$$

was proved by Bennis and Mahdou in [11] and by Emmanouil in [24] using different method.

(2) For any ring R, the equality

$$\sup\{\mathrm{Gfd}_R(M) \mid M \in R\text{-Mod}\} = \sup\{\mathrm{Gfd}_R(M) \mid M \in R^{\mathrm{op}}\text{-Mod}\}$$

was proved in [17]. Note from Šaroch and Šťovíček [40, Theorem 4.11] that any ring is *left and right GF-closed* (i.e., the class of all Gorenstein flat left or right R-modules is closed under extensions). Such an equality can be also obtained by Bouchiba [12, Theorem 6(2)], as noted in [45].

It is well-known that $\mathrm{wgldim}(R) \le \mathrm{gldim}(R)$ for any ring R. The corresponding inequality

$$\text{G-wgldim}(R) \le \text{G-gldim}(R)$$

for a right coherent ring R was proved in [17, Corollary 3.5]. Recently, the firstly named author and coauthors [44] improved the result.

Lemma 16 ([44, Theorem 3.7 and Remark 3.12]). *Let R be a ring.*

(1) *There is an inequality* $\text{G-wgldim}(R) \le \text{G-gldim}(R)$.
(2) *The equality* $\text{G-wgldim}(R) = \text{G-gldim}(R)$ *(resp.* $\text{G-wgldim}(R) = \text{G-gldim}(R^{op})$*) holds true if R is left perfect (resp. right perfect).*

We end this section by the next lemma.

Lemma 17. *Let R be a right coherent ring or a ring with* $\text{G-wgldim}(R) < \infty$ *(in particular the case* $\text{G-gldim}(R) < \infty$*). Then $\mathscr{GP}(R) = \mathscr{GF}(R)$ if and only if R is left perfect.*

Proof. The "only if" part holds by [18, Proposition 3.1]. For the "if" part, we suppose that R is left perfect. Then one has $\mathscr{P}(R) = \mathscr{F}(R)$. If R is a right coherent ring, then $\mathscr{GP}(R) = \mathscr{GF}(R)$ holds by [43, Proposition 3.5]; in the other case, i.e., R is a ring with $\text{G-wgldim}(R) < \infty$, the equality follows by [45, Theorems 2.3 and 2.9]. □

3. (Co)tilting classes and Gorenstein modules

In this section, we will give a thorough-paced answer to Questions 1 and 2 (from the introduction). We start with the following definitions.

Definition 18. A class $\mathscr{X}$ of R-modules is called *definable* if $\mathscr{X}$ is closed under pure submodules, direct products and direct limits.

Remark 19. Note that a class $\mathscr{X}$ of R-modules is definable if and only if it is closed under products, pure epimorphic images and pure submodules. Indeed, if $\mathscr{X}$ is definable, then clearly it is closed under products and pure submodules. It is also closed under pure epimorphic images by [9, Proposition 4.3(3)]. Conversely, suppose that $\mathscr{X}$ is closed under products, pure epimorphic images and pure submodules. Since any direct limit of a family of R-modules is a pure epimorphic image of the direct sum of such a family of R-modules and any direct sum of a family of R-modules is a pure submodule of the direct product of such a family of R-modules, $\mathscr{X}$ is also closed under direct limits, and hence is definable.

Definition 20. An R-module M is said to be *of type FP_∞* [3, 13] or *compact* [9] if M possesses a projective resolution consisting of finitely generated R-modules.

Definition 21. A class $\mathscr{X}$ of R-modules is of *finite type* if $\mathscr{X} = {}^\perp(\mathscr{S}^\perp)$, where $\mathscr{S}$ is a set of R-modules of type FP_∞. If furthermore the set $\mathscr{S}$ consists of R-modules of type FP_∞ with finite projective dimension, then we call that the class $\mathscr{X} = {}^\perp(\mathscr{S}^\perp)$ is of *strongly finite type*.

Remarks 22.

(1) Clearly any class $\mathscr{X}$ of R-modules of strongly finite type is of finite type. Note that there exists a class of R-modules of finite type which is not of strongly finite type. We also note that Bazzoni, Göbel and Trlifaj in [9, 32] called that a class $\mathscr{X}$ is "of finite type", is just of strongly finite type in our sense.
(2) Let $\mathscr{X}$ be a class of R-modules which is of finite type. Then there is a complete and hereditary cotorsion pair $(\mathscr{X}, \mathscr{X}^\perp)$ cogenerated by some set $\mathscr{S}$ of R-modules of type FP_∞. The cotorsion pair is said to be of *finite type*. Similarly, we have the notion of that a cotorsion pair is of *strongly finite type*.

Definition 23.

(1) *An R-module T is called* tilting *if there is an integer $n \geq 0$ such that T is an n-tilting R-module, that is, T satisfies the following:*

(T1) $\mathrm{pd}_R(M) \leq n$.

(T2) $\mathrm{Ext}_R^{i\geq 1}(T, T^{(J)}) = 0$ *for all set J.*

(T3) *There is an exact sequence of R-modules* $0 \to R \to X^0 \to X^1 \to \cdots \to X^n \to 0$ *with each* $X^i \in \mathrm{Add}\, T$.

(2) *A class $\mathscr{X}$ of R-modules is called* tilting *if $\mathscr{X}$ is an n-tilting class for some $n \geq 0$, that is, there is an n-tilting R-module T such that $\mathscr{X} = \{T\}^{\perp}$.*

Definition 24.

(1) *An R-module T is called* cotilting *if there is an integer $n \geq 0$ such that T is an n-cotilting R-module, that is, T satisfies the following:*

(CT1) $\mathrm{id}_R(M) \leq n$.

(CT2) $\mathrm{Ext}_R^{i\geq 1}(T^J, T) = 0$ *for all set J.*

(CT3) *There is an exact sequence of R-modules* $0 \to X_n \to \cdots \to X_1 \to X_0 \to W \to 0$ *with each* $X_i \in \mathrm{Prod}\, T$ *and W an injective cogenerator.*

(2) *A class $\mathscr{X}$ of R-modules is called* cotilting *if $\mathscr{X}$ is an n-cotilting class for some $n \geq 0$, that is, there is an n-cotilting R-module T such that $\mathscr{X} = {}^{\perp}\{T\}$.*

Remarks 25.

(1) Obviously 0-tilting (resp. 0-cotilting) R-modules coincide with projective generators (resp. injective cogenerators). Thus, 0-tilting and 0-cotilting classes of R-modules are just R-Mod.
(2) It is known from [32, Theorems 5.1.14 and 8.1.9] that any tilting or cotilting class is definable.
(3) Following [9, Proposition 3.7(1)], we know that a class $\mathscr{X}$ (and hence a cotorsion pair $(\mathscr{X}, \mathscr{X}^{\perp})$) of R-modules is tilting if and only if it is of strongly finite type.
(4) If given a tilting (resp. cotilting class $\mathscr{X}$), then we have a complete and hereditary cotorsion pair $(\mathscr{X}, \mathscr{X}^{\perp})$ (resp. $({}^{\perp}\mathscr{X}, \mathscr{X})$) which is cogenerated (resp. generated) by the tilting (resp. cotilting) R-module T. The corresponding cotorsion pair $(\mathscr{X}, \mathscr{X}^{\perp})$ (resp. $({}^{\perp}\mathscr{X}, \mathscr{X})$) is called *tilting* (resp. *cotilting*).

Let $n \geq 0$ be an integer. Recall from [9] that a class $\mathscr{X}$ of R-modules is *closed under n-submodules* (resp. *closed under n-images*) provided that any R-module M is in $\mathscr{X}$ whenever there is an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{n-1}$ (resp. $X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$) of R-modules with each $X_i \in \mathscr{X}$.

By virtue of [9, Theorem 6.1], we know that a class $\mathscr{X}$ of R-modules is cotilting if and only if $\mathscr{X}$ is is definable, projectively resolving and there is an integer $n \geq 0$ such that $\mathscr{X}$ is closed under n-submodules; we know from [9, Theorem 6.1] that a definable class $\mathscr{X}$ of R-modules is tilting if and only if $\mathscr{X}$ is injective coresolving, special preenveloping and there is an integer $n \geq 0$ such that $\mathscr{X}$ is closed under n-images. These classification results for (co)tilting classes and their proof lead us to obtain the next two lemmas, which play an important role in the proof Theorems 33, 34 and 35.

Lemma 26. *Let R be a ring and $\mathscr{X}$ a class of R-modules. Then the following are equivalent:*

(1) *$\mathscr{X}$ is cotilting.*
(2) *$\mathscr{X}$ is definable, projectively resolving and* $\sup\{\mathscr{X}\text{-}\mathrm{pd}_R(M) \mid M \in R\text{-}\mathrm{Mod}\} < \infty$.
(3) *$\mathscr{X}$ is definable, projectively resolving and $\mathscr{X}$-$\mathrm{pd}_R(M) < \infty$ for any R-module M.*

Proof. Let $\mathscr{X}$ be a cotilting or definable, and projectively resolving class of R-modules. Then $\mathscr{X}$ is special precovering by the proof of [9, Theorem 6.1]. In other words, there is a complete and hereditary cotorsion pair $(\mathscr{X},\mathscr{X}^{\perp})$ in this case. Thus, (2) ⇔ (3) holds by Lemma 10. Moreover, combining [9, Theorem 6.1] with the implication (1) ⇔ (3) in Lemma 10, one can obtain (1) ⇔ (2). □

Lemma 27. *Let R be a ring and $\mathscr{X}$ a class of R-modules. Then the following are equivalent:*

(1) *$\mathscr{X}$ is tilting.*
(2) *$\mathscr{X}$ is definable, injectively coresolving, special preenveloping and there is a positive integer n such that $\mathscr{X}$ is closed under n-images.*
(3) *$\mathscr{X}$ is definable, injectively coresolving, special preenveloping and* $\mathscr{X}\text{-id}_R(M)<\infty$ *for any R-module M.*
(4) *$\mathscr{X}$ is definable, injectively coresolving, special preenveloping and* $\sup\{\mathscr{X}\text{-pd}_R(M)\mid M\in R\text{-Mod}\}<\infty$.

Proof. Note from Remak 25(2) that any tilting class of R-modules is always definable. So (1) ⇔ (2) holds by [9, Theorem 6.3]. Now let $\mathscr{X}$ be a class satisfying any one of (2), (3) and (4). Then there is a complete and hereditary cotorsion pair $({}^{\perp}\mathscr{X},\mathscr{X})$. Thus, (3) ⇔ (4) follows from Lemma 12 and (2) ⇔ (3) comes from Lemma 11. □

Remarks 28.

(1) Note that the condition "$\mathscr{X}$ is tilting" (resp. "$\mathscr{X}$ is cotilting") implies the one "$\mathscr{X}\text{-pd}_R(M)<\infty$ for any R-module M" (resp. "$\mathscr{X}\text{-id}_R(M)<\infty$ for any R-module M") was proved in [2, Lemma 2.2(a)] (resp. [2, Lemma 2.2(b)]).
(2) According to [9, Proposition 7.2], we know that "special preenveloping" in (2), (3) and (4) of Lemma 27 can not be omitted.

We know from [40, Proposition 4.13] that the class $\mathscr{GF}(R)$ is definable if and only if it is closed under products, and that these equivalent conditions for R deduces that R is right coherent. The next three results can be viewed as a continuation of such facts.

Lemma 29. *Let R be a ring with* $\text{G-wgldim}(R)<\infty$. *Then the following are equivalent:*

(1) *The class $\mathscr{GF}(R)$ is definable.*
(2) *The class $\mathscr{GF}(R)$ is closed under arbitrary direct products.*
(3) *R is right coherent.*

Proof. Note that the implication (1) ⇔ (2) ⇒ (3) holds by [40, Proposition 4.13].

Let's prove (3) ⇒ (2). Let $\{(G_j)_{j\in J}\}$ be a family of Gorenstein flat R-modules. Then for each $j\in J$, there is an exact sequence of R-modules

$$0\longrightarrow G_j\longrightarrow F_j^0\longrightarrow F_j^1\longrightarrow\cdots$$

with each $F_j^i\in\mathscr{F}(R)$. Hence, one can obtain an exact sequence

$$0\longrightarrow\prod_{j\in J}G_j\longrightarrow\prod_{j\in J}F_j^0\longrightarrow\prod_{j\in J}F_j^1\longrightarrow\cdots$$

of R-modules with each $\prod_{j\in J}F_j^i\in\mathscr{F}(R)$ since R is right coherent. Therefore, $\prod_{j\in J}G_j$ is Gorenstein flat by [45, Theorem 2.9]. □

Lemma 30. *Consider the following conditions for a ring R:*

(1) *The class $\mathscr{GP}(R)$ is definable.*
(2) *The class $\mathscr{GP}(R)$ is closed under arbitrary direct products.*
(3) *R is right coherent and left perfect.*
(4) *R is a right coherent ring such that $\mathscr{GP}(R)=\mathscr{GF}(R)$.*
(5) *The class $\mathscr{GF}(R)$ is definable and $\mathscr{GP}(R)=\mathscr{GF}(R)$.*

Then (5) ⇒ (1) ⇒ (2) ⇒ (3) ⇔ (4) *and* (1)–(5) *are equivalent if* $\text{G-wgldim}(R)<\infty$.

Proof.

(5) ⇒ (1) ⇒ (2). It is clear.

(3) ⇔ (4). It holds by Lemma 17.

(2) ⇒ (3). Suppose that the class $\mathscr{GP}(R)$ is closed under arbitrary direct products. Then of course $\prod_{j\in J} P_j$ is Gorenstein projective for any family $\{(P_j)_{j\in J}\}$ of projective R-modules. Thus, by the definition, there is a short exact sequence of R-modules

$$0 \longrightarrow \prod_{j\in J} P_j \longrightarrow Q \longrightarrow G \longrightarrow 0$$

with Q projective and G Gorenstein projective. Thus $\mathrm{Ext}^1_R(G, \prod_{j\in J} P_j) \cong \prod_{j\in J} \mathrm{Ext}^1_R(G, P_j) = 0$. This induces that the short sequence is split, and so $\prod_{j\in J} P_j$ is projective. It follows that R is right coherent and left perfect.

Now let G-wgldim$(R) < \infty$. In order to see that (1)–(5) are equivalent, it remains to show:

(3) ⇒ (5). For this, we assume that R is right coherent and left perfect. Then $\mathscr{GP}(R) = \mathscr{GF}(R)$ by Lemma 17. Besides, the class $\mathscr{GF}(R)$ is definable due to Lemma 29. □

Let R be a left coherent ring and M an R^{op}-module. Then there is a $\mathrm{Hom}_{R^{\mathrm{op}}}(-,\mathscr{F}(R^{\mathrm{op}}))$-exact complex

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with all $F_i \in \mathscr{F}(R^{\mathrm{op}})$ and a $\mathrm{Hom}_{R^{\mathrm{op}}}(-,\mathscr{F}(R^{\mathrm{op}}))$-exact complex

$$0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with all $F^i \in \mathscr{F}(R^{\mathrm{op}})$ (these complexes are called *left and right $\mathscr{F}(R^{\mathrm{op}})$-resolution* of M respectively, here we need not require the exactness of the complexes). It follows from [26, Definition 8.2.13] that $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$ is *left balanced* by $\mathscr{F}(R^{\mathrm{op}}) \times \mathscr{F}(R^{\mathrm{op}})$. This may construct left derived functors of $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$, denoted by $\mathrm{Ext}_m^{\mathscr{F}(R^{\mathrm{op}})}(-,-)$, which can be computed using a right $\mathscr{F}(R^{\mathrm{op}})$-resolution of the first variable or a left $\mathscr{F}(R^{\mathrm{op}})$-resolution of the second variable.

Proposition 31. *Consider the following conditions for a ring R:*

(1) *The class $\mathscr{GI}(R)$ is definable.*
(2) *The class $\mathscr{GI}(R)$ is closed under pure submodules and pure epimorphic images.*
(3) *The class $\mathscr{GI}(R)$ is closed under arbitrary sums.*
(4) *R is left Noetherian.*
(5) *An R-module $M \in \mathscr{GI}(R)$ if and only if $M^+ \in \mathscr{GF}(R^{\mathrm{op}})$.*
(6) *An R-module M is in $\mathscr{GI}(R)$ if and only if so is M^{++}.*

Then (6) ⇐ (5) ⇒ (2) ⇔ (1) ⇒ (3) ⇒ (4) *and* (1)–(6) *are equivalent if* G-wgldim$(R) < \infty$.

Proof.

(5) ⇒ (2). Assume that an R-module M is in $\mathscr{GI}(R)$ if and only if M^+ is in $\mathscr{GF}(R^{\mathrm{op}})$. To see (2), let $0 \to A \to N \to B \to 0$ be a pure short exact sequence of R-modules with N Gorenstein injective. Then there is a split short exact sequence of R^{op}-modules $0 \to B^+ \to N^+ \to A^+ \to 0$. By the "only if part" of the assumption, N^+ is in $\mathscr{GF}(R^{\mathrm{op}})$. Note from [10, Corollary 2.6] that the class $\mathscr{GF}(R^{\mathrm{op}})$ is closed under any direct summands since any ring is (left and) right GF-closed by [40, Theorem 4.11]. Thus, both A and B are in $\mathscr{GF}(R^{\mathrm{op}})$. It then follows from the "if part" of the assumption that A and B are Gorenstein injective.

(2) ⇔ (1). It holds by Remark 19 since the class $\mathscr{GI}(R)$ is always closed under arbitrary direct products (see [33, Theorem 2.6]).

(1) ⇒ (3). It is clear since any direct sum is a certain direct limits.

(3) ⇒ (4). Suppose that the class $\mathcal{GI}(R)$ is closed under arbitrary direct sums. Then of course $\coprod_{j\in J} I_j$ is Gorenstein injective for any family $\{(I_j)_{j\in J}\}$ of injective R-modules. Thus, by the definition, there is a short exact sequence of R-modules

$$0 \longrightarrow G \longrightarrow H \longrightarrow \coprod_{j\in J} I_j \longrightarrow 0$$

with H injective and G Gorenstein injective. It follows that $\mathrm{Ext}^1_R(\coprod_{j\in J} I_j, G) \cong \prod_{j\in J} \mathrm{Ext}^1_R(I_j, G) = 0$. This gives that the short sequence is split, and so, $\coprod_{j\in J} I_j$ is injective. Thus, R is left Noetherian.

(5) ⇒ (6). Assume that an R-module M is in $\mathcal{GI}(R)$ if and only if M^+ is in $\mathcal{GF}(R^{\mathrm{op}})$. Then R is left Noetherian by what we have proved (i.e. (5) ⇒ (2) ⇔ (1) ⇒ (3) ⇒ (4)). In particular, R is left coherent. So, for any R-module M, one has $M \in \mathcal{GI}(R)$ if and only if $M^{++} \in \mathcal{GI}(R)$ since $M^{++} \in \mathcal{GI}(R) \Leftrightarrow M^+ \in \mathcal{GF}(R^{\mathrm{op}})$ by [33, Theorem 3.6].

Now let $\text{G-wgldim}(R) < \infty$. In order to prove that (1)–(6) are equivalent, it remains to show the implications (4) ⇒ (5) and (6) ⇒ (5).

(6) ⇒ (5). Suppose that R is a ring over which any R-module M is in $\mathcal{GI}(R)$ if and only if so is M^{++}. Since R is a right GF-closed ring with $\text{G-wgldim}(R) = \text{G-wgldim}(R^{\mathrm{op}}) < \infty$, [12, Theorem 4] yields that, for any R-module M, $M^{++} \in \mathcal{GI}(R)$ if and only if $M^+ \in \mathcal{GF}(R^{\mathrm{op}})$. Now the result follows.

(4) ⇒ (5). Suppose that R is left Noetherian. For any $M \in \mathcal{GI}(R)$, there is an exact sequence of R-modules

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow M \longrightarrow 0$$

with each I_i injective. This yields another exact sequence of R^{op}-modules

$$0 \longrightarrow M^+ \longrightarrow (I_0)^+ \longrightarrow (I_1)^+ \longrightarrow \cdots$$

with all $(I_i)^+$ flat since R is left Noetherian. Thus, M^+ is Gorenstein flat by [45, Theorem 2.9] and the assumption $\text{G-wgldim}(R) < \infty$. Conversely, let $M^+ \in \mathcal{GF}(R^{\mathrm{op}})$. Note that R is left Noetherian. By [33, Theorem 3.6] there is a $\mathrm{Hom}_{R^{\mathrm{op}}}(-,\mathcal{F}(R^{\mathrm{op}}))$-exact exact sequence of R^{op}-modules

$$\mathbf{F} = 0 \longrightarrow M^+ \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with all F^i flat. Since $\mathcal{I}(R)$ is covering, there exists a $\mathrm{Hom}_R(\mathcal{I}(R),-)$-exact complex of R-modules

$$\mathbf{E} = \cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow M \longrightarrow 0$$

with all I_i injective. This enables us to obtain a $\mathrm{Hom}_{R^{\mathrm{op}}}(-,\mathcal{F}(R^{\mathrm{op}}))$-exact complex of R^{op}-modules

$$\mathbf{E}^+ = 0 \longrightarrow M^+ \longrightarrow (I_0)^+ \longrightarrow (I_1)^+ \longrightarrow \cdots$$

with all $(I_i)^+$ flat. Using the facts $\mathrm{Ext}^{\mathcal{F}(R^{\mathrm{op}})}_{i\geq 1}(R_R, M^+) = 0$ and $\mathrm{Ext}^{\mathcal{F}(R^{\mathrm{op}})}_{0}(R_R, M^+) \cong M^+$ which are guaranteed by the exact sequence $\mathbf{F}$, we get that the complex $\mathbf{E}^+$ is exact, and then so is $\mathbf{E}$. Thus, $M \in \mathcal{GI}(R)$ by [45, Theorem 2.6] and the assumption $\text{G-wgldim}(R) < \infty$. □

Remark 32. Recall from Li, Wang, Geng and Hu [36] that a ring R is *left Gorenstein hereditary* if $\text{G-gldim}(R) \leq 1$ (so $\text{G-wgldim}(R) \leq 1$ by Lemma 16(1)). According to [36, Theorem 1.2], we know that a left Gorenstein hereditary ring is left Noetherian if and only if an R-module M is Gorenstein injective if and only M^+ is in $\mathcal{GF}(R^{\mathrm{op}})$. Note that this result is a special case of Proposition 31.

We are now in a position to give our main theorems below, which answer Questions 1 and 2 (from the introduction) thoroughly.

Theorem 33. *The following are equivalent for any ring R:*

1. *The class $\mathcal{GF}(R)$ is cotilting.*
2. *R is a right coherent ring with* $\text{G-wgldim}(R) < \infty$.
3. *R is a right coherent ring such that* $\text{Gfd}_R(M) < \infty$ *for any R-module M.*

Proof.

(1) ⇒ (2). Assume that $\mathcal{GF}(R)$ is cotilting. Then by Lemma 26, $\mathcal{GF}(R)$ is definable and admits $\text{G-wgldim}(R) = \sup\{\text{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\} < \infty$. Furthermore, R is right coherent by Lemma 29.

(2) ⇒ (1). Suppose that R is a right coherent ring with $\text{G-wgldim}(R) < \infty$. Then $\mathcal{GF}(R)$ is definable by Lemma 29. Note that $\mathcal{GF}(R)$ is always projectively resolving via [40, Corollary 4.12]. Thus, Lemma 26 yields that $\mathcal{GF}(R)$ is cotilting.

(2) ⇔ (3). According to [40, Corollary 4.12], we know that $(\mathcal{GF}(R), \mathcal{GF}(R)^{\perp})$ is a complete and hereditary cotorsion pair. Thus the result comes from Lemma 10. □

Theorem 34. *The following are equivalent for any ring R:*

1. *The class $\mathcal{GP}(R)$ is cotilting.*
2. *R is a right coherent and left perfect ring with* $\text{G-gldim}(R) < \infty$.
3. *The class $\mathcal{GF}(R)$ is cotilting and $\mathcal{GP}(R) = \mathcal{GF}(R)$.*
4. *R is a right coherent and left perfect ring such that* $\text{Gpd}_R(M) < \infty$ *for any R-module M.*

Proof.

(3) ⇒ (1). It is trivial.

(1) ⇒ (2). Assume that $\mathcal{GP}(R)$ is cotilting. Then by Lemma 26, $\mathcal{GP}(R)$ is definable and admits $\text{G-gldim}(R) = \sup\{\text{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\} < \infty$, and so $\text{G-wgldim}(R) \le \text{G-gldim}(R) < \infty$ by Lemma 16(1). At the same time, R is right coherent and left perfect by Lemma 30.

(2) ⇒ (3). Suppose that R is a right coherent and left perfect ring with $\text{G-gldim}(R) < \infty$. This happens if and only if R is a right coherent and left perfect ring with $\text{G-wgldim}(R) < \infty$ by Lemma 16(2). So $\mathcal{GF}(R)$ is cotilting due to Theorem 33. Moreover, the equality $\mathcal{GP}(R) = \mathcal{GF}(R)$ holds by Lemma 17.

(2) ⇔ (4). Suppose that R is a right coherent and left perfect ring. Then $\mathcal{GP}(R) = \mathcal{GF}(R)$ by Lemma 17. Now [40, Corollary 4.12] yields that $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp})$ is a complete and hereditary cotorsion pair. Hence the result follows from Lemma 10. □

Theorem 35. *The following are equivalent for any ring R:*

1. *The class $\mathcal{GI}(R)$ is tilting.*
2. *R is left Noetherian ring with* $\text{G-gldim}(R) < \infty$.
3. *The class $\mathcal{GF}(R^{\text{op}})$ is a cotilting class and an R-module M is in $\mathcal{GI}(R)$ if and only if so is M^{++}.*
4. *R is a left Noetherian ring such that* $\text{Gid}_R(M) < \infty$ *for any R-module M.*

Proof.

(1) ⇒ (2). Assume that $\mathcal{GI}(R)$ is tilting. Then by Lemma 27, $\mathcal{GI}(R)$ is definable and $\text{G-gldim}(R) = \sup\{\text{Gid}_R(M) \mid M \text{ is an } R\text{-module}\} < \infty$, which deduces that $\text{G-wgldim}(R) < \infty$ by Lemma 16(1). Note that R is also left Noetherian by Proposition 31.

(2) ⇒ (3). Suppose that R is a left Noetherian ring with G-gldim$(R) < \infty$. This happens if and only if R is a left Noetherian ring with G-wgldim(R) = G-wgldim$(R^{\mathrm{op}}) < \infty$ by [12, Theorem 7]. It follows from Theorem 33 that $\mathcal{GF}(R^{\mathrm{op}})$ is cotilting. Furthermore, the other assertion in (3) holds by Proposition 31.

(3) ⇒ (1). Suppose that the following two conditions hold:

(I) The class $\mathcal{GF}(R^{\mathrm{op}})$ is cotilting, and
(II) An R-module M is in $\mathcal{GI}(R)$ if and only if M^{++} is in $\mathcal{GI}(R)$.

By Theorem 33, the condition (I) yields that R is left coherent ring with G-wgldim(R) = G-wgldim $(R^{\mathrm{op}}) < \infty$. Then according to Proposition 31, the condition (II) yields that R is also left Noetherian. Whence, $\mathcal{GI}(R)$ is definable again by Proposition 31. Notice further that $\mathcal{GI}(R)$ is always special preenveloping and injectively coresolving by [40, Theorem 5.6], it is tilting due to Lemma 27.

(2) ⇔ (4). It holds by Lemma 12 since $({}^{\perp}\mathcal{GI}(R), \mathcal{GI}(R))$ is a complete and hereditary cotorsion pair (see[40, Theorem 5.6]). □

Remarks 36. Let $n \geq 0$ be an integer. By the proofs in Theorems 33, 34 and 35, one see that

(1) The class $\mathcal{GF}(R)$ is n-cotilting if and only if R is a right coherent ring with G-wgldim$(R) \leq n$.
(2) The class $\mathcal{GP}(R)$ is n-cotilting if and only if R is a right coherent and left perfect ring with G-gldim$(R) \leq n$.
(3) The class $\mathcal{GI}(R)$ is n-tilting if and only if R is a left Noetherian ring with G-gldim$(R) \leq n$ if and only if the class $\mathcal{GF}(R^{\mathrm{op}})$ is n-cotilting and an R-module is in $\mathcal{GI}(R)$ if and only if so is M^{++}.

4. Applications

This section is divided into four subsections, by which some applications of Theorems 33, 34 and 35 are given.

4.1. *Characterizations of Gorenstein rings and Ding–Chen rings*

Recall that a ring R is *Gorenstein* [34] (resp. *Ding–Chen* [22, 29]) if R is an n-Gorenstein ring (resp. n-FC ring) for some nonnegative integer n, i.e., R is a two-sided Noetherian (resp. two-sided coherent) ring with self-injective (resp. self-FP-injective) dimension at most n on both sides. In particular, 0-Gorenstein ring and 0-FC ring is just *QF ring* and *FC ring*, respectively. Recall that an Artin algebra R is *Gorenstein* if it is Gorenstein as a ring. In particular, 0-Gorenstein Artin algebra is exactly *self-injective Artin algebra*.

As the first application of Theorems 33, 34 and 35, this subsection is devoted to give some new characterizations for Gorenstein rings (including Gorenstein Artin algebras) and Ding–Chen rings. Firstly, we characterize Gorenstein rings via the class $\mathcal{GI}(-)$ being tilting.

Theorem 37. *The following are equivalent for any ring R:*

(1) *R is Gorenstein.*
(2) *R is a right Noetherian ring such that the class $\mathcal{GI}(R)$ is tilting.*
(3) *R is a left Noetherian ring such that the class $\mathcal{GI}(R^{\mathrm{op}})$ is tilting.*
(4) *Both the classes $\mathcal{GI}(R)$ and $\mathcal{GI}(R^{\mathrm{op}})$ are tilting.*

In particular, a commutative ring R is Gorenstein if and only if the class $\mathcal{GI}(R)$ is tilting.

Proof. According to Theorem 35, we know that the condition (2) (resp. (3)) happens if and and only if R is a two-sided Noetherian ring with G-gldim$(R) < \infty$ (resp. G-gldim$(R^{\text{op}}) < \infty$), and that the condition (4) happens if and and only if R is a two-sided Noetherian ring with G-gldim$(R) < \infty$ and G-gldim$(R^{\text{op}}) < \infty$. Thus, the implication (1) ⇔ (2) (resp. (1) ⇔ (3) and (1) ⇔ (4)) follows from [17, Remark 3.11].

The last statement is an immediate consequence of the implication (1) ⇔ (4). □

Corollary 38. *Let R be a two-sided Noetherian ring. Then the following are equivalent:*

(1) *R is Gorenstein.*
(2) *Any one of the classes $\mathcal{GI}(R)$ and $\mathcal{GI}(R^{\text{op}})$ is tilting.*
(3) *Any one of the classes $\mathcal{GF}(R)$ and $\mathcal{GF}(R^{\text{op}})$ is cotilting.*

Proof.

(1) ⇔ (2). It holds by the proof of (1) ⇔ (4) in Theorem 37.

(1) ⇔ (3). Using the assumption and Theorem 33, one has the condition (3) happens if and and only if G-wgldim$(R) < \infty$ or G-wgldim$(R^{\text{op}}) < \infty$. Thus, the results follows by [17, Remark 3.11] (see also [45, Lemma 1.2]). □

As mentioned in the introduction, Angeleri Hügel, Herbera and Trlifaj in [3, Theorem 3.4] proved that a two-sided Noetherian ring R is Gorenstein if and only if both the classes $\mathcal{GI}(R)$ and $\mathcal{GI}(R^{\text{op}})$ are tilting, equivalently, if and only if the class $\mathcal{GI}(R)$ is tilting and the class $\mathcal{GF}(R)$ is cotilting. Note that both Corollary 38 and the equivalence of (1) ⇔ (4) in Theorem 37 provide a slight improvement of [3, Theorem 3.4]. On the other hand, the coming example shows that a general ring R satisfying that the class $\mathcal{GI}(R)$ is tilting and the class $\mathcal{GF}(R)$ is cotilting may not be Gorenstein.

Example 39. Let $S = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. Then by Small [41], S is right Noetherian but not left Noetherian ring with gldim$(S) = 1$ and gldim$(S^{\text{op}}) = 2$. Now we consider the ring $R = S^{\text{op}}$. It is then seen that R is left Noetherian but not right Noetherian ring with gldim$(R) = 2$ and gldim$(R^{\text{op}}) = 1$. Note that gldim$(R^{\text{op}}) = 1$ shows that R is right hereditary, and hence right coherent. Thus, R is left Noetherian and right coherent ring with G-gldim$(R) \le$ gldim$(R) = 2$ and G-wgldim$(R) \le$ wgldim$(R) \le$ gldim$(R^{\text{op}}) = 1$. It follows from Theorems 33 and 35 that the class $\mathcal{GI}(R)$ is tilting and the class $\mathcal{GF}(R)$ is cotilting. However, R is not Gorenstein since it is not right Noetherian.

Secondly, we characterize Ding–Chen rings via the class $\mathcal{GF}(-)$ being cotilting.

Theorem 40. *Let R be a ring. Then the following are equivalent:*

(1) *R is Ding–Chen.*
(2) *R is a left coherent ring such that the class $\mathcal{GF}(R)$ is cotilting.*
(3) *R is a right coherent ring such that the class $\mathcal{GF}(R^{\text{op}})$ is cotilting.*
(4) *Both the classes $\mathcal{GF}(R)$ and $\mathcal{GF}(R^{\text{op}})$ are cotilting.*

In particular, a commutative ring R is Gorenstein if and only if the class $\mathcal{GF}(R)$ is cotilting.

Proof. According to Theorem 33, we know that the condition(2) (resp. (3)) happens if and and only if R is a two-sided coherent ring with G-wgldim$(R) < \infty$ (resp. G-wgldim$(R^{\text{op}}) < \infty$), and that the condition (4) happens if and and only if R is a two-sided coherent ring with G-wgldim$(R) < \infty$ and G-wgldim$(R^{\text{op}}) < \infty$. Thus, the implication (1) ⇔ (2) (resp. (1) ⇔ (3) and (1) ⇔ (4)) follows from [17, Remark 3.11].

The last statement is an immediate consequence of the implication (1) ⇔ (4). □

Note that the proof of Theorem 40 implies that

Corollary 41. *Let R be a two-sided coherent ring. Then the following are equivalent:*

(1) *R is Ding–Chen.*
(2) *Any one of the classes $\mathscr{GF}(R)$ and $\mathscr{GF}(R^{\mathrm{op}})$ is cotilting.*

Finally, we characterize Gorenstein Artin algebras via the class $\mathscr{GP}(-)$ being cotilting.

Theorem 42. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is a Gorenstein Artin algebra.*
(2) *The class $\mathscr{GP}(R)$ is cotilting.*
(3) *The class $\mathscr{GF}(R)$ is cotilting and $\mathscr{GF}(R) = \mathscr{GP}(R)$.*
(4) *The class $\mathscr{GI}(R)$ forms a tilting class such that an R-module M is in $\mathscr{GP}(R)$ if and only if M^+ is in $\mathscr{GI}(R)$.*

Proof.

(3) ⇒ (2). It is clear.

(2) ⇒ (1). Suppose that the class $\mathscr{GP}(R)$ is cotilting. Then by Theorem 34, this happens if and only if R is a right coherent and left perfect ring with G-gldim$(R) < \infty$. Note that the right coherence and left perfectness of R imply that the class $\mathscr{P}(R)$ is closed under direct products. It follows from [15, Theorem 3.4] that R is Artinian since R is commutative, and hence, R is an Artin algebra. Note from [17, Remark 3.11] that any Artin algebra with finite Gorenstein weak global dimension is Gorenstein. In particular, R is a Gorenstein Artin algebra.

(1) ⇒ (4). Assume that R is a Gorenstein Artin algebra. Then R is a two-sided Artinian (hence a two-sided Noetherian) ring with G-gldim$(R) < \infty$. It follows from Theorem 35 that the class $\mathscr{GI}(R)$ is tilting. Meanwhile, as R is right coherent and left perfect, one has $\mathscr{GF}(R) = \mathscr{GP}(R)$ by Lemma 17. Consequently, [33, Theorem 3.6] yields that an R-module M is in $\mathscr{GP}(R)$ if and only if M^+ is in $\mathscr{GI}(R^{\mathrm{op}}) = \mathscr{GI}(R)$ (as R is commutative).

(4) ⇒ (3). Suppose that the following two conditions hold:

(I) The class $\mathscr{GI}(R)$ is tilting, and
(II) An R-module is in $\mathscr{GP}(R)$ if and only if M^+ is in $\mathscr{GI}(R)$.

Since R is commutative, it follows from Theorems 35 and 33 that the condition (I) induces that the class $\mathscr{GF}(R) = \mathscr{GF}(R^{\mathrm{op}})$ is cotilting, and that R is (right) coherent. Thus, combining [33, Theorem 3.6] with the condition (II), one has $\mathscr{GF}(R) = \mathscr{GP}(R)$. This completes the proof. □

Let $n \geq 0$ be an integer and R a perfect and n-FC ring. If $n = 0$, then R is 0-Gorenstein by [21, Corollary 3.7]. Otherwise, for $n \geq 1$, let $R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix}$. Then [43, Example 3.4] showed that R is a perfect and hereditary (hence perfect and 1-FC) ring which is not n-Gorenstein for any $n \geq 0$. We note that such a ring R is not commutative.

The following corollary shows that any commutative perfect and n-FC rings are always Gorenstein.

Corollary 43. *Let R be a commutative ring. Then the following are equivalent:*

(1) *R is a Gorenstein Artin algebra.*
(2) *R is Ding–Chen and perfect.*

Proof.

(1) ⇒ (2). It is obvious.

(2) ⇒ (1). Assume that R is Ding–Chen and perfect. Note that R is commutative, it suffices to show that the class $\mathscr{GF}(R)$ is cotilting and $\mathscr{GF}(R) = \mathscr{GP}(R)$ by Theorem 42. On one hand, the class $\mathscr{GF}(R)$ is cotilting by Theorem 40 since R is Ding–Chen. On the other hand, the equality $\mathscr{GF}(R) = \mathscr{GP}(R)$ holds by Lemma 17 as R is right coherent and left perfect. Thus, the result follows. □

We end this subsection with some remarks.

Remark 44. Let $n \geq 0$ be an integer. By the proofs in Theorems 37, 40 and 42, as well as Corollaries 38 and 41, one can see that

(1) A ring R is n-Gorenstein if and only if both the classes $\mathscr{GI}(R)$ and $\mathscr{GI}(R^{\mathrm{op}})$ are n-tilting. In particular, a commutative (or two-sided Noetherian) ring R is n-Gorenstein if and only if the class $\mathscr{GI}(R)$ is n-tilting.
(2) A ring R is n-FC if and only if both the classes $\mathscr{GF}(R)$ and $\mathscr{GF}(R^{\mathrm{op}})$ are n-cotilting. In particular, a commutative (or two-sided coherent) ring R is n-FC if and only if the class $\mathscr{GF}(R)$ is n-cotilting.
(3) A commutative ring R is an n-Gorenstein Artin algebra if and only if the class $\mathscr{GP}(R)$ is n-cotilting.

4.2. *Characterizations of Gorenstein modules via finitely generated modules*

It is well-known that injective (resp. flat) modules can be characterized via finitely generated modules by vanishing of the functor Ext (resp. Tor). As the second application of Theorems 33, 34 and 35, we will obtain a Gorenstein version of the characterizations (see Theorem 45 and Lemma 47). As a result, we prove that left Noetherian rings with finite left Gorenstein global dimension satisfy First Finitistic Dimension Conjecture (see Corollary 46), and prove a result related to a question posed by Bazzoni [9, Question 1(1)] (see Theorem 48).

Let us firstly consider the characterizations of Gorenstein injective modules via finitely generated modules by vanishing of the functor Ext. Let R be a Gorenstein ring. Then Enochs and Jenda [27, Theorem 2.5] proved that an R-module M is Gorenstein injective if and only if $\mathrm{Ext}^i_R(L, M) = 0$ for all $i > 0$ and all countably generated R-modules L with $\mathrm{pd}_R(L) < \infty$. [3, Corollary 3.5(1)] tells us that this characterization of Gorenstein injective modules can be relaxed as "an R-module M is Gorenstein injective if and only if $\mathrm{Ext}^i_R(L, M) = 0$ for all $i > 0$ and all finitely generated R-modules L with $\mathrm{pd}_R(L) < \infty$".

In what follows, we denote by $\widehat{\mathscr{P}}$ the class consisting of all R-modules with finite projective dimensions. By the proof of [32, Corollary 7.1.13(a)] and by noting from Chen [16, Lemma 5.1] that there is a hereditary and complete cotorsion pair $(\widehat{\mathscr{P}}, \mathscr{GI}(R))$ whenever R is of $\mathrm{G\text{-}gldim}(R) < \infty$, one can see that, to obtain the above characterization, the Gorenstein condition can be relaxed to "left Noetherian rings with finite left Gorenstein global dimension". The added value of the next result is to show that, in order to obtain the characterization "for all R-module M, M is Gorenstein injective if and only if $\mathrm{Ext}^i_R(L, M) = 0$ for all $i > 0$ and all finitely generated R-modules L with $\mathrm{pd}_R(L) < \infty$", the ring R must be left Noetherian rings with finite left Gorenstein global dimension.

Theorem 45. *The following are equivalent for any ring R:*

(1) *R is a left Noetherian ring with* $\mathrm{G\text{-}gldim}(R) < \infty$.
(2) *An R-module M is Gorenstein injective if and only if* $\mathrm{Ext}^i_R(L, M) = 0$ *for all $i > 0$ and all R-modules L of type FP_∞ with* $\mathrm{pd}_R(L) < \infty$.

(3) *An R-module M is Gorenstein injective if and only if* $\mathrm{Ext}^i_R(L,M)=0$ *for all* $i>0$ *and all finitely generated R-modules L with* $\mathrm{pd}_R(L)<\infty$.

(4) *An R-module M is Gorenstein injective if and only if* $\mathrm{Ext}^i_R(L,M)=0$ *for all* $i>0$ *and all finitely presented R-modules L with* $\mathrm{pd}_R(L)<\infty$.

Furthermore, if any one of the above conditions is satisfied, then the cotorsion pair $(\widehat{\mathscr{P}},\mathscr{GI}(R))$ *is of strongly finite type.*

Proof. For any ring R, note from [40, Theorem 5.6] that $({}^{\perp}\mathscr{GI}(R),\mathscr{GI}(R))$ forms a hereditary and complete cotorsion pair. So, the condition (1) is equivalent to that "the complete and hereditary cotorsion pair $({}^{\perp}\mathscr{GI}(R),\mathscr{GI}(R))$ is of strongly finite type". It follows from Remark 25(3) that the condition (2) is further equivalent to that "the class $\mathscr{GI}(R)$ is tilting". Thus, (1) ⇔ (2) holds by Theorem 35.

On the other hand, one concludes from [19, Lemma 9.2.7] that $\mathrm{Ext}^{i\geq 1}_R(L,G)=0$ for all Gorenstein injective R-modules G and all R-modules L with $\mathrm{pd}_R(L)<\infty$. Meanwhile, it is trivial that any modules of type FP_∞ is finite presented and that any finite presented modules are finitely generated. Whence, one has (2) ⇔ (3) ⇔ (4).

Now suppose that R is a left Noetherian ring with $\text{G-gldim}(R)<\infty$. Then the proof of (1) ⇔ (2) above shows that the complete and hereditary cotorsion pair $({}^{\perp}\mathscr{GI}(R),\mathscr{GI}(R))$ is of strongly finite type. But [16, Lemma 5.1] tells us that ${}^{\perp}\mathscr{GI}(R)=\widehat{\mathscr{P}}$. □

In what follows, for any class $\mathscr{X}$ of R-modules, we denote by $\mathscr{X}^{<\omega}$ the subclass of $\mathscr{X}$ consisting of all modules of type FP_∞.

Recall that for any ring R, the *big finitistic dimension* of R is defined as

$$\mathrm{FPD}(R)=\sup\{\mathrm{pd}_R(M)\mid M \text{ is an } R\text{-module with } \mathrm{pd}_R(M)<\infty\},$$

and the *little finitistic dimension* of R is defined as

$$\mathrm{fpd}(R)=\sup\{\mathrm{pd}_R(M)\mid M \text{ is a finitely generated } R\text{-module with } \mathrm{pd}_R(M)<\infty\}.$$

For a ring R, "First Finitistic Dimension Conjecture" and "Second Finitistic Dimension Conjecture" calim $\mathrm{FPD}(R)=\mathrm{fpd}(R)$ and $\mathrm{fpd}(R)<\infty$ respectively. It is a famous result that "First Finitistic Dimension Conjecture" and "Second Finitistic Dimension Conjecture" vanish for Gorenstein rings (see [3, Theorem 3.2] and [32, Theorem 7.1.12]). The next corollary shows that the Gorenstein condition can be relaxed to "left Noetherian rings with finite left Gorenstein global dimension" and to "rings with finite left Gorenstein global dimension", respectively. Note that a ring R is Gorenstein if and only if R is a two-sided Noetherian ring with $\text{G-gldim}(R)<\infty$ (see [17, Remark 3.11]); see Example 39 for the existence of a left Noetherian ring R with $\text{G-gldim}(R)<\infty$ which is not Gorenstein.

Corollary 46. *Let R be a ring with* $\text{G-gldim}(R)<\infty$. *Then "Second Finitistic Dimension Conjecture" vanishes. If R is also left Noetherian, then "First Finitistic Dimension Conjecture" vanishes as well.*

Proof. Let R be a left Noetherian ring with $\text{G-gldim}(R)<\infty$ and $\widehat{\mathscr{P}}$ be as above. Then by Theorem 45, the pair $(\widehat{\mathscr{P}},\mathscr{GI}(R))$ is of strongly finite type. It follows that $\widehat{\mathscr{P}}={}^{\perp}((\widehat{\mathscr{P}}^{<\omega})^{\perp})$. Thus, [32, Corollary 3.2.4] yields that every R-module L with $\mathrm{pd}_R(L)<\infty$ (i.e., $L\in\widehat{\mathscr{P}}$) is a summand of some R-module in $\widehat{\mathscr{P}}^{<\omega}$. Note that $\widehat{\mathscr{P}}^{<\omega}$ is just the class of all finitely generated R-modules with finite projective dimension since R is left Noetherian. Hence, $\mathrm{FPD}(R)\leq\mathrm{fpd}(R)$ and so $\mathrm{FPD}(R)=\mathrm{fpd}(R)$. □

Let R be a left Noetherian ring with $\text{G-gldim}(R)<\infty$. By Theorem 45, Gorenstein injective R-modules can be characterized via finitely generated modules by vanishing of the functor Ext. Now we consider the behaviors of Gorenstein projective and Gorenstein flat modules.

Lemma 47. *Let R be a left Noetherian ring with* G-gldim$(R^{\mathrm{op}}) < \infty$. *Then the following are equivalent for any R^{op}-module N:*

(1) *N is Gorenstein flat.*
(2) $\mathrm{Tor}_i^R(N, L) = 0$ *for all $i > 0$ and all finite generated R-modules L with* $\mathrm{pd}_R(L) < \infty$.
(3) $\mathrm{Ext}_R^i(L, N^+) = 0$ *for all $i > 0$ and all finite generated R-modules L with* $\mathrm{pd}_R(L) < \infty$.
(4) $\mathrm{Tor}_i^R(N^{++}, L) = 0$ *for all $i > 0$ and all finite generated R-modules L with* $\mathrm{pd}_R(L) < \infty$.

Furthermore, if R is also a left Noetherian and right perfect ring with G-gldim$(R^{\mathrm{op}}) < \infty$*, then the above conditions are equivalent to*

(5) *M is Gorenstein projective.*

Proof.

(1) ⇔ (3). Since R is left Notherian, one has an R^{op}-module N is Gorenstein flat if and only if N^+ is Gorenstein injective by [33, Theorem 3.6]. So the result holds by Theorem 45 as G-gldim$(R) < \infty$.

(2) ⇔ (3). It holds by the isomorphism $\mathrm{Tor}_i^R(N, L)^+ \cong \mathrm{Ext}_R^i(L, N^+)$ and the faithful property of the functor $(-,-)^+$.

(3) ⇔ (4). It follows from the isomorphism $\mathrm{Ext}_R^i(L, N^+)^+ \cong \mathrm{Tor}_i^R(N^{++}, L)$ (since R is left Noetherian) and the faithful property of the functor $(-,-)^+$.

(1) ⇔ (5). Suppose that R is a left Noetherian and left perfect ring with G-gldim$(R) < \infty$. Then the equivalence follows from Lemma 17. □

Note that there are dual notions of that classes of modules are of (strongly) finite type. Recall that a class $\mathscr{X}$ of R^{op}-modules is of *cofinite type* (resp. *strongly cofinite type*) if there exists a set $\mathscr{S}$ consists of R^{op}-modules of type FP_∞ (resp. with finite projective dimension) such that $\mathscr{X} = \{M \in R^{\mathrm{op}}\text{-}\mathrm{Mod} \mid \mathrm{Tor}_{i\geq 1}^R(M, S) = 0, \forall S \in \mathscr{S}\}$ (we refer to the readers that Bazzoni, Göbel and Trlifaj in [9, 32] called that a class $\mathscr{X}$ is "of cofinite type", is just of strongly cofinite type in our sense). Let $\mathscr{X}$ be a class of R^{op}-modules which is of strongly cofinite type. Then according to [32, Definition 8.1.11 and Proposition 8.1.12], $\mathscr{X}$ is always cotilting. However, there exists a cotilting class which is not of strongly cofinite type (see [32, Example 8.2.13]). Furthermore, it is an open question whether left Noetherian rings admits cotilting classes of right modules which are not of strongly cofinite type (see [9, Question 1(1)]). The next result shows that such a question has an affirmative answer in the Gorenstein homological algebra.

Theorem 48. *Let R be a left Noetherian ring. Then the following are equivalent:*

(1) *The class $\mathscr{GF}(R^{\mathrm{op}})$ (resp. $\mathscr{GP}(R^{\mathrm{op}})$) is cotilting.*
(2) *The class $\mathscr{GF}(R^{\mathrm{op}})$ (resp. $\mathscr{GP}(R^{\mathrm{op}})$) is of strongly cofinite type.*

Proof.

(2) ⇒ (1). It holds by [32, Definition 8.1.11 and Proposition 8.1.12].

(1) ⇒ (2). Suppose that the class $\mathscr{GF}(R^{\mathrm{op}})$ (resp. $\mathscr{GP}(R^{\mathrm{op}})$) is cotilting. Then, in view of Theorem 33 (resp. 34), this will happen if and only if R is a ring with G-wgldim$(R^{\mathrm{op}}) < \infty$ (resp. a right perfect ring with G-gldim$(R^{\mathrm{op}}) < \infty$) as R is left Noetherian. Notice further that G-gldim(R^{op}) = G-wgldim(R^{op}) by [12, Theorem 7], and that an R-module is of type FP_∞ if and only if it is finitely generated, again since R is left Noetherian. Thus, the result follows from Lemma 47. □

By virtue of [3, Theorem 2.2] (or [32, Theorems 5.2.23 and 8.1.14]), we know that, over any ring R, there is a bijective correspondence between tilting classes of R-modules (resp. class of R^{op}-modules of strongly cofinite type), and resolving subcategories $\mathscr{S}$ consisting of those R-modules of type FP_∞ with finite projective dimension. The correspondence is given by the mutually inverse assignments

$$\mathscr{X} \longmapsto ({}^{\perp}\mathscr{X})^{<\omega} \text{ and } \mathscr{S} \longmapsto \mathscr{S}^{\perp} \ (\textit{resp.} \quad \mathscr{X} \longmapsto ({}^{\top}\mathscr{X})^{<\omega} \text{ and } \mathscr{S} \longmapsto \mathscr{S}^{\top}).$$

Here $\mathscr{S}^{\top} = \{M \in R^{\text{op}}\text{-Mod} \mid \text{Tor}^R_{i\geq 1}(M,S) = 0, \forall S \in \mathscr{S}\}$ and ${}^{\top}\mathscr{X}$ is defined by dually.

Let $(\mathscr{X},\mathscr{Y})$ be a complete and hereditary cotorsion pair of R-modules. Recall that $(\mathscr{X},\mathscr{Y})$ is a *projective cotorsion pair* (resp. an *injective cotorsion pair*) if $(\mathscr{X} \cap \mathscr{Y}) = \mathscr{P}(R)$ (resp. $(\mathscr{X} \cap \mathscr{Y}) = \mathscr{I}(R)$). Gillespie in [30] studied the lattices of projective and injective cotorsion pairs respectively. According to [30, Theorems 5.2 and 5.4] we know that $\mathscr{X} \subseteq \mathscr{GP}(R)$ (resp. $\mathscr{Y} \subseteq \mathscr{GI}(R)$) whenever $(\mathscr{X},\mathscr{Y})$ is a projective cotorsion pair (resp. an injective cotorsion pair). In other words, in the lattices of projective (resp. injective) cotorsion pairs, the one induced by Gorenstein projective (resp. Gorenstein injective) modules is a maximal element.

Motivated by Gillespie's results, it is natural to consider which role does the class of Gorenstein projective (resp. Gorenstein injective) modules play in the collections of tilting (resp. cotilting) classes. We end the subsection by the next result, building from the above facts in [3, Theorem 2.2] (or [32, Theorems 5.2.23 and 8.1.14]), which shows that, under some certain conditions, in the lattice of tilting (resp. cotilting) classes, the class of Gorenstein projective (resp. Gorenstein injective) modules is a minimal element.

Proposition 49. *Let R be a ring.*

(1) *If the class $\mathscr{GI}(R)$ is tilting, then it is the smallest tilting class, in the sense of that $\mathscr{GI}(R) \subseteq \mathscr{X}$ for all tilting class of R-modules.*
(2) *If the class $\mathscr{GF}(R^{\text{op}})$ (resp. $\mathscr{GP}(R^{\text{op}})$) is cotilting over a left Noetherian ring R, then it is the smallest class of strongly cofinite type in the similar sense.*

Proof.

(1). By Theorem 35, the class $\mathscr{GI}(R)$ is tilting if and only if R is a left Noetherian ring with $\text{G-gldim}(R) < \infty$. Then the result holds by [32, Theorems 5.2.23] and Theorem 45.

(2). Suppose that R is left Noetherian. By Theorem 48 and its proof, one has the class $\mathscr{GF}(R^{\text{op}})$ (resp. $\mathscr{GP}(R^{\text{op}})$) is cotilting if and only if the class $\mathscr{GF}(R^{\text{op}})$ (resp. $\mathscr{GP}(R^{\text{op}})$) is of strongly cofinite type, or equivalently, if and only if R is a (resp. right perfect) ring with $\text{G-gldim}(R) < \infty$. Then the result is an immediate consequence of [32, Theorems 8.1.14] and Lemma 47. □

4.3. *(Co)tilting property for the classes of classical homological modules*

In this subsection, as the third application of Theorems 33, 34 and 35, we will consider when the classes $\mathscr{P}(R)$ and $\mathscr{F}(R)$ are cotilting and when the class $\mathscr{I}(R)$ is tilting as follows.

Proposition 50. *Let R be a ring. Then the following are equivalent:*

(1) *The class $\mathscr{F}(R)$ is cotilting.*
(2) *The class $\mathscr{GF}(R)$ is cotilting and $\mathscr{F}(R) = \mathscr{GF}(R)$.*
(3) *R is a right coherent ring with* $\text{wgldim}(R) < \infty$.
(4) *R is a right coherent ring such that $\text{fd}_R(M) < \infty$ for any R-module M.*

Proof.

(2) ⇒ (1). It is trivial.

(1) ⇒ (3). Assume that the class $\mathscr{F}(R)$ is cotilting. Then by Lemma 26, $\mathscr{F}(R)$ is definable and $\mathrm{wgldim}(R) = \sup\{\mathrm{fd}_R(M) \mid M \text{ is an } R\text{-module}\} < \infty$. In particular, $\mathscr{F}(R)$ is closed under arbitrary direct products, and hence R is also right coherent.

(3) ⇒ (2). Suppose that R is a right coherent ring with $\mathrm{wgldim}(R) < \infty$. Then of course R is a right coherent ring with $\text{G-wgldim}(R) < \infty$. It follows from Theorem 33 that the class $\mathscr{GF}(R)$ is cotilting. Meanwhile, $\mathrm{wgldim}(R) < \infty$ implies that $\mathscr{F}(R) = \mathscr{GF}(R)$ (see the proof of [10, Proposition 2.2(2)]).

(3) ⇔ (4). It holds by Lemma 10, using the complete hereditary cotorsion pair $(\mathscr{F}(R), \mathscr{F}(R)^{\perp})$. □

Proposition 51. *Let R be a ring. Then the following are equivalent:*

(1) *The class* $\mathscr{P}(R)$ *is cotilting.*
(2) *The class* $\mathscr{GP}(R)$ *is cotilting and* $\mathscr{P}(R) = \mathscr{GP}(R)$.
(3) *R is a right coherent and left perfect ring with* $\mathrm{gldim}(R) < \infty$.
(4) *R is a right coherent and left perfect ring such that* $\mathrm{pd}_R(M) < \infty$ *for any R-module M.*

Proof.

(2) ⇒ (1). It is obvious.

(1) ⇒ (3). Assume that the class $\mathscr{P}(R)$ is cotilting. Then by Lemma 26, $\mathscr{P}(R)$ is definable and $\mathrm{gldim}(R) = \sup\{\mathrm{pd}_R(M) \mid M \text{ is an } R\text{-module}\} < \infty$. In particular, $\mathscr{P}(R)$ is closed under arbitrary direct products, and hence R is also right coherent and left perfect.

(3) ⇒ (2). Suppose that R is a right coherent and left perfect ring with $\mathrm{gldim}(R) < \infty$. Then of course R is a right coherent and left perfect ring with $\text{G-gldim}(R) < \infty$. It follows from Theorem 34 that the class $\mathscr{GP}(R)$ is cotilting. Meanwhile, $\mathrm{gldim}(R) < \infty$ implies that $\mathscr{GP}(R) = \mathscr{P}(R)$ (see the proof of [33, Proposition 2.27)]).

(3) ⇔ (4). It holds by Lemma 10, using the trivial cotorsion pair $(\mathscr{P}(R), R\text{-Mod})$. □

Proposition 52. *Let R be a ring. Then the following are equivalent:*

(1) *The class* $\mathscr{I}(R)$ *is tilting.*
(2) *The class* $\mathscr{GI}(R)$ *is cotilting and* $\mathscr{I}(R) = \mathscr{GI}(R)$.
(3) *R is left Noetherian ring with* $\mathrm{gldim}(R) < \infty$.
(4) *R is a left Noetherian ring such that* $\mathrm{fd}_R(M) < \infty$ *for any R-module M.*

Proof. By dual of Proposition 51, we leave it to the readers. □

4.4. *(Co)silting property for classes of classical and Gorensrein homological modules*

As the last application of Theorems 33, 34 and 35, we will consider when the classes $\mathscr{GP}(R)$, $\mathscr{GF}(R)$, $\mathscr{P}(R)$ and $\mathscr{F}(R)$ are cosilting and when the classes $\mathscr{GI}(R)$ and $\mathscr{I}(R)$ are silting, which induces some characterizations of Dedekind and Prüfer domains.

Definition 53.

(1) *For a morphism σ between projective R-modules, we denote by* $\mathscr{D}_\sigma$ *the class of R-modules*

$$\mathscr{D}_\sigma = \{M \in R\text{-Mod} \mid \mathrm{Hom}_R(\sigma, M) \text{ is surjective}\}.$$

(2) *An R-module T is* silting *if it admits a projective presentation* $P_1 \xrightarrow{\sigma} P_0 \to T \to 0$ *such that* $\mathrm{Gen}\, T = \mathscr{D}_\sigma$. *The class* $\mathrm{Gen}\, T$ *is then called a* silting class *of R-modules.*

(3) *For a morphism τ between injective R-modules, we denote by $\mathscr{C}_\tau$ the class of R-modules*

$$\mathscr{C}_\tau = \{M \in R\text{-Mod} \mid \mathrm{Hom}_R(M,\tau) \text{ is surjective}\}.$$

(4) *An R-module C is cosilting if it admits an injective copresentation $0 \to T \to E_0 \xrightarrow{\tau} E_1$ such that* $\mathrm{Cogen}\, T = \mathscr{C}_\tau$*. The class* $\mathrm{Cogen}\, T$ *is then called a* cosilting class *of R-modules.*

Recall that a class $\mathscr{X}$ of R-modules is *torsion* (resp. *torsionfree*) if $\mathscr{X}$ is closed under epimorphic images, extensions and coproducts (resp. under submodules, extensions and products). We know from [5, Corollary 3.5 and Proposition 3.10] (see also [4, Remarks in p. 4135]) that any silting class of R-modules is definable and torsion; from [1, Corollary 3.9] that a class $\mathscr{X}$ of R-modules is cosilting if and only if $\mathscr{X}$ is definable and torsionfree.

It was shown in [32, Lemma 6.1.2] (resp. [32, Lemma 8.2.2]) that a class $\mathscr{X}$ of modules is 1-tilting (resp. 1-cotilting) if and only if there is a 1-tilting (resp. 1-cotilting) module T such that $\mathscr{X} = \mathrm{Gen}\, T$ (resp. $\mathscr{X} = \mathrm{Cogen}\, T$). By [1, Example 2.4(1) and (3)] (resp. [14, Example 3.3(a) and (c)]) the inclusions {1-tilting modules} ⊆ {silting modules} and {1-cotilting modules} ⊆ {cosilting modules} are strict. It is then a routine to check that the inclusions {1-tilting classes of modules} ⊆ {silting classes of modules} and {1-cotilting classes of modules} ⊆ {cosilting classes of modules} are strict as well.

However, we will show that the silting (resp. cosilting) and 1-tilting (resp. 1-cotilting) property of the class $\mathscr{GI}(R)$ (resp. the classes $\mathscr{GP}(R)$ and $\mathscr{GF}(R)$) coincide.

Proposition 54. *Let R be a ring. Then the following are equivalent:*

(1) *The class $\mathscr{GF}(R)$ is cosilting.*
(2) *The class $\mathscr{GF}(R)$ is 1-cotilting.*
(3) *R is a right coherent ring with* $\mathrm{G}\text{-wgldim}(R) \le 1$.

Proof.

(3) ⇒ (2). It follows from Remark 36(1).

(2) ⇒ (1). It is obvious.

(1) ⇒ (3). Assume that the class $\mathscr{GF}(R)$ is cosilting. Then by [1, Corollary 3.9], the class $\mathscr{GF}(R)$ is definable and torsionfree. Thus, one has

$$\mathrm{G}\text{-wgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\} \le 1$$

since $\mathscr{GF}(R)$ is closed under submodules. Furthermore, R is right coherent by Lemma 29. □

Proposition 55. *Let R be a ring. Then the following are equivalent:*

(1) *The class $\mathscr{GP}(R)$ is cosilting.*
(2) *The class $\mathscr{GF}(R)$ is cosilting and $\mathscr{GP}(R) = \mathscr{GF}(R)$.*
(3) *The class $\mathscr{GP}(R)$ is 1-cotilting.*
(4) *The class $\mathscr{GF}(R)$ is 1-cotilting and $\mathscr{GP}(R) = \mathscr{GF}(R)$.*
(5) *R is a right coherent and left perfect ring with* $\mathrm{G}\text{-gldim}(R) \le 1$.

Proof.

(3) ⇔ (4) ⇔ (5). It follows from Remark 36(2).

(4) ⇒ (2) ⇒ (1). It is clear.

(1) ⇒ (5). Assume that the class $\mathscr{GP}(R)$ is cosilting. Then by [1, Corollary 3.9], the class $\mathscr{GP}(R)$ is definable and torsionfree. So one gets that

$$\mathrm{G}\text{-gldim}(R) = \sup\{\mathrm{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\} \le 1$$

since $\mathscr{GP}(R)$ is closed under submodules. It follows from Lemma 16(1) that G-wgldim$(R) \le 1$. Thus, R is right coherent and left perfect due to Lemma 30. □

Proposition 56. *Let R be a ring. Then the following are equivalent:*

(1) *The class $\mathscr{GI}(R)$ is silting.*
(2) *The class $\mathscr{GF}(R^{\text{op}})$ is cosilting and an R-module M is in $\mathscr{GI}(R)$ if and only if so is M^{++}.*
(3) *The class $\mathscr{GI}(R)$ is 1-tilting.*
(4) *The class $\mathscr{GF}(R^{\text{op}})$ is 1-cotilting and an R-module M is in $\mathscr{GI}(R)$ if and only if so is M^{++}.*
(5) *R is a left Noetherian ring with* G-gldim$(R) \le 1$.

Proof.

(3) ⇔ (5) ⇔ (5). It follows from Remark 36(3).

(4) ⇒ (2) and (3) ⇒ (1). These implications are trivial.

(1) ⇒ (5). Assume that the class $\mathscr{GI}(R)$ is silting. Then by [1, Corollary 3.9], the class $\mathscr{GI}(R)$ is definable and torsion. Now, one obtains that

$$\text{G-gldim}(R) = \sup\{\text{Gid}_R(M) \mid M \text{ is an } R\text{-module}\} \le 1$$

as the class $\mathscr{GI}(R)$ is closed under epimorphic images. Again, G-wgldim$(R) \le 1$ via Lemma 16(1). Now R is also left Noetherian by Proposition 31.

(2) ⇒ (5). Suppose that the class $\mathscr{GF}(R^{\text{op}})$ is cosilting and an R-module M is in $\mathscr{GI}(R)$ if and only if so is M^{++}. According to Proposition 54, the first statement of the assumption yields that R is a left coherent ring with G-wgldim(R) = G-wgldim$(R^{\text{op}}) \le 1$. Now the second statement of the assumption yields that R is left Noetherian by Proposition 31. In addition, we conclude that G-gldim(R) = G-wgldim$(R) \le 1$ by [12, Theorem 7]. □

Using Propostions 50, 51 and 52, and applying the arguments used in the proof of Propositions 54, 55 and 56, respectively, one can obtain

Proposition 57. *Let R be a ring. Then the following are equivalent:*

(1) *The class $\mathscr{F}(R)$ is cosilting.*
(2) *The class $\mathscr{F}(R)$ is 1-cotilting.*
(3) *R is a right coherent ring with* wgldim$(R) \le 1$.

Proposition 58. *Let R be a ring. Then the following are equivalent:*

(1) *The class $\mathscr{P}(R)$ is cosilting.*
(2) *The class $\mathscr{P}(R)$ is 1-cotilting.*
(3) *R is a right coherent and left perfect ring with* gldim$(R) \le 1$.

Proposition 59. *Let R be a ring. Then the following are equivalent:*

(1) *The class $\mathscr{I}(R)$ is silting.*
(2) *The class $\mathscr{I}(R)$ is 1-tilting.*
(3) *R is a left Noetherian ring with* gldim$(R) \le 1$.

Recall that a ring R is *Dedekind* (resp. Prüfer) if R is a hereditary (resp. semi-hereditary) domain. Here a (possibly not communicative) *hereditary* (resp. *semi-hereditary*) ring is defined as the one such that every left and right ideal (resp. finitely generated left and right ideal) is projective. It is well-known that any hereditary (resp. semi-hereditary) ring R has gldim$(R) \le 1$ (resp. wgldim$(R) \le 1$).

According to [32, Theorems 6.2.15, 6.2.19, 6.2.22, 8.2.9 and 8.2.12], we know that both the (co)tilting modules and classes over a Dedekind (resp. Prüfer) domain have a nice description. We will end the paper by characterizing Dedekind (resp. Prüfer) domain using some special (co)tilting classes.

Theorem 60. *Let R be a domain. Then the following are equivalent:*

(1) *R is Prüfer.*
(2) *The class $\mathscr{F}(R)$ is 1-cotilting.*
(3) *The class $\mathscr{F}(R)$ is cosilting.*
(4) *The class $\mathscr{G}\mathscr{F}(R)$ is 1-cotilting and $\mathscr{G}\mathscr{F}(R) = \mathscr{F}(R)$.*
(5) *The class $\mathscr{G}\mathscr{F}(R)$ is cosilting and $\mathscr{G}\mathscr{F}(R) = \mathscr{F}(R)$.*

Proof.

(4) ⇒ (2) and (5) ⇒ (3). These implications are trivial.

(2) ⇒ (3). It holds by Proposition 57.

(3) ⇒ (1). Suppose that R is a domain such that the class $\mathscr{F}(R)$ is cosilting. Then by Proposition 57, R is a coherent domain with $\text{wgldim}(R) \leq 1$, and so R is a semi-hereditary domain by [35, Theorem 4.67]. Thus, R is Prüfer.

(1) ⇒ (4) and (1) ⇒ (5). Assume that R is a Prüfer domain. Then $\text{wgldim}(R) \leq 1$, which implies that $\mathscr{G}\mathscr{F}(R) = \mathscr{F}(R)$. In addition, R is coherent by [35, Theorem 4.67]. So the class $\mathscr{G}\mathscr{F}(R)$ is 1-cotilting (resp. cosilting) due to Proposition 57. □

Theorem 61. *Let R be a domain. Then the following are equivalent:*

(1) *R is Dedekind.*
(2) *The class $\mathscr{I}(R)$ is 1-tilting.*
(3) *The class $\mathscr{I}(R)$ is silting.*
(4) *The class $\mathscr{G}\mathscr{I}(R)$ is 1-tilting and $\mathscr{G}\mathscr{I}(R) = \mathscr{I}(R)$.*
(5) *The class $\mathscr{G}\mathscr{I}(R)$ is silting and $\mathscr{G}\mathscr{I}(R) = \mathscr{I}(R)$.*

Proof.

(4) ⇒ (2) and (5) ⇒ (3). These implications are trivial.

(2) ⇒ (3). It follows by Proposition 59.

(3) ⇒ (1). Suppose that R is a domain such that the class $\mathscr{I}(R)$ is silting. Then by Proposition 59, R is a domain with $\text{gldim}(R) \leq 1$, and so R is a hereditary domain by [39, Theorem 4.23]. That is, R is Dedekind.

(1) ⇒ (4) and (1) ⇒ (5). Suppose that R is a Dedekind domain. Then $\text{gldim}(R) \leq 1$, which implies that $\mathscr{G}\mathscr{I}(R) = \mathscr{I}(R)$. Furthermore, R is Noetherian by [39, Corollary 4.26]. Consequently, the class $\mathscr{G}\mathscr{I}(R)$ is 1-tilting (resp. silting) via Proposition 59. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] L. Angeleri Hügel, "On the abundance of silting modules", in *Surveys in representation theory of algebras*, American Mathematical Society, 2018, pp. 1–23.

[2] L. Angeleri Hügel and F. U. Coelho, "Infinitely generated tilting modules of finite projective dimension", *Forum Math.* **13** (2001), no. 2, pp. 239–250.

[3] L. Angeleri Hügel, D. Herbera and J. Trlifaj, "Tilting modules and Gorenstein rings", *Forum Math.* **18** (2006), no. 2, pp. 211–229.

[4] L. Angeleri Hügel and M. Hrbek, "Silting modules over commutative rings", *Int. Math. Res. Not.* **2017** (2017), no. 13, pp. 4131–4151.

[5] L. Angeleri Hügel, F. Marks and J. Vitória, "Silting modules", *Int. Math. Res. Not.* **2016** (2016), no. 4, pp. 1251–1284.

[6] L. Angeleri Hügel, J. Šaroch and J. Trlifaj, "On the telescope conjecture for module categories", *J. Pure Appl. Algebra* **212** (2008), no. 2, pp. 297–310.

[7] L. Angeleri Hügel and J. Trlifaj, "Tilting theory and the finitistic dimension conjectures", *Trans. Am. Math. Soc.* **354** (2002), no. 11, pp. 4345–4358.

[8] M. Auslander and M. Bridger, *Stable module theory*, American Mathematical Society, 1969, p. 146.

[9] S. Bazzoni, "When are definable classes tilting and cotilting classes?", *J. Algebra* **320** (2008), no. 12, pp. 4281–4299.

[10] D. Bennis, "Rings over which the class of Gorenstein flat modules is closed under extensions", *Commun. Algebra* **37** (2009), no. 3, pp. 855–868.

[11] D. Bennis and N. Mahdou, "Global Gorenstein dimensions", *Proc. Am. Math. Soc.* **138** (2010), no. 2, pp. 461–465.

[12] S. Bouchiba, "On Gorenstein flat dimension", *J. Algebra Appl.* **14** (2015), no. 6, article no. 1550096 (18 pages).

[13] D. Bravo, J. Gillespie and M. Hovey, "The stable module category of a general ring", 2014. https://arxiv.org/abs/1405.5768.

[14] S. Breaz and F. Pop, "Cosilting modules", *Algebr. Represent. Theory* **20** (2017), no. 5, pp. 1305–1321.

[15] S. U. Chase, "Direct products of modules", *Trans. Am. Math. Soc.* **97** (1960), pp. 457–473.

[16] X.-W. Chen, "Homotopy equivalences induced by balanced pairs", *J. Algebra* **324** (2010), no. 10, pp. 2718–2731.

[17] L. W. Christensen, S. Estrada and P. Thompson, "Gorenstein weak global dimension is symmetric", *Math. Nachr.* **294** (2021), no. 11, pp. 2121–2128.

[18] L. W. Christensen, S. Estrada and P. Thompson, "Five theorems on Gorenstein global dimensions", in *Algebra and coding theory. Virtual conference in honor of Tariq Rizvi. Noncommutative rings and their applications VII, Université d'Artois, Lens, France, July 5–7, 2021. Virtual conference on quadratic forms, rings and codes, Université d'Artois, Lens, France, July 8, 2021*, American Mathematical Society, 2023, pp. 67–78.

[19] L. W. Christensen, H.-B. Foxby and H. Holm, "Derived category methods in commutative algebra", 2022. preprint.

[20] Z. Di, J. Wei, X. Zhang and J. Chen, "Tilting subcategories with respect to cotorsion triples in abelian categories", *Proc. R. Soc. Edinb., Sect. A, Math.* **147** (2017), no. 4, pp. 703–726.

[21] N. Q. Ding and J. Chen, "The flat dimensions of injective modules", *Manuscr. Math.* **78** (1993), no. 2, pp. 165–177.

[22] N. Ding and J. Chen, "Coherent rings with finite self-FP-injective dimension", *Commun. Algebra* **24** (1996), no. 9, pp. 2963–2980.

[23] N. Ding, Y. Li and L. Mao, "Strongly Gorenstein flat modules", *J. Aust. Math. Soc.* **86** (2009), no. 3, pp. 323–338.

[24] I. Emmanouil, "On the finiteness of Gorenstein homological dimensions", *J. Algebra* **372** (2012), pp. 376–396.

[25] E. E. Enochs and O. M. G. Jenda, "Gorenstein injective and projective modules", *Math. Z.* **220** (1995), no. 4, pp. 611–633.

[26] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, Walter de Gruyter, 2000, pp. xii+339.

[27] E. E. Enochs and O. M. G. Jenda, "Gorenstein injective modules and Ext", *Tsukuba J. Math.* **28** (2004), no. 2, pp. 303–309.

[28] E. E. Enochs, O. M. G. Jenda and B. Torrecillas, "Gorenstein flat modules", *J. Nanjing Univ., Math. Biq.* **10** (1993), no. 1, pp. 1–9.

[29] J. Gillespie, "Model structures on modules over Ding–Chen rings", *Homology Homotopy Appl.* **12** (2010), no. 1, pp. 61–73.

[30] J. Gillespie, "Gorenstein complexes and recollements from cotorsion pairs", *Adv. Math.* **291** (2016), pp. 859–911.

[31] J. Gillespie, "On Ding injective, Ding projective and Ding flat modules and complexes", *Rocky Mt. J. Math.* **47** (2017), no. 8, pp. 2641–2673.

[32] R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras of modules*, Walter de Gruyter, 2006.

[33] H. Holm, "Gorenstein homological dimensions", *J. Pure Appl. Algebra* **189** (2004), no. 1-3, pp. 167–193.

[34] Y. Iwanaga, "On rings with finite self-injective dimension", *Commun. Algebra* **7** (1979), no. 4, pp. 393–414.

[35] T. Y. Lam, *Lectures on modules and rings*, Springer, 1999, pp. xxiv+557.

[36] Y. Li, J. Wang, Y. Geng and J. Hu, "Noncommutative G-hereditary rings", *J. Algebra Appl.* **20** (2021), no. 4, article no. 2150052 (13 pages).

[37] P. Moradifar and J. Šaroch, "Finitistic dimension conjectures via Gorenstein projective dimension", *J. Algebra* **591** (2022), pp. 15–35.

[38] P. Moradifar and S. Yassemi, "Infinitely generated Gorenstein tilting modules", *Algebr. Represent. Theory* **25** (2022), no. 6, pp. 1389–1427.

[39] J. J. Rotman, *An introduction to homological algebra*, Academic Press Inc., 1979, pp. xi+376.

[40] J. Šaroch and J. Šťovíček, "Singular compactness and definability for Σ-cotorsion and Gorenstein modules", *Sel. Math., New Ser.* **26** (2020), no. 2, article no. 23 (40 pages).

[41] L. W. Small, "An example in Noetherian rings", *Proc. Natl. Acad. Sci. USA* **54** (1965), pp. 1035–1036.

[42] J. Wang, Y. Li and J. Hu, "When the kernel of a complete hereditary cotorsion pair is the additive closure of a tilting module", *J. Algebra* **530** (2019), pp. 94–113.

[43] J. Wang, "Ding projective dimension of Gorenstein flat modules", *Bull. Korean Math. Soc.* **54** (2017), no. 6, pp. 1935–1950.

[44] J. Wang, G. Yang, Q. Shao and X. Zhang, "On Gorenstein global and Gorenstein weak global dimensions", *Colloq. Math.* **174** (2023), no. 1, pp. 45–67.

[45] J. Wang and X. Zhang, "On rings with finite Gorenstein weak global dimension", *J. Algebra Appl.* **22** (2023), no. 5, article no. 2350111 (15 pages).

[46] J. Wei, "Relative singularity categories, Gorenstein objects and silting theory", *J. Pure Appl. Algebra* **222** (2018), no. 8, pp. 2310–2322.