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A note on quantum subgroups of free quantum groups

Une note sur les sous-groupes quantiques des groupes quantiques libres

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Abstract. In this short note, quantum subgroups in finite free products of the Pontryagin duals of free unitary quantum groups are classified. They correspond to pairs of a subgroup Γ and a subset S of the free group $\mathbb{F}_n$ such that S is Γ-invariant, containing Γ, and connected in the Cayley graph of $\mathbb{F}_n$.

Résumé. Dans cette courte note, les sous-groupes quantiques dans les produits libres finis des duaux de Pontryagin des groupes quantiques unitaires libres sont classifiés. Ils correspondent à des paires formées d'un sous-groupe Γ et d'un sous-ensemble S du groupe libre $\mathbb{F}_n$ tel que S est Γ-invariant, contenant Γ, et connexe dans le graphe de Cayley de $\mathbb{F}_n$.

Keywords. free unitary quantum group, quantum subgroup, Cayley graph.

Mots-clés. groupe quantique unitaire libéré, sous-groupe quantique, graphe de Cayley.

2020 Mathematics Subject Classification. 46L67, 05C25.

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1. Introduction

In this note, we classify quantum subgroups in the Pontryagin duals of free unitary quantum groups or, more generally, in their finite free products. Associated with $Q \in GL_N(\mathbb{C})$ for $N \geq 2$, van Daele–Wang [4] defined a compact quantum group called a free unitary quantum group, which shall be denoted by U_Q^+. It is constructed by forgetting the commutativity relations for the coefficients of a unitary matrix. Variations of this construction have provided abundant examples of compact quantum groups.

For general conventions and fundamental facts on compact quantum groups and tensor categories, we refer to [3]. For a compact quantum group G, its finite dimensional unitary representations form a rigid semisimple C*-tensor category with a simple unit object, denoted by $\operatorname{Rep} G$. We write $\operatorname{Irr} G$ for the set of all isomorphism classes of simple objects in $\operatorname{Rep} G$. We write $\widehat{G}$ for the Pontryagin dual of G, which is a discrete quantum group.

The notion of free product for (the C*-algebras of) discrete quantum groups was studied by Wang [5]. The main object of this note is the free product $*_{j=1}^{n}\widehat{U}_{Q_j}^{+}$ for $n \in \mathbb{Z}_{\geq 1}$, $Q_j \in GL(N_j, \mathbb{C})$, $N_j \geq 2$, defined by the Pontryagin dual $\widehat{F}$ of the compact quantum group F given by $C_u(F) = *_{j=1}^{n} C_u(U_{Q_j}^{+})$ with a canonical comultiplication. This quantum group $\widehat{F}$ has the following remarkable property analogous to the free group $\mathbb{F}_n$ with n generators: for any compact quantum group G, any n-tuple of unitary representations $(u_j)_{j=1}^{n}$ of G with $\dim_{\mathbb{C}} u_j = N_j$ and $Q_j^{*-1} u_j^{\top} Q_j^{*}$ being a unitary uniquely induces a unital $*$-homomorphism $C_u(F) \to C_u(G)$ preserving comultiplications. Quantum subgroups of $\widehat{F}$ has been recently classified by Freslon–Weber [2] when $\widehat{F} = \widehat{U}_Q^{+}$, together with rich structural results for such quantum subgroups. We classify quantum subgroups of $\widehat{F}$ in general, based on a simple observation relating $\operatorname{Irr} F$ to the path space of the Cayley graph of $\mathbb{F}_n$. Our classification exhibits to what extent more quantum subgroups exist than mere subgroups of free groups and thus could be regarded as a complementary result to [2] towards the quantum analogue of Kurosh's theorem.

2. Classification

Note that the classification of quantum subgroups in $\widehat{F}$ is equivalent to the classification of idempotent complete full rigid C*-tensor subcategories in $\operatorname{Rep} F$, as a consequence of Woronowicz's Tannaka–Krein duality. Since the latter is a purely categorical problem and in fact relies only on the fusion rule of $\operatorname{Rep} F$, we work with $\operatorname{Rep} F$ rather than $\widehat{F}$ and restrict to a fixed skeleton of $\operatorname{Rep} F$ for simplicity.

The theory of unitary representations of U_Q^{+} was investigated by Banica [1]. We recall its fusion rule. Consider the free product of two copies of monoids $M := \{\alpha^r \mid r \in \mathbb{Z}_{\geq 0}\} * \{\beta^r \mid r \in \mathbb{Z}_{\geq 0}\} \cong \mathbb{Z}_{\geq 0} * \mathbb{Z}_{\leq 0}$. By abusing notation, we write $[a^r] := \alpha^r$ and $[a^{-r}] := \beta^r$ in M for $r \in \mathbb{Z}_{\geq 0}$. Then, we can identify $\operatorname{Irr} U_Q^{+}$ with M so that the unit object is $[a^0]$, $\overline{[a^r]} = [a^{-r}]$ for $r \in \mathbb{Z}$, $\overline{(xy)} = \overline{y}\,\overline{x}$ for $x, y \in M$, and the tensor product is determined recursively by

$$(x[a^r]) \otimes ([a^s]y) = \begin{cases} x[a^r][a^s]y = x[a^{r+s}]y & \text{if } rs = 1, \\ x[a^r][a^s]y \oplus (x \otimes y) & \text{if } rs = -1 \end{cases} \qquad (x, y \in M, r, s \in \{\pm 1\}),$$

with the aid of the fact that any element in M is expressed as a word of $[a^1]$ and $[a^{-1}]$.

Combined with [5], we can canonically identify $\operatorname{Irr} F$ with the monoid M^{*n} as a set. For $1 \leq j \leq n$, we put α_j, β_j for the generators of the jth component $M = \{\alpha_j^r \mid r \in \mathbb{Z}_{\geq 0}\} * \{\beta_j^r \mid r \in \mathbb{Z}_{\geq 0}\}$ in M^{*n}, and a_j for the generator of the jth component $\mathbb{Z} \cong \{a_j^r \mid r \in \mathbb{Z}\}$ in $\mathbb{F}_n$. We shall write $[a_j^r] := \alpha_j^r$ and $[a_j^{-r}] := \beta_j^r$ in M^{*n} for $r \in \mathbb{Z}_{\geq 0}$.

Consider the Cayley graph $\mathscr{G}$ of $\mathbb{F}_n$, where the set of its oriented edges is $\{(g, g a_j^{\pm 1}) \mid g \in \mathbb{F}_n, 1 \leq j \leq n\}$. Let $\operatorname{Path}(\mathscr{G})$ be the set of all finite paths in $\mathscr{G}$ starting from $1_{\mathbb{F}_n}$, possibly with turning-backs. We can identify $\operatorname{Path}(\mathscr{G})$ with M^{*n} by assigning each path of the form $(g_0, g_1, \ldots, g_k) \in \mathbb{F}_n^{k+1}$ with $k \geq 0$, $g_0 = 1_{\mathbb{F}_n}$, $g_l = g_{l-1} a_{j_l}^{r_l}$, $1 \leq j_l \leq n$, $r_l = \pm 1$ for $1 \leq l \leq k$ to $[a_{j_1}^{r_1}][a_{j_2}^{r_2}] \cdots [a_{j_k}^{r_k}] \in M^{*n}$. Then, extending our convention of $[a_j^r]$ for $r \in \mathbb{Z}$, we define $[g] \in M^{*n}$ for $g \in \mathbb{F}_n$ by the shortest path from $1_{\mathbb{F}_n}$ to g via the identification $M^{*n} = \operatorname{Path}(\mathscr{G})$. From now, the symbol k, j and r, possibly accompanied with indices, stand for elements in $\mathbb{Z}_{\geq 0}$, $\mathbb{Z} \cap [1, n]$, and $\{\pm 1\}$ respectively, unless clarified otherwise. Note that any element in $\operatorname{Irr} F = M^{*n}$ can be expressed in the form $[a_{j_1}^{r_1}][a_{j_2}^{r_2}] \cdots [a_{j_k}^{r_k}]$ without redundancy.

Remark. We reinterpret the fusion rule of $\operatorname{Rep} F$ in terms of $\mathscr{G}$. Take any $e, e' \in \operatorname{Path}(\mathscr{G})$, whose endpoints are denoted by $g, g' \in \mathbb{F}_n$, respectively. We write ge' for the path from g to gg' given by the left translation of $\mathbb{F}_n$ on $\mathscr{G}$ and $ee' \in \operatorname{Path}(\mathscr{G})$ for the concatenation of e and ge'. Note that ee' coincides with the product of the monoid M^{*n}. We set $e_0 := e$, $e'_0 := e'$ and define $e_{l+1}, e'_{l+1} \in \operatorname{Path}(\mathscr{G}) = M^{*n}$ recursively on $l \in \mathbb{Z}_{\geq 0}$ so that $e_{l+1}[a_j^r] = e_l$ and $[a_j^{-r}]e'_{l+1} = e'_l$ as long as such j and r exist. If we regard $e, e' \in \operatorname{Irr} F$, the tensor product $e \otimes e'$ is the direct sum of all paths of the form $e_l e'_l \in \operatorname{Path}(\mathscr{G}) = \operatorname{Irr} F$. Also, when we write $\bar{e} \in \operatorname{Path}(\mathscr{G})$ for the path from $1_{\mathbb{F}_n}$ to g^{-1} given by reversing e, the involution $e \mapsto \bar{e}$ corresponds to the conjugate operation on $\operatorname{Irr} F$. If we put $V(e) \subset \mathbb{F}_n$ for the set of all vertices appearing in e, we have $V(e_l e'_l) \subset V(ee') = V(e) \cup gV(e')$ for e_l and e'_l above, and $V(\bar{e}) = g^{-1}V(e)$.

Our main result is as follows.

Main Theorem. *For any pair (Γ, S) of a subgroup $\Gamma \subset \mathbb{F}_n$ and a left Γ-invariant subset $S \subset \mathbb{F}_n$ containing Γ and connected in the Cayley graph $\mathscr{G}$, there is an idempotent complete full rigid C*-tensor subcategory of* $\operatorname{Rep} F$ *such that the set of its irreducible objects equals*

$$\left\{[a_{j_1}^{r_1}][a_{j_2}^{r_2}]\cdots[a_{j_k}^{r_k}] \in \operatorname{Irr} F \,\middle|\, a_{j_1}^{r_1} a_{j_2}^{r_2}\cdots a_{j_k}^{r_k} \in \Gamma, a_{j_1}^{r_1} a_{j_2}^{r_2}\cdots a_{j_l}^{r_l} \in S \text{ for all } 0 \le l \le k\right\}. \tag{I}$$

Conversely, any idempotent complete full rigid C-tensor subcategory of* $\operatorname{Rep} F$ *is of this form for a pair (Γ, S) with the conditions above, which must be necessarily unique.*

Proof. It is not hard to see the first half of the statement with the aid of equations on $V(e)$ in the remark above. We show the latter half. Note that (Γ, S) can be reconstructed from (I) since S is connected and contains the unit element. This shows the uniqueness. Let $\mathscr{C}$ be an idempotent complete full rigid C*-tensor subcategory of $\operatorname{Rep} F$ and I be the set of irreducible objects in $\mathscr{C}$. For any $[a_{j_1}^{r_1}]\cdots[a_{j_k}^{r_k}] \in I \setminus \{[1_{\mathbb{F}_n}]\}$, inductively on $l = 1, \dots, k$, we see

$$[a_{j_1}^{r_1}\cdots a_{j_l}^{r_l}][a_{j_{l+1}}^{r_{l+1}}]\cdots[a_{j_k}^{r_k}] \le \left([a_{j_1}^{r_1}\cdots a_{j_{l-1}}^{r_{l-1}}][a_{j_{l-1}}^{-r_{l-1}}\cdots a_{j_1}^{-r_1}]\right) \otimes \left([a_{j_1}^{r_1}\cdots a_{j_{l-1}}^{r_{l-1}}][a_{j_l}^{r_l}]\cdots[a_{j_k}^{r_k}]\right) \in \mathscr{C}, \tag{1}$$

and

$$[a_{j_1}^{r_1}\cdots a_{j_l}^{r_l}][a_{j_l}^{-r_l}\cdots a_{j_1}^{-r_1}] \le \left([a_{j_1}^{r_1}\cdots a_{j_l}^{r_l}][a_{j_{l+1}}^{r_{l+1}}]\cdots[a_{j_k}^{r_k}]\right) \otimes \left([a_{j_k}^{-r_k}]\cdots[a_{j_{l+1}}^{-r_{l+1}}][a_{j_l}^{-r_l}\cdots a_{j_1}^{-r_1}]\right) \in \mathscr{C}. \tag{2}$$

Indeed, (1) for $l = 1$ is trivial, and (2) for l can be obtained by tensoring (1) for l with its conjugate, while (1) for $l+1$ can be seen by tensoring (2) for l and (1) for l.

Clearly, $S := \{a_{j_1}^{r_1}\cdots a_{j_l}^{r_l} \mid [a_{j_1}^{r_1}]\cdots[a_{j_k}^{r_k}] \in I, 0 \le l \le k\} \subset \mathbb{F}_n$ is connected in $\mathscr{G}$ as every element in S can be connected to $1_{\mathbb{F}_n}$ by a path within S, and $\Gamma := \{a_{j_1}^{r_1}\cdots a_{j_k}^{r_k} \mid [a_{j_1}^{r_1}]\cdots[a_{j_k}^{r_k}] \in I\}$ is a subgroup of $\mathbb{F}_n$ contained in S. Then, (1) and (2) for $l = k$ imply $[g] \in I$ for all $g \in \Gamma$ and $[g][g^{-1}] \in I$ for all $g \in S$, respectively. For all $g \in \Gamma$ and $h \in S$, it holds $[g][h][h^{-1}] \in I$ as a direct summand of $[g] \otimes ([h][h^{-1}])$ and thus $gh \in S$. By construction, the set defined by (I) contains I. They coincide because for any $[a_{j_1}^{r_1}]\cdots[a_{j_k}^{r_k}] \in \operatorname{Irr} F$ satisfying the conditions in (I),

$$\begin{aligned}[a_{j_1}^{r_1}]\cdots[a_{j_k}^{r_k}] &\le \left([a_{j_1}^{r_1}]\cdots[a_{j_k}^{r_k}][a_{j_k}^{-r_k}\cdots a_{j_1}^{-r_1}]\right) \otimes [a_{j_1}^{r_1}\cdots a_{j_k}^{r_k}] \\ &\le \left([a_{j_1}^{r_1}][a_{j_1}^{-r_1}]\right) \otimes \left([a_{j_1}^{r_1} a_{j_2}^{r_2}][a_{j_2}^{-r_2} a_{j_1}^{-r_1}]\right) \otimes \cdots \otimes \left([a_{j_1}^{r_1}\cdots a_{j_k}^{r_k}][a_{j_k}^{-r_k}\cdots a_{j_1}^{-r_1}]\right) \otimes [a_{j_1}^{r_1}\cdots a_{j_k}^{r_k}] \in \mathscr{C}. \quad \square\end{aligned}$$

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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