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Some fundamental properties of the approximate synchronization by groups for a coupled system of wave equations with internal controls

Quelques propriétés fondamentales de la synchronisation approchée par groupes pour un système couplé d'équations des ondes avec contrôles internes

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Abstract. In this paper, we show that the independence of approximately synchronizable state u by p-groups with respect to applied controls, the linear independence of the components, the non extensibility of the approximate synchronization by p-groups as well as the necessity of the condition of C_p-compatibility, all these properties are the consequence of the minimality of Kalman's rank condition and vice versa. These results reveal the role of Kalman rank conditions on control problems from different aspects, and further develop the synchronization theory.

Résumé. Dans cette note, nous montrons que l'indépendance de l'état de synchronisation approchée par rapport aux contrôles, la non extensibilité de la synchronisation approchée, l'indépendance linéaire des composants de l'état de synchronisation approchée ainsi que la condition de C_p-compatibilité, toutes ces propriétés sont la conséquence de la minimalité de la condition du rang de Kalman.

Keywords. Kalman's rank condition, approximate synchronization by groups, system of wave equations.

Mots-clés. Condition du rang de Kalman, synchronisation approchée par groupes, système d'équations d'ondes.

2020 Mathematics Subject Classification. 93B05, 93C20, 35L53.

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1. Introduction

The research on the synchronization for a coupled system of wave equations is a progressive field with broad application prospects. The related topic is closely connected with the control theory. We introduced the concepts of exact synchronization in [5] and approximate synchronization in [6] on partial differential equations. The relevant results about synchronization achieved only through boundary controls were collected in the monograph [7], published by Birkhäuser Publishing House in 2019. In [8,9], we carried on the study of synchronization using locally distributed controls.

For the sake of a concise writing and the reader's convenience, we briefly review the main results. Let A be a matrix of order N and D be a full column-rank matrix of order $N \times M$. Both A and D are composed of constant entries. Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ. We consider the following problem for the variable $U = (u^{(1)}, \dots, u^{(N)})^T$:

$$\begin{cases} U'' - \Delta U + AU = D\chi_\omega H & \text{in } (0, +\infty) \times \Omega, \\ U = 0 & \text{on } (0, +\infty) \times \Gamma \end{cases} \tag{1}$$

associated with the initial condition:

$$t = 0: \quad U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in } \Omega, \tag{2}$$

where $H = (h^{(1)}, \dots, h^{(M)})^T$ denotes the internal control and χ_ω is the characteristic function of the subdomain $\omega \subset \Omega$.

The following well-posedness can be shown by the classic approach in [1,3,11].

Proposition 1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ and ω be a subdomain of Ω. For any given initial data $(\widehat{U}_0, \widehat{U}_1) \in \big(H_0^1(\Omega) \times L^2(\Omega)\big)^N$ and any given function $H \in \big(L^1_{\text{loc}}\big(0, +\infty; L^2(\Omega)\big)\big)^M$, system* (1) *admits a unique weak solution:*

$$U \in \Big(C^0_{\text{loc}}\big([0, +\infty); H_0^1(\Omega)\big) \cap C^1_{\text{loc}}\big([0, +\infty); L^2(\Omega)\big)\Big)^N \tag{3}$$

with continuous dependence.

System (1) is approximately controllable at the time $T > 0$ if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in \big(H_0^1(\Omega) \times L^2(\Omega)\big)^N$, there exists a sequence of internal controls $\{H_n\}_{n\in\mathbb{N}} \in \big(L^2\big(0, +\infty; L^2(\Omega)\big)\big)^M$ with compact support in $[0, T]$, such that the sequence $\{U_n\}_{n\in\mathbb{N}}$ of the corresponding solutions to system (1) satisfies

$$U_n \longrightarrow 0 \quad \text{in } \Big(C^0_{\text{loc}}\big([T, +\infty); H_0^1(\Omega)\big) \cap C^1_{\text{loc}}\big([T, +\infty); L^2(\Omega)\big)\Big)^N \tag{4}$$

as $n \to +\infty$.

Theorem 2 ([8, Theorem 13]). *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ and ω be a subdomain of Ω. If system* (1) *is approximately controllable at the time $T > 0$, then we necessarily have the following Kalman's rank condition:*

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N. \tag{5}$$

Conversely, assume that Kalman's rank condition (5) *holds, then system* (1) *is approximately controllable at the time $T > 0$, provided that $T > 2d(\Omega)$, where $d(\Omega)$ denotes the geodesic diameter of Ω.*

Let $p \geqslant 1$ be an integer and

$$0 = n_0 < n_1 < \cdots < n_p = N \tag{6}$$

be a partition with $n_r - n_{r-1} \geqslant 2$ for $1 \leqslant r \leqslant p$. We re-arrange the components of the state variable U into p groups

$$(u^{(1)}, \dots, u^{(n_1)}),\ (u^{(n_1+1)}, \dots, u^{(n_2)}),\ \dots,\ (u^{(n_{p-1}+1)}, \dots, u^{(n_p)}). \tag{7}$$

System (1) is approximately synchronizable by p-groups in the consensus sense at the time $T>0$, if for any given initial data $(\widehat{U}_0, \widehat{U}_1)$ in $\big(H_0^1(\Omega)\times L^2(\Omega)\big)^N$, there exists a sequence $\{H_n\}_{n\in\mathbb{N}}$ of controls in $\big(L^2\big(0,+\infty; L^2(\Omega)\big)\big)^M$ with compact support in $[0,T]$, such that the sequence $\{U_n\}_{n\in\mathbb{N}}$ of the corresponding solutions to system (1) satisfies

$$u_n^{(k)} - u_n^{(l)} \longrightarrow 0 \quad \text{in } C^0_{\mathrm{loc}}\big([T,+\infty); H_0^1(\Omega)\big)\cap C^1_{\mathrm{loc}}\big([T,+\infty); L^2(\Omega)\big) \tag{8}$$

as $n\to+\infty$ for all $n_{r-1}+1\leqslant k,l\leqslant n_r$ and $1\leqslant r\leqslant p$.

Let S_r be a full row-rank matrix of order $(n_r-n_{r-1}-1)\times(n_r-n_{r-1})$:

$$S_r = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1\leqslant r\leqslant p. \tag{9}$$

We define the $(N-p)\times N$ matrix C_p of synchronization by p-groups as

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \tag{10}$$

Then (8) can be equivalently written as

$$C_p U_n \longrightarrow 0 \quad \text{in } \Big(C^0_{\mathrm{loc}}\big([T,+\infty); H_0^1(\Omega)\big)\cap C^1_{\mathrm{loc}}\big([T,+\infty); L^2(\Omega)\big)\Big)^{N-p} \tag{11}$$

as $n\to+\infty$.

Assume that A satisfies the condition of C_p-compatibility:

$$A\,\mathrm{Ker}(C_p)\subseteq \mathrm{Ker}(C_p). \tag{12}$$

By [7, Proposition 2.15], there exists a matrix A_p of order $(N-p)$, such that

$$C_pA = A_pC_p. \tag{13}$$

Applying C_p to system (1) and setting $Y_p = C_pU$ and $D_p = C_pD$, we get the following reduced system:

$$\begin{cases} Y_p'' - \Delta Y_p + A_pY_p = D_p\chi_\omega H & \text{in } (0,+\infty)\times\Omega, \\ Y_p = 0 & \text{on } (0,+\infty)\times\Gamma. \end{cases} \tag{14}$$

Thus the approximate internal synchronization by p-groups of system (1) is transformed into the approximate internal controllability of the reduced system (14). Moreover, we have the following result.

Theorem 3 ([9, Theorem 6.3]). *Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with smooth boundary Γ and ω be a subdomain of Ω. Let $T>2d(\Omega)$, where $d(\Omega)$ denotes the geodesic diameter of Ω. Assume that A satisfies the condition of C_p-compatibility* (12)*, then system* (1) *is approximately synchronizable by p-groups if and only if*

$$\mathrm{rank}\big(D_p, A_pD_p, \ldots, A_p^{N-p-1}D_p\big) = N-p, \tag{15}$$

or equivalently,

$$\mathrm{rank}\big(C_p(D, AD, \ldots, A^{N-1}D)\big) = N-p. \tag{16}$$

Furthermore, if there exist some functions $u_1,\ldots,u_p$ such that

$$u_n^{(k)} \longrightarrow u_r \quad \text{in } C^0_{\mathrm{loc}}\big([T,+\infty); H_0^1(\Omega)\big)\cap C^1_{\mathrm{loc}}\big([T,+\infty); L^2(\Omega)\big) \tag{17}$$

as $n \to +\infty$ for all $n_{r-1}+1 \leqslant k \leqslant n_r$ and $1 \leqslant r \leqslant p$, system (1) will be called approximately synchronizable by p-groups in the pinning sense, and $u = (u_1, \dots, u_p)^T$ will be called approximately synchronizable state by p-groups.

Let

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \dots, e_p\}, \tag{18}$$

where

$$e_r = (0, \dots, 0, \overset{(n_{r-1}+1)}{1}, \dots, \overset{(n_r)}{1}, 0, \dots, 0)^T, \quad 1 \leqslant r \leqslant p. \tag{19}$$

Then (17) can be equivalently written as

$$U_n \longrightarrow \sum_{r=1}^{p} u_r e_r \quad \text{in } \Big(C^0_{\mathrm{loc}}\big([T, +\infty); H^1_0(\Omega)\big) \cap C^1_{\mathrm{loc}}\big([T, +\infty); L^2(\Omega)\big)\Big)^N \tag{20}$$

as $n \to +\infty$.

Clearly, the approximate synchronization in the pinning sense implies that in the consensus sense.

Theorem 4 ([9, Theorem 7.3]). *Assume that system* (1) *is approximately synchronizable by p-groups under the minimal rank condition*

$$\mathrm{rank}(D, AD, \dots, A^{N-1}D) = N - p. \tag{21}$$

Then it is approximately synchronizable by p-groups in the pinning sense, and the approximately synchronizable state by p-groups $u = (u_1, \dots, u_p)^T$ is independent of applied controls. Furthermore the matrix A satisfies the condition of C_p-compatibility (12).

Through in-depth analysis, it can be found that due to the use of internal controls, more deep-going results on the approximate synchronization by groups can be obtained. In Section 2, we will first characterize the feasibility of Kalman's rank condition (21) via the diagonalization by blocks of A on a suitable basis of type $\mathrm{Im}(C_p^T) \oplus V$. Then in Section 3, we will show that the independence of approximately synchronizable state by groups with respect to applied controls, the linear independence of the components of the approximately synchronizable state by groups, and the possibility of the extensibility of approximate synchronization etc., all these important properties are the consequence of the minimality of Kalman's rank condition (21). Thus, we will give a complete answer to these fundamental questions, which have plagued us for a long time.

2. Algebraic structure of synchronization

In this section we will describe the minimality of Kalman's rank condition (21) by the algebraic structure of the matrix A.

We first recall the following basic property on Kalman's matrix.

Lemma 5 ([6, Lemma 2.5]). *Let $d \geqslant 0$ be an integer. Kalman's rank condition*

$$\mathrm{rank}(D, AD, \dots, A^{N-1}D) = N - d \tag{22}$$

holds if and only if d is the dimension of the largest subspace V, which is invariant for A^T and contained in $\mathrm{Ker}(D^T)$. *Moreover, V is given by*

$$V = \mathrm{Ker}(D, AD, \dots, A^{N-1}D)^T. \tag{23}$$

Theorem 6. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ and ω be a subdomain of Ω. Assume that system* (1) *is approximately synchronizable by p-groups. Then the following equivalences hold:*

(a) *the matrix A satisfies Kalman's rank condition* (21)*;*

(b) *the subspace* $\text{Im}(C_p^T)$ *is invariant for* A^T *and admits a supplement* V*, on which the projection of system* (1) *is independent of applied controls;*

(c) *the subspace* $\text{Im}(C_p^T)$ *is invariant for* A^T *and admits a supplement* V*, which is contained in* $\text{Ker}(D^T)$ *and invariant for* A^T.

Proof. (a)⟹(b). By Theorem 4, A satisfies the condition of C_p-compatibility (12), namely, $\text{Im}(C_p^T)$ is invariant for A^T. Moreover, by [9, Proposition 6.2], we have

$$V \cap \text{Im}(C_p^T) = \{0\}, \tag{24}$$

where V denotes the largest subspace invariant for A^T and contained in $\text{Ker}(D^T)$. By Lemma 5, the rank condition (21) implies that V has the dimension p. Noting that $\dim \text{Im}(C_p^T) = N - p$, V is a supplement of $\text{Im}(C_p^T)$.

Let $V = \text{Span}\{\mathscr{E}_1, \ldots, \mathscr{E}_p\}$. There exists reals $c_{rs}(1 \leqslant r, s \leqslant d)$ such that

$$A^T \mathscr{E}_r = \sum_{s=1}^{p} c_{rs} \mathscr{E}_s \quad \text{and} \quad D^T \mathscr{E}_r = 0, \quad r = 1, \ldots, p. \tag{25}$$

For $r = 1, \ldots, p$, applying $\mathscr{E}_r^T$ to system (1) and setting $\psi_r = \mathscr{E}_r^T U$, we get a system of projection:

$$\begin{cases} \psi_r'' - \Delta \psi_r + \sum_{s=1}^{p} c_{rs} \psi_s = 0 & \text{in } (0, +\infty) \times \Omega, \\ \psi_r = 0 & \text{on } (0, +\infty) \times \Gamma, \end{cases} \tag{26}$$

which is independent of applied controls.

(b)⟹(c). By the decomposition $\mathbb{R}^N = V \oplus \text{Im}(C_p^T)$, for any given $\mathscr{E} \in V$ we have

$$A^T \mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2, \quad \mathscr{E}_1 \in V, \, \mathscr{E}_2 \in \text{Im}(C_p^T).$$

Applying $\mathscr{E}^T$ to (1), we get

$$\begin{cases} \mathscr{E}^T U'' - \Delta \mathscr{E}^T U + \mathscr{E}_1^T U + \mathscr{E}_2^T U = \mathscr{E}^T D \chi_\omega H & \text{in } (0, +\infty) \times \Omega, \\ \mathscr{E}^T U = 0 & \text{on } (0, +\infty) \times \Gamma. \end{cases} \tag{27}$$

Since the projections $\mathscr{E}^T U$ and $\mathscr{E}_1^T U$ are independent of applied controls H, and the mapping $H \to U$ is linear, it follows that

$$\mathscr{E}_2^T U = \mathscr{E}^T D \chi_\omega H, \quad \forall \, H \in \left(L^2\left(0, +\infty; L^2(\Omega)\right)\right)^M. \tag{28}$$

Let $H = D^T \mathscr{E} h$ with $h \in L^2\big(0, +\infty; L^2(\Omega)\big)$. By Proposition 1, the mapping

$$\mathscr{R}\colon h \longrightarrow U \tag{29}$$

is continuous from $L^2_{\text{loc}}\big(0, +\infty; L^2(\Omega)\big)$ to $\big(C^0_{\text{loc}}\big(0, +\infty; H^1_0(\Omega)\big) \cap C^1_{\text{loc}}\big(0, +\infty; L^2(\Omega)\big)\big)^N$. By [10, Theorem 4.1], or [12, Corollary 5, p. 86], the embedding

$$C^0\big([T, 2T]; H^1_0(\Omega)\big) \cap C^1\big([T, 2T]; L^2(\Omega)\big) \longrightarrow L^2\big(T, 2T; L^2(\Omega)\big) \tag{30}$$

is compact, so is the mapping $\mathscr{R}$ from $L^2\big(0, T; L^2(\Omega)\big)$ to $(L^2\big(0, T; L^2(\Omega)\big))^N$. It follows from (28) with $H = D^T \mathscr{E} h$ that

$$\|D^T \mathscr{E}\|^2 \|h\|_{L^2(0,T;L^2(\Omega))} \leqslant \|\mathscr{E}_2^T\| \|\mathscr{R} h\|_{(L^2(0,T;L^2(\Omega)))^N}. \tag{31}$$

The compactness of $\mathscr{R}$ leads to $D^T \mathscr{E} = 0$, namely, $V \subseteq \text{Ker}(D^T)$.

Now, writing $\mathscr{E}_2 = C_p^T x$, it follows from (28) that

$$t \geqslant 0\colon \quad \mathscr{E}_2^T U = x^T C_p U = 0. \tag{32}$$

Since the reduced system (14) is approximately controllable, the state variable $Y_p = C_p U$ can approximately touch any given target in $\big(H^1_0(\Omega) \times L^2(\Omega)\big)^{N-p}$. Then it follows from (32) that $x = 0$, then $\mathscr{E}_2 = 0$. Therefore, the subspace V is invariant for A^T.

(c)$\Longrightarrow$(a). Noting that $\dim(V) = p$, by Lemma 5, we get

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) \leqslant N - p, \tag{33}$$

which together with (16) implies (21). □

The approximate synchronization by p-groups under Kalman's rank condition (21) requires that A^T should be diagonalizable by blocks on the basis: $\mathbb{R}^N = \operatorname{Im}(C_p^T) \bigoplus V$. The following result confirms the converse.

Proposition 7. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary Γ and ω be a subdomain of Ω. Assume that $\operatorname{Im}(C_p^T)$ admits a supplement V, both $\operatorname{Im}(C_p^T)$ and V being invariant for A^T. Define D by $\operatorname{Ker}(D^T) = V$. Then, system* (1) *is approximately synchronizable by p-groups with Kalman's rank condition* (21).

Proof. Noting $\dim(V) = p$, Lemma 5 implies (21) and

$$\operatorname{Im}(C_p^T) \cap \operatorname{Ker}(D, AD, \dots, A^{N-1}D)^T = \operatorname{Im}(C_p^T) \cap V = \{0\}. \tag{34}$$

It follows that

$$\operatorname{rank} C_p(D, AD, \dots, A^{N-1}D) = \operatorname{rank}(C_p) = N - p. \tag{35}$$

Thus by Theorem 3, we get the approximate synchronization by p-groups of system (1). □

3. Stability of approximate internal synchronization by groups

In this section, we will examine the minimality of Kalman's rank condition (21) from the point of view of controllability. We show that the independence of the approximately synchronizable state by groups with respect to applied controls, the linear independence of the components of the approximately synchronizable state by groups, and the non-extensibility of approximate synchronization, these properties are all the consequences of the minimality of Kalman's rank condition (21), and vice versa.

We first construct the extension matrix. For $1 \leqslant i \leqslant m$, let λ_i be the eigenvalues of A^T and denote by

$$\mathscr{E}_{i0} = 0, \quad A^T \mathscr{E}_{ij} = \lambda_i \mathscr{E}_{ij} + \mathscr{E}_{i,j-1}, \quad 1 \leqslant j \leqslant d_i, \tag{36}$$

the corresponding Jordan chain (see [4]).

Assume that A satisfies the condition of C_p-compatibility (12), namely, $\operatorname{Im}(C_p^T)$ is invariant for A^T. Then[1] the Jordan chain of A^T contained in $\operatorname{Im}(C_p^T)$ can be extended by adding root vectors to a Jordan chain in $\mathbb{C}^N$. Then we can find a set of indices:

$$I = \left\{ i : \mathscr{E}_{i\alpha_i} \in \operatorname{Im}(C_p^T) \text{ with } 1 \leqslant \alpha_i \leqslant d_i \right\} \tag{37}$$

such that

$$\operatorname{Im}(C_p^T) = \bigoplus_{i \in I} \operatorname{Span}\left\{ \mathscr{E}_{i1}, \dots, \mathscr{E}_{i\alpha_i} \right\}. \tag{38}$$

Define the extension matrix of order $(N - q) \times N$ by

$$\operatorname{Im}(C_q^T) = \bigoplus_{i \in I} \operatorname{Span}\left\{ \mathscr{E}_{i1}, \dots, \mathscr{E}_{id_i} \right\} \quad \text{with} \quad \operatorname{Ker}(C_q) = \operatorname{Span}\{\epsilon_1, \dots, \epsilon_q\}. \tag{39}$$

where

$$q = N - \sum_{i \in I} d_i. \tag{40}$$

We say that system (1) is approximately C_q-synchronizable if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in \left(H_0^1(\Omega) \times L^2(\Omega)\right)^N$, there exists a sequence $\{H_n\}_{n \in \mathbb{N}}$ of internal controls in $\left(L^2\left(0, +\infty; L^2(\Omega)\right)\right)^M$

[1] This property can be verified after a basis transformation, and $\operatorname{Im}(C_p^T)$ will be called strongly A-marked in [2].

with compact support in $[0,T]$, such that the sequence $\{U_n\}_{n\in\mathbb{N}}$ of the corresponding solutions to system (1) satisfies

$$C_qU_n \longrightarrow 0 \quad \text{in } \Big(C^0_{\text{loc}}\big([T,+\infty);H^1_0(\Omega)\big)\cap C^1_{\text{loc}}\big([T,+\infty);L^2(\Omega)\big)\Big)^{N-q} \tag{41}$$

as $n\to+\infty$. Furthermore, if there exists some functions $v_1,\dots,v_q$ such that

$$U_n \longrightarrow \sum_{s=1}^{q} v_s\epsilon_s \quad \text{in } \Big(C^0_{\text{loc}}\big([T,+\infty);H^1_0(\Omega)\big)\cap C^1_{\text{loc}}\big([T,+\infty);L^2(\Omega)\big)\Big)^{N} \tag{42}$$

as $n\to+\infty$, system (1) will be called approximately C_q-synchronizable in the pining sense.

Lemma 8. *Let C_q be a full row-rank matrix of order $N\times(N-q)$, such that* $\text{Ker}(C_q)\subseteq\text{Ker}(C_p)$ *and $C_qC_q^T=I$. We have*

$$C_p = C_pC_q^TC_q. \tag{43}$$

Furthermore, $C_pC_q^T$ is a full row-rank matrix of order $(N-p)\times(N-q)$, such that

$$\text{Ker}(C_pC_q^T) = C_q\,\text{Ker}(C_p). \tag{44}$$

Theorem 9. *Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with smooth boundary Γ and ω be a subdomain of Ω. Let A satisfy the condition of C_p-compatibility* (12). *The following assertions are equivalent:*

(a) *system* (1) *is approximately synchronizable by p-groups under Kalman's rank condition* (21)*;*
(b) *system* (1) *is approximately synchronizable by p-groups in the pinning sense, and the approximately synchronizable state by p-groups $u=(u_1,\dots,u_p)^T$ is independent of applied controls;*
(c) *system* (1) *is approximately synchronizable by p-groups in the pinning sense and the components $u_1,\dots,u_p$ of the approximately synchronizable state by p-groups are linearly independent;*
(d) *system* (1) *is approximately synchronizable by p-groups in the pinning sense, and this property cannot be extended to an approximate C_q-synchronization with $q<p$.*

Proof. (a)⟹(b). Let $V=\text{Span}\{\mathscr{E}_1,\dots,\mathscr{E}_p\}$ denote the largest subspace invariant for A^T and contained in $\text{Ker}(D^T)$. Let $\text{Im}(Q_p)=V^\perp$. Noting (24), by [7, Proposition 2.8], $\text{Im}(C_p^T)$ and $V^\perp$ are bi-orthonormal and $\text{Ker}(C_p)\oplus\text{Im}(Q_p)=\mathbb{R}^N$. Without loss of generality, we may assume that $C_pQ_p=I_{N-p}$ and $E_r^Te_s=\delta_{rs}$ for all $1\leqslant r,s\leqslant p$. A straightforward computation gives

$$U_n = \sum_{r=1}^{p}\psi_re_r + Q_pC_pU_n, \tag{45}$$

where U_n denotes the solution to system (1) with H replaced by H_n and U replaced by U_n respectively. The projection $\psi_r=\mathscr{E}_r^TU_n$ for $r=1,\dots,p$ are governed by the homogeneous system (26), therefore, independent of applied controls. Noting (11), we get (17) with $u_r=\psi_r$ for $r=1,\dots,p$.

(b)⟹(c). Assume by absurd that there exist p reals $c_1,\dots,c_p$ not all zero, such that

$$t\geqslant T:\qquad \sum_{r=1}^{p}c_r(n_r-n_{r-1})u_r=0 \tag{46}$$

for any given initial data $(\widehat{U}_0,\widehat{U}_1)$ in the space $\big(H^1_0(\Omega)\times L^2(\Omega)\big)^N$. Define the row vector c_{N-p+1} by

$$c_{N-p+1}=c_1e_1^T+\cdots+c_pe_p^T. \tag{47}$$

We construct the extended matrix of order $(N-p+1)\times N$ by

$$C_{p-1}=\begin{pmatrix}C_p\\ c_{N-p+1}\end{pmatrix}. \tag{48}$$

By (17) and (46), system (1) possesses the approximate synchronization by $(p-1)$-groups in the pinning sense. Furthermore, by [9, Theorem 8.7], system (1) possesses the approximate C_q-synchronization with $q < p$. On the other hand, A satisfies the condition of C_q-compatibility (12) with C_p replaced by C_q, by [7, Proposition 2.15], there exists a matrix A_q of order $(N-q)$, such that $C_qA = A_qC_q$. Setting $Y_q = C_qU$, the approximate C_q-synchronization implies the approximate controllability of the following reduced system:

$$\begin{cases} Y_q'' - \Delta Y_q + A_qY_q = C_qD\chi_\omega H & \text{in } (0,+\infty)\times\Omega, \\ Y_q = 0 & \text{on } (0,+\infty)\times\Gamma. \end{cases} \tag{49}$$

By Lemma 8, $C_pC_q^T$ is a full row-rank matrix of order $(N-p)\times(N-q)$. Using (13) and (44), we get

$$C_pC_q^TA_q\operatorname{Ker}(C_pC_q^T) = C_pC_q^TA_qC_q\operatorname{Ker}(C_p) = C_pC_q^TC_qA\operatorname{Ker}(C_p). \tag{50}$$

Next, by (12) and (43), we get

$$C_pC_q^TC_qA\operatorname{Ker}(C_p) = C_pA\operatorname{Ker}(C_p) \subseteq C_p\operatorname{Ker}(C_p) = \{0\}. \tag{51}$$

Then, it follows that

$$A_q\operatorname{Ker}(C_pC_q^T) \subseteq \operatorname{Ker}(C_pC_q^T), \tag{52}$$

namely, A_q satisfy the condition of $C_pC_q^T$-compatibility. By [7, Proposition 2.15] with C_p replaced by $C_pC_q^T$, there exists a matrix A_{pq} of order $(N-p)$, such that

$$C_pC_q^TA_q = A_{pq}C_pC_q^T. \tag{53}$$

Applying the matrix $C_pC_q^T$ to system (49), we get

$$\begin{cases} C_pC_q^TY_q'' - \Delta C_pC_q^TY_q + A_{pq}C_pC_q^TY_q = 0 & \text{in } (T,+\infty)\times\Omega, \\ C_pC_q^TY_q = 0 & \text{on } (T,+\infty)\times\Gamma. \end{cases} \tag{54}$$

With the initial data at $t = T$,

$$t = T: \quad C_pC_q^TY_q = 0, \quad C_pC_q^TY_q' = 0, \tag{55}$$

in the homogeneous system (54), we get

$$t \geqslant T: \quad C_pC_q^TY_q = C_pC_q^TY_q' = 0. \tag{56}$$

Now we return to the reduced system (49) with the initial data $(\widehat{U}_0,\widehat{U}_1) = (0,0)$. By the approximate controllability, for any given functions $\big(Y_q(T),Y_q'(T)\big) \in \big(H_0^1(\Omega)\times L^2(\Omega)\big)^{N-q}$, there exists a sequence $\{H_n\}_{n\in\mathbb{N}}$ of internal controls, such that the sequence $\{U_n\}_{n\in\mathbb{N}}$ of the corresponding solutions satisfies

$$t = T: \quad C_q(U_n,U_n') \longrightarrow (Y_q,Y_q') \quad \text{as } n\to+\infty, \tag{57}$$

Applying $C_pC_q^T$ to (57) and noting (43), we get

$$t = T: \quad C_p(U_n,U_n') \longrightarrow C_pC_q^T(Y_q,Y_q') \quad \text{as } n\to+\infty, \tag{58}$$

which together with (56) gives

$$C_p(U_n,U_n') \longrightarrow 0 \quad \text{in } \Big(C^0_{\text{loc}}\big([T,+\infty);H_0^1(\Omega))\times L^2(\Omega)\big)\Big)^{N-p} \tag{59}$$

as $n\to+\infty$.

On the other hand, by the independence of $(u_1,\dots,u_p)^T$ with respect to applied controls, we deduce that $(u_1,\dots,u_p)^T \equiv 0$, which together with (59) gives

$$(U_n,U_n') \longrightarrow 0 \quad \text{in } \Big(C^0_{\text{loc}}\big([T,+\infty);H_0^1(\Omega)\times L^2(\Omega)\big)\Big)^{N} \tag{60}$$

as $n\to+\infty$. Passing to the limit in (57) as $n\to+\infty$, we get

$$t = T: \quad Y_q = Y_q' = 0. \tag{61}$$

Thus, we get $\mathrm{Ker}(C_pC_q^T) = \{0\}$, which together with (44) gives

$$C_q\,\mathrm{Ker}(C_p) = \mathrm{Ker}(C_pC_q^T) = \{0\}, \tag{62}$$

namely, $\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_q)$, which contradicts the condition $q < p$.

(c)$\Longrightarrow$(d). Assume that system (1) can be extended to an approximate C_q-synchronization. Applying C_q to (17), we get

$$t \geqslant T: \quad \sum_{r=1}^{p} C_q e_r u_r = 0 \tag{63}$$

for all the initial data $(\widehat{U}_0, \widehat{U}_1) \in \left(H_0^1(\Omega) \times L^2(\Omega)\right)^N$. Since $\mathrm{Ker}(C_q) \subset \mathrm{Ker}(C_p)$, there exists some $r(1 \leqslant r \leqslant p)$ such that $C_q e_r \neq 0$. This contradicts the linear independence of the function $u_1, \dots, u_p$.

(d)$\Longrightarrow$(a). Since system (1) is not approximately C_q-synchronizable with $q < p$, we have $C_q = C_p$ and $p = q$ in (39). It follows that

$$\mathrm{Im}(C_p^T) = \bigoplus_{i \in I} \mathrm{Span}\{\mathscr{E}_{i1}, \dots, \mathscr{E}_{id_i}\}. \tag{64}$$

Noting (16), we can write

$$\mathrm{rank}(D, AD, \dots, A^{N-1}D) = N - p' \quad \text{with } p' \leqslant p. \tag{65}$$

Let $V = \mathrm{Ker}(D, AD, \dots, A^{N-1}D)^T$ denote the largest subspace invariant for A^T and contained in $\mathrm{Ker}(D^T)$. Moreover, noting (64) and (24), we have

$$V \cap \bigoplus_{i \in I} \mathrm{Span}\{\mathscr{E}_{i1}, \dots, \mathscr{E}_{id_i}\} = V \cap \mathrm{Im}(C_p^T) = \{0\}. \tag{66}$$

Then, we have

$$V \subseteq \bigoplus_{i \in I^c} \mathrm{Span}\{\mathscr{E}_{i1}, \dots, \mathscr{E}_{id_i}\}. \tag{67}$$

Noting (64) and $\dim \mathrm{Im}(C_p^T) = N - p$, we have

$$\dim \bigoplus_{i \in I^c} \mathrm{Span}\{\mathscr{E}_{i1}, \dots, \mathscr{E}_{id_i}\} = p. \tag{68}$$

If $\dim(V) = p' < p$, there exists at least an integer $i_0 \in I^c$ and $\alpha_{i_0} < d_{i_0}$, such that

$$\mathscr{E}_{i_0 1}, \mathscr{E}_{i_0 2}, \dots, \mathscr{E}_{i_0 \alpha_{i_0}-1} \in V \qquad \text{and} \qquad \mathscr{E}_{i_0 \alpha_{i_0}} \notin V. \tag{69}$$

For $j = 1, \dots, m-1$, applying the vectors $\mathscr{E}_{i_0 j}^T$ to system (1) and setting $\psi_j = \mathscr{E}_{i_0 j}^T U$, we get a system on the sub-chain $\mathscr{E}_{i_0 1}, \mathscr{E}_{i_0 2}, \dots, \mathscr{E}_{i_0 \alpha_{i_0}-1}$:

$$\psi_0 = 0, \quad \begin{cases} \psi_j'' - \Delta\psi_j + \lambda_{i_0}\psi_j + \psi_{j-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \psi_j = 0 & \text{on } (0, +\infty) \times \Gamma. \end{cases} \tag{70}$$

Furthermore, applying $\mathscr{E}_{i_0 \alpha_{i_0}}^T$ to system (1) and setting $\psi_{\alpha_{i_0}} = \mathscr{E}_{i_0 \alpha_{i_0}}^T U$, we get an inhomogeneous system:

$$\begin{cases} \psi_{\alpha_{i_0}}'' - \Delta\psi_{\alpha_{i_0}} + \lambda_{i_0}\psi_{\alpha_{i_0}} + \psi_{\alpha_{i_0}-1} = \mathscr{E}_{i_0 \alpha_{i_0}}^T D\chi_\omega H & \text{in } (0, +\infty) \times \Omega, \\ \psi_{\alpha_{i_0}} = 0 & \text{on } (0, +\infty) \times \Gamma. \end{cases} \tag{71}$$

Define a function θ by

$$\begin{cases} \theta'' - \Delta\theta + \lambda_{i_0}\theta + \psi_{\alpha_{i_0}-1} = 0 & \text{in } (0, +\infty) \times \Omega, \\ \theta = 0 & \text{on } (0, +\infty) \times \Gamma. \end{cases} \tag{72}$$

The function $\psi_{\alpha_{i_0}-1}$ is independent of applied controls, so is the function θ. Then, inserting the new variable

$$\eta = \psi_{\alpha_{i_0}} - \theta \tag{73}$$

into system (71), we get the following self-closed system:

$$\begin{cases}\eta'' - \Delta\eta + \lambda_{i_0}\eta = \mathscr{E}^T_{i_0\alpha_{i_0}} D\chi_\omega H & \text{in } (0,+\infty)\times\Omega,\\ \eta = 0 & \text{on } (0,+\infty)\times\Gamma.\end{cases} \tag{74}$$

Thus, setting

$$\widetilde{A}_p = \begin{pmatrix} A_p & 0\\ 0 & \lambda_{i_0}\end{pmatrix}, \quad \widetilde{D}_p = \begin{pmatrix} C_p D\\ \mathscr{E}^T_{i_0\alpha_{i_0}} D\end{pmatrix}, \quad \widetilde{Y}_p = \begin{pmatrix} Y_p\\ \eta\end{pmatrix}, \tag{75}$$

and combining the two systems (14) and (74) together, we get an extended system:

$$\begin{cases}\widetilde{Y}_p'' - \Delta\widetilde{Y}_p + \widetilde{A}_p\widetilde{Y}_p = \widetilde{D}_p\chi_\omega H & \text{in } (0,+\infty)\times\Omega,\\ \widetilde{Y}_p = 0 & \text{on } (0,+\infty)\times\Gamma.\end{cases} \tag{76}$$

Let us admit for the moment the following rank condition:

$$\operatorname{rank}(\widetilde{D}_p, \widetilde{A}_p\widetilde{D}_p, \ldots, \widetilde{A}_p^{N-p}\widetilde{D}) = N - p + 1. \tag{77}$$

By Theorem 2 with p replaced by $(p-1)$, system (76) is approximately controllable. Then, for any given initial data $(\widehat{U}_0, \widehat{U}_1)$, there exists a sequence of controls $\{H_n\}_{n\in\mathbb{N}}$, such that the sequence of corresponding solutions $\{\widetilde{Y}_{pn}\}_{n\in\mathbb{N}}$ to system (76) satisfies

$$t = T: \quad \widetilde{Y}_{pn} = \begin{pmatrix} C_p U_n\\ \mathscr{E}^T_{i_0\alpha_{i_0}} U_n - \eta\end{pmatrix} \longrightarrow \begin{pmatrix} 0\\ -\eta\end{pmatrix} \tag{78}$$

as $n\to\infty$, namely,

$$t = T: \quad C_p U_n \longrightarrow 0 \quad \text{and} \quad \mathscr{E}^T_{i_0\alpha_{i_0}} U_n \longrightarrow 0 \tag{79}$$

as $n\to+\infty$. Noting (14) with $Y_{pn} = C_p U_n$ and that H_n has the compact support in $[0, T]$, we deduce from the first condition in (79) that

$$t \geqslant T: \quad C_p U_n \longrightarrow 0 \quad \text{as } n\to+\infty. \tag{80}$$

Then the approximate pinning synchronization by p-groups implies the existence of functions $u_1, \ldots, u_p$, such that

$$U_n \longrightarrow \sum_{r=1}^{p} e_r u_r \quad \text{in } \Big(C^0_{\text{loc}}\big([T,+\infty); H^1_0(\Omega)\big)\cap C^1_{\text{loc}}\big([T,+\infty); L^2(\Omega)\big)\Big)^N \tag{81}$$

as $n\to+\infty$. Passing to the limit in the second condition in (79), we get

$$t = T: \quad \sum_{r=1}^{p}\mathscr{E}^T_{i_0\alpha_{i_0}} e_r u_r = 0. \tag{82}$$

By [13, Theorem 3.3], $u = (u_1, \ldots, u_p)^T$ fills the subspace $\big(H^1_0(\Omega)\times L^2(\Omega)\big)^p$, then it follows from (82) that

$$\mathscr{E}^T_{i_0\alpha_{i_0}} e_1 = 0, \ \ldots, \ \mathscr{E}^T_{i_0\alpha_{i_0}} e_p = 0, \tag{83}$$

namely, $\mathscr{E}_{i_0\alpha_{i_0}} \in \{\operatorname{Ker}(C_p)\}^\perp = \operatorname{Im}(C_p^T)$. Noting (64) and $i_0 \in I^c$, this contradicts the condition $p' < p$.

Finally, we show the rank condition (77). Let

$$\widetilde{x}_p = \begin{pmatrix} x_p\\ y\end{pmatrix} \in \mathbb{R}^{N-p+1} \quad \text{with } x_p \in \mathbb{R}^{N-p} \text{ and } y \in \mathbb{R},$$

such that $\widetilde{A}_p^T\widetilde{x}_p = \lambda_{i_0}\widetilde{x}_p$, namely,

$$\begin{cases} D^T(C_p^T x_p + y\mathscr{E}_{i_0\alpha_{i_0}}) = 0,\\ A_p^T x_p = \lambda_{i_0} x_p.\end{cases} \tag{84}$$

We claim that $y = 0$. Otherwise, without loss of generality, we may take $y = 1$. Using (13), (36), (3) and the first formula in (84), we successively get

$$\begin{cases} A^T(\mathscr{E}_{i_0\alpha_{i_0}} + C_p^T x_p) = \mathscr{E}_{i_0\alpha_{i_0}-1} + \lambda_{i_0}(\mathscr{E}_{i_0\alpha_{i_0}} + C_p^T x_p) \subseteq V + \lambda_{i_0}(\mathscr{E}_{i_0\alpha_{i_0}} + C_p^T x_p), \\ \mathscr{E}_{i_0\alpha_{i_0}} + C_p^T x_p \in \operatorname{Ker}(D^T). \end{cases} \tag{85}$$

Since V is also invariant for A^T and contained in $\operatorname{Ker}(D^T)$, so is the subspace

$$W = V \bigoplus \operatorname{Span}\{\mathscr{E}_{i_0\alpha_{i_0}} + C_p^T x_p\}. \tag{86}$$

Noting (24) and (3), we get

$$V \cap \operatorname{Span}\{\mathscr{E}_{i_0\alpha_{i_0}} + C_p^T x_p\} \subseteq V \cap \operatorname{Span}\{\mathscr{E}_{i_0\alpha_{i_0}}\} + V \cap \operatorname{Im}(C_p^T) = \{0\}. \tag{87}$$

Thus, by Lemma 5, we get

$$\operatorname{rank}(D, AD, \ldots, A^{N-1}D) \leqslant N - \dim(W) = N - (p' + 1), \tag{88}$$

which contradicts (65). Thus, inserting $y = 0$ in (84), x_p is an eigenvector of A_p^T contained in $\operatorname{Ker}(D_p^T)$. On the other hand, condition (15) implies that the largest subspace invariant for A_p^T and contained in $\operatorname{Ker}(D_p^T)$ is reduced to $\{0\}$, then $x_p = 0$, namely, $\widetilde{x}_p = 0$. Consequently, the largest subspace invariant for $\widetilde{A}_p^T$ and contained in $\operatorname{Ker}(\widetilde{D}_p^T)$ is reduced to $\{0\}$, which gives Kalman's rank condition (77). □

The present work mainly concentrates on the internal controllability and synchronization for the problem with Dirichlet boundary condition. However, the approaches can be easily adapted to the problem with Neumann boundary condition, and some other problems. In particular, the sufficiency of Kalman's rank condition (21) for the approximate synchronization by p-groups is only used in the implication (d) $\Longrightarrow$ (a), and the other ones have an abstract character, then can be applicable to the approximate synchronization by groups for the problems with boundary controls considered in [7].

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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