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Comptes Rendus
Mathématique

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Volume 362 (2024), p. 1379-1388

Online since: 14 November 2024

https://doi.org/10.5802/crmath.649

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www.centre-mersenne.org — e-ISSN : 1778-3569

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Comptes Rendus. Mathématique
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https://doi.org/10.5802/crmath.649

Research article / *Article de recherche*
Functional analysis, Geometry and Topology / *Analyse fonctionnelle, Géométrie et Topologie*

Maximal exponent of the Lorentz cones

Exposant maximal des cônes de Lorentz

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Abstract. We show that the maximal exponent (i.e., the minimum number of iterations required for a primitive map to become strictly positive) of the n-dimensional Lorentz cone is equal to n. As a byproduct, we show that the optimal exponent in the quantum Wielandt inequality for qubit channels is equal to 3.

Résumé. Nous démontrons que l'exposant maximal (c'est-à-dire le nombre minimal d'itérations requises pour qu'une application primitive devienne strictement positive) du cône de Lorentz de dimension n est égal à n. Nous montrons également que l'exposant optimal dans l'inégalité de Wielandt quantique pour des canaux agissant sur un qubit est égal à 3.

Keywords. Lorentz cone, Maximal exponent, Quantum Wielandt inequality.

Mots-clés. Cône de Lorentz, exposant maximal, inégalité de Wielandt quantique.

2020 Mathematics Subject Classification. 52A20, 51M04.

Funding. The first author was supported by ANR (France) under the grant ESQuisses (ANR-20-CE47-0014-01). The second author was awarded a scholarship from China Scholarship Council (CSC) to study in France as a visiting PhD Student.

Manuscript received 12 December 2023, revised 19 February 2024 and 24 April 2024, accepted 6 May 2024.

Our main object of study is the n-dimensional *Lorentz cone* (also known as second-order cone, quadratic cone, or ice-cream cone), which is the cone $\mathscr{L}_n \subset \mathbf{R}^n$ defined as

$$\mathscr{L}_n = \left\{(x_1,\dots,x_n) \in \mathbf{R}^n \;:\; x_n \geqslant \left(x_1^2+\cdots+x_{n-1}^2\right)^{1/2}\right\}.$$

We denote by $\operatorname{int}(\mathscr{L}_n)$ the interior of $\mathscr{L}_n$. We say that a linear map $\Psi : \mathbf{R}^n \to \mathbf{R}^n$ is *$\mathscr{L}_n$-positive* if $\Psi(\mathscr{L}_n) \subset \mathscr{L}_n$, *strictly $\mathscr{L}_n$-positive* if $\Psi(\mathscr{L}_n \setminus \{0\}) \subset \operatorname{int}(\mathscr{L}_n)$ and *$\mathscr{L}_n$-primitive* if it is $\mathscr{L}_n$-positive and if there exists an integer $k \geqslant 1$ such that Ψ^k is strictly $\mathscr{L}_n$-positive. If Ψ is $\mathscr{L}_n$-primitive, the smallest such k is called the *$\mathscr{L}_n$-primitivity index* of Ψ and denoted $\gamma(\Psi)$. The main result of this paper is the following theorem.

Theorem 1. *Let $n \geqslant 1$. If $\Psi : \mathbf{R}^n \to \mathbf{R}^n$ is $\mathscr{L}_n$-primitive, then $\gamma(\Psi) \leqslant n$. Moreover, there is a $\mathscr{L}_n$-primitive map $\Psi : \mathbf{R}^n \to \mathbf{R}^n$ such that $\gamma(\Psi) = n$.*

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* Corresponding author

As we explain later, this theorem can be seen as the affine or projective analogue of the following classical result by Pták [9]: if A is an $n \times n$ matrix, then $\rho(A) = \|A\|$ if and only if $\|A^n\| = \|A\|^n$ (we denote by $\rho(A)$ and $\|A\|$ respectively the spectral radius and operator norm of A).

The paper is organized as follows. Section 1 contains background and connects to related works, as well as a reformulation of Theorem 1 involving affine self-maps of the Euclidean ball. The bound $\gamma(\Psi) \leqslant n$ is proved in Section 2 and the sharpness of this inequality follows from the example constructed in Section 3. Finally, when specialized to $n = 4$, our result has an implication in quantum information theory which we develop in Section 4.

1. Introduction

1.1. *Cones, maximal exponent*

We work in a finite-dimensional real vector space V. A subset $\mathscr{C} \subset V$ is said to be a *convex cone* if, for every $x, y \in \mathscr{C}$ and $s, t \in \mathbf{R}_+$, we have $sx + ty \in \mathscr{C}$. A cone $\mathscr{C}$ is said to be *proper* if it is closed, salient (i.e., $\mathscr{C} \cap (-\mathscr{C}) = \{0\}$) and generating (i.e., $\mathscr{C} - \mathscr{C} = V$).

We extend to the setting of a proper cone $\mathscr{C} \subset V$ the concepts of positivity and primitivity defined earlier for the Lorentz cones. A linear map $\Psi : V \to V$ is *$\mathscr{C}$-positive* if $\Psi(\mathscr{C}) \subset \mathscr{C}$, *strictly $\mathscr{C}$-positive* if $\Psi(\mathscr{C} \setminus \{0\}) \subset \operatorname{int}(\mathscr{C})$ and *$\mathscr{C}$-primitive* if it is $\mathscr{C}$-positive and if there exists an integer $k \geqslant 1$ such that Ψ^k is strictly $\mathscr{C}$-positive. If Ψ is $\mathscr{C}$-primitive, the smallest such k is called the *$\mathscr{C}$-primitivity index* of Ψ and denoted $\gamma(\mathscr{C}, \Psi)$.

The *maximal exponent* of $\mathscr{C}$, denoted $\gamma(\mathscr{C})$, is the supremum of $\gamma(\mathscr{C}, \Psi)$ over all $\mathscr{C}$-primitive maps $\Psi : V \to V$. With this notation, the statement of Theorem 1 reads as the equality $\gamma(\mathscr{L}_n) = n$.

Given vector spaces V and V', two proper cones $\mathscr{C} \subset V$ and $\mathscr{C}' \subset V'$ are said to be *isomorphic* if there exists a linear bijection $f : V \to V'$ such that $f(\mathscr{C}) = \mathscr{C}'$. It is simple to check that two isomorphic cones have the same maximal exponent.

We use the finite-dimensional version of the Krein–Rutman theorem (see [3, Theorem 19.2]): every $\mathscr{C}$-positive operator Ψ has an eigenvector $x \in \mathscr{C}$ associated to the eigenvalue $\rho(\Psi)$ (the spectral radius of Ψ). If moreover Ψ is $\mathscr{C}$-primitive, then necessarily $\rho(\Psi) > 0$ (otherwise Ψ would be nilpotent, contradicting $\mathscr{C}$-primitivity) and $x \in \operatorname{int}(\mathscr{C})$.

1.2. *Duality*

If $\mathscr{C} \subset V$ is a cone, its *dual cone* is the cone in the dual vector space V^* defined as

$$\mathscr{C}^* = \{f \in V^* \,:\, \langle f, x \rangle \geqslant 0 \text{ for every } x \in \mathscr{C}\}.$$

If $\mathscr{C}$ is proper, then $\mathscr{C}^*$ is also proper. The bipolar theorem asserts that $(\mathscr{C}^*)^* = \mathscr{C}$ provided we identify the double dual space V^{**} with V. The Lorentz cone $\mathscr{L}_n$ is self-dual: if we identify the vector space $\mathbf{R}^n$ with its own dual using the standard inner product, then $\mathscr{L}_n^* = \mathscr{L}_n$.

A *sole* of a proper cone $\mathscr{C}$ is a set of the form $\{x \in \mathscr{C} \,:\, f(x) = \alpha\}$, where $f \in \operatorname{int}(\mathscr{C}^*)$ and $\alpha > 0$. If K is a sole of $\mathscr{C}$, then K is compact and $\mathscr{C} = \{\lambda x \,:\, x \in K, \lambda \geqslant 0\}$.

We have the relation

$$\operatorname{int}(\mathscr{C}) = \{x \in V \,:\, \langle f, x \rangle > 0 \text{ for every } f \in \mathscr{C}^* \setminus \{0\}\}.$$

Given a linear map $\Psi : V \to V$, we have the equivalences

$$\Psi \text{ is } \mathscr{C}\text{-positive} \iff \langle f, \Psi(x) \rangle \geqslant 0 \text{ for every } x \in \mathscr{C}, f \in \mathscr{C}^*$$

$$\Psi \text{ is strictly} \mathscr{C}\text{-positive} \iff \langle f, \Psi(x) \rangle > 0 \text{ for every } x \in \mathscr{C} \setminus \{0\}, f \in \mathscr{C}^* \setminus \{0\}.$$

It is clear from these formulas that Ψ is $\mathscr{C}$-positive (resp. strictly $\mathscr{C}$-positive, resp. $\mathscr{C}$-primitive) if and only if the adjoint map $\Psi^* : V^* \to V^*$ is $\mathscr{C}^*$-positive (resp. strictly $\mathscr{C}^*$-positive, resp. $\mathscr{C}^*$-primitive). Moreover the cones $\mathscr{C}$ and $\mathscr{C}^*$ have the same maximal exponents.

1.3. *Affine maximal exponent*

Let K be a convex body (i.e., a compact convex set of full dimension) in a finite-dimensional affine space W. An affine map $\Phi : W \to W$ is said to be *K-positive* if $\Phi(K) \subset K$, *strictly K-positive* if $\Phi(K) \subset \mathrm{int}(K)$ and *K-primitive* if it is K-positive and if there exists a positive integer k such that Φ^k is strictly K-positive. If Φ is K-primitive, the smallest integer k with this property is called the *affine K-primitivity index* of Φ and denoted $\gamma_{\mathrm{aff}}(K,\Phi)$. The *affine maximal exponent* of K, denoted $\gamma_{\mathrm{aff}}(K)$, is the supremum of $\gamma_{\mathrm{aff}}(K,\Phi)$ over all K-primitive affine maps $\Phi : W \to W$.

If X is a finite-dimensional normed space with unit ball B, observe that a linear map $T : X \to X$ is B-positive (resp. strictly B-positive) if and only if it has operator norm $\leqslant 1$ (resp. < 1). Moreover, T is B-primitive if and only if it has operator norm $\leqslant 1$ and spectral radius < 1. The supremum of $\gamma(B,T)$ over B-primitive linear maps T has been studied in the Banach space literature as the *critical exponent* of the normed space X. We refer to [10] for a survey on critical exponents.

Given affine spaces W and W', two convex bodies $K \subset W$ and $K' \subset W'$ are said to be *affinely isomorphic* if there exists an affine bijection $f : W \to W'$ such that $f(K) = K'$. It is simple to check that two affinely isomorphic cones have the same affine maximal exponent.

The next proposition states that the maximal exponent of a cone is the supremum of affine maximal exponents of its soles. While this statement is folklore, we could note locate it in the literature and include a proof.

Proposition 2. *Let $\mathscr{C} \subset V$ be a proper cone. Then*

$$\gamma(\mathscr{C}) = \sup_{K \text{ sole of } \mathscr{C}} \gamma_{\mathrm{aff}}(K). \tag{1}$$

Proof. Given $f \in \mathrm{int}(\mathscr{C}^*)$ and $\alpha > 0$, consider the affine hyperplane $W = \{x \in V \ : \ f(x) = \alpha\}$ and the sole of $\mathscr{C}$ given by $K = \mathscr{C} \cap W$. Any affine map $\Phi : W \to W$ can be extended uniquely into a linear map $\Psi : V \to V$. Moreover, the affine map Φ is K-positive (resp., strictly K-positive, K-primitive) if and only if the linear map Ψ is $\mathscr{C}$-positive (resp., strictly $\mathscr{C}$-positive, $\mathscr{C}$-primitive). We have therefore $\gamma_{\mathrm{aff}}(K,\Phi) = \gamma(\mathscr{C},\Psi)$ and the inequality $\geqslant$ in equation (1) follows by taking supremum over K and Φ.

Conversely, let $\Psi : V \to V$ be a $\mathscr{C}$-primitive map. Its spectral radius $\rho(\Psi)$ is nonzero and we may assume by rescaling that $\rho(\Psi) = 1$. By the Krein–Rutman theorem, the adjoint map Ψ^*, which is $\mathscr{C}^*$-primitive, admits an eigenvector $f \in \mathrm{int}(\mathscr{C}^*)$ for the eigenvalue 1. Consider the affine hyperplane $W = \{x \in V \ : \ f(x) = 1\}$ and the sole $K = \mathscr{C} \cap W$. Since $\Psi(W) \subset W$, the linear map Ψ induces by restriction a K-primitive affine map $\Phi : W \to W$. As before, we have $\gamma_{\mathrm{aff}}(K,\Phi) = \gamma(\mathscr{C},\Psi)$ and the inequality $\leqslant$ in equation (1) follows by taking supremum over Ψ. □

We denote by B_n the unit ball of the standard Euclidean space $\mathbf{R}^n$. Any sole of the Lorentz cone $\mathscr{L}_{n+1}$ is affinely isomorphic to B_n. By Proposition 2, Theorem 1 can be equivalently stated as follows.

Theorem 3. *For every integer $n \geqslant 1$, we have $\gamma_{\mathrm{aff}}(B_n) = n+1$.*

Sections 2 and 3 are devoted to the proof of Theorem 3: in Section 2 we prove that any B_n-primitive affine map $\Psi : \mathbf{R}^n \to \mathbf{R}^n$ satisfies $\gamma_{\mathrm{aff}}(B_n,\Psi) \leqslant n+1$, and in Section 3 we construct an example showing that this inequality is sharp.

1.4. *Related works*

The question of computing the maximal exponent of the Lorentz cone does not seem to have been considered in the literature and our main contribution is to fill this gap.

The study of maximal exponents of cones can be traced back to the classical result by Wielandt [16] which asserts that the maximal exponent of the cone $\mathbf{R}_+^n$ equals $(n-1)^2+1$ (Wielandt's original proof was only published posthumously in [14]). The maximal exponents of polyhedral cones have been studied in detail in the series of papers [5–7]. We also mention that there exist proper cones for which the maximal exponent is infinite (see [6, Section 6]).

Our result is closely related to Pták's theorem [9] stating that the critical exponent of the n-dimensional Euclidean space ℓ_2^n equals n. This means that if Φ is a linear contraction on ℓ_2^n with spectral radius < 1, then its nth iteration Φ^n maps the unit ball into its interior. Our Theorem 3 shows that for affine maps, one more iteration is necessary and sufficient to achieve this property.

Another cone of interest is the cone M_n^+ of $n \times n$ positive semidefinite matrices with complex entries. The connection with our work is that for $n = 2$, this cone is isomorphic to the Lorentz cone $\mathscr{L}_4$. The study of the maximal exponent of M_n^+ is relevant in quantum information theory in the context of the quantum Wielandt inequality which we review in Section 4.

2. Upper bound on the maximal exponent

Throughout this section and the following one, we fix an integer $n \geqslant 1$ and we use the terminology "positive", "strictly positive" and "primitive" to mean "B_n-positive", "strictly B_n-positive" and "B_n-primitive". We denote by $S^{n-1} = \partial B_n$ the unit sphere in the Euclidean space $\mathbf{R}^n$. Given a subset $X \subset \mathbf{R}^n$, we denote by $\operatorname{aff}(X)$ the affine subspace generated by X. We start with a simple lemma.

Lemma 4. *If an affine map $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ is positive and nonconstant, then $\Phi(\operatorname{int}(B_n)) \subset \operatorname{int}(B_n)$.*

Proof. Take $x \in \operatorname{int}(B_n)$ and assume by contradiction that $\Phi(x) \in S^{n-1}$. Let V be a open ball centered at x and contained in B_n. For every $y \in V$, the point $z = 2x - y$ is in V and we have $x = \frac{y+z}{2}$, hence $\Phi(x) = \frac{\Phi(y)+\Phi(z)}{2}$. Since $\Phi(x)$ is an extreme point of B_n, it follows that $\Phi(y) = \Phi(z) = \Phi(x)$. The affine function Φ is constant on V hence constant on $\mathbf{R}^n = \operatorname{aff}(V)$, leading to a contradiction. □

Given a positive map $\Phi : \mathbf{R}^n \to \mathbf{R}^n$, we introduce the set

$$C(\Phi) = S^{n-1} \cap \Phi(S^{n-1}). \tag{2}$$

A subset $A \subset S^{n-1}$ is said to be a *subsphere* if it satisfies the relation $A = S^{n-1} \cap \operatorname{aff}(A)$. We say that a subset of $\mathbf{R}^n$ is an *ellipsoid* if it is a linear image of B_n. The following observation is fundamental to our proof. In the three-dimensional case, it appears in [2, Proposition IV.6].

Lemma 5. *Let $\mathscr{E}$ be an ellipsoid such that $\mathscr{E} \subset B_n$. Then $\mathscr{E} \cap S^{n-1}$ is a subsphere.*

Proof. Assume first that $\mathscr{E}$ is origin-symmetric. In this case, there is an orthonormal basis $(x_1, \ldots, x_n)$ and numbers $\lambda_1, \ldots, \lambda_n$ in $[0,1]$ such that

$$\mathscr{E} = \left\{ \sum_{i=1}^{n} \lambda_i t_i x_i \ : \ (t_1, \ldots, t_n) \in B_n \right\}.$$

It is simple to check that $\mathscr{E} \cap S^{n-1}$ equals $F \cap S^{n-1}$, where $F \subset \mathbf{R}^n$ is the linear subspace spanned by $\{x_i \ : \ \lambda_i = 1\}$. This proves the lemma under the extra hypothesis that $\mathscr{E}$ is origin-symmetric.

Assume now that $\mathscr{E}$ is a general ellipsoid. If $\operatorname{card}(\mathscr{E} \cap S^{n-1}) \leqslant 1$, then $\mathscr{E} \cap S^{n-1}$ is a subsphere. Otherwise, $\mathscr{E} \cap S^{n-1}$ contains two distinct elements x and x'. Since the group $PO(1,n)$ of projective automorphisms of B_n acts transitively on the set of lines intersecting

$\operatorname{int}(B_n)$ [12, Theorem 3.1.6], we may find a projective transformation $\Theta : B_n \to B_n$ sending x and x' to a pair of antipodal points. The ellipsoid $\mathscr{F} = \Theta(\mathscr{E})$ intersects S^{n-1} in two antipodal points and is therefore origin-symmetric. Since Θ preserves subspheres and $\mathscr{E} \cap S^{n-1} = \Theta^{-1}(\mathscr{F} \cap S^{n-1})$, we conclude by reducing to the origin-symmetric case. □

We now show that a primitive affine map $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ satisfies $\gamma_{\text{aff}}(B_n, \Phi) \leqslant n+1$. If Φ is constant equal to $x \in B_n$, then necessarily $x \in \operatorname{int}(B_n)$ (otherwise Φ would not be primitive) and therefore $\gamma_{\text{aff}}(B_n, \Phi) = 1$. We now assume that Φ is nonconstant.

Given an integer $k \geqslant 0$, we set $A_k = C(\Phi^k)$. Since Φ is nonconstant, it follows from Lemma 4 that $A_{k+1} \subset A_k$. Assume that $A_{k+1} = A_k$ for some $k \geqslant 0$. Consider an element $x \in A_{k+1}$. There exists $y \in S^{n-1}$ such that $x = \Phi^{k+1}(y)$. The point $\Phi^k(y)$ belongs to A_k, hence to A_{k+1}, and therefore we have $\Phi^k(y) = \Phi^{k+1}(z)$ for some $z \in S^{n-1}$. It follows that $x = \Phi^{k+2}(z)$ and thus that x belongs to A_{k+2}. We proved that $A_{k+2} = A_{k+1} = A_k$ and therefore, by induction, $A_l = A_k$ for every $l \geqslant k$. Since Φ is primitive, it follows that $A_l = \emptyset$ for every $l \geqslant k$.

Let $N = \gamma_{\text{aff}}(B_n, \Phi)$ be the affine primitivity index of Φ. The previous paragraph shows that

$$\emptyset = A_N \subsetneq A_{N-1} \subsetneq \cdots \subsetneq A_2 \subsetneq A_1 \subsetneq A_0 = S^{n-1}.$$

By Lemma 5, each set A_k is a subsphere. If two subspheres A, A' satisfy $A \subsetneq A'$, then we have $\operatorname{aff}(A) \subsetneq \operatorname{aff}(A')$ and therefore $\dim \operatorname{aff}(A) < \dim \operatorname{aff}(A')$. The chain of inequalities

$$0 \leqslant \dim \operatorname{aff}(A_{N-1}) < \cdots < \dim \operatorname{aff}(A_2) < \dim \operatorname{aff}(A_1) < \dim \operatorname{aff}(A_0) = n$$

implies that $N \leqslant n+1$.

3. A map with large maximal exponent

Our goal is to give an example of an affine map $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ which is primitive and such that Φ^n is not strictly positive. Such a map satisfies $\gamma_{\text{aff}}(B_n, \Phi) \geqslant n+1$ and, together with the result from Section 2, allows us to conclude that $\gamma_{\text{aff}}(B_n) = n+1$.

Given an angle $\theta \in [-\pi/2, \pi/2]$, we denote by $E_{n,\theta}$ the "circle of latitude θ" defined as

$$E_{n,\theta} = \{(x_1, \ldots, x_n) \in S^{n-1} \,:\, x_n = \sin\theta\}.$$

Our first lemma shows that affine positive maps may send any circle of positive latitude to any circle of higher latitude.

Lemma 6. *Let $0 < \alpha < \beta < \pi/2$ and set $\lambda = \frac{\cos\beta}{\cos\alpha}$, $\mu = \frac{\tan\alpha}{\tan\beta}$. Define a map $\Psi : \mathbf{R}^n \to \mathbf{R}^n$ by the formula*

$$\Psi : (x_1, \ldots, x_n) \longmapsto \left(\lambda x_1, \ldots, \lambda x_{n-1}, \lambda\mu x_n + \sqrt{(1-\lambda^2)(1-\mu^2)}\right).$$

(a) *The map Ψ is a positive affine bijection.*
(b) *If $x \in E_{n,\alpha}$, then $\Psi(x) \in E_{n,\beta}$.*
(c) *If $x, y \in E_{n,\alpha}$, then $\|\Psi(x) - \Psi(y)\| = \lambda \|x - y\|$.*
(d) *If $x \in B_n$ is such that $\Psi(x) \in S^{n-1}$, then $x \in E_{n,\alpha}$.*

Proof. It is immediate to check that Ψ is affine and bijective, as well as property (c). Property (b) follows from the formula $\sin\beta = \lambda\mu\sin\alpha + \sqrt{(1-\lambda^2)(1-\mu^2)}$. To check positivity of Ψ, it suffices to show that $\|\Psi(x)\| \leqslant 1$ for any $x \in S^{n-1}$. Let $\theta \in [-\pi/2, \pi/2]$ be the latitude of x, i.e., such that $x \in E_{n,\theta}$. We compute

$$\begin{aligned} 1 - \|\Psi(x)\|^2 &= 1 - \lambda^2 \cos^2\theta - \left(\lambda\mu\sin\theta + \sqrt{(1-\lambda^2)(1-\mu^2)}\right)^2 \\ &= \left(\lambda\sqrt{1-\mu^2}\sin\theta - \mu\sqrt{1-\lambda^2}\right)^2 \\ &= \lambda^2(1-\mu^2)(\sin\theta - \sin\alpha)^2. \end{aligned}$$

The positivity of Ψ, together with property (d), follow from this formula. □

Lemma 7. *Let $A = (a_{ij})$ be a $n \times n$ positive definite symmetric matrix satisfying $a_{ii} = 1$ for every i in $\{1, \dots, n\}$. There is a number $\alpha \in (0, \pi/2)$ and vectors $x_1, \dots, x_n \in E_{n,\alpha}$ such that, for every i, j in $\{1, \dots, n\}$*

$$a_{ij} = \langle x_i, x_j \rangle.$$

Proof. It is well-known [4, Corollary 7.2.11] that we can find $y_1, \dots, y_n \in S^{n-1}$ such that $a_{ij} = \langle y_i, y_j \rangle$. Since A is invertible, the vectors $y_1, \dots, y_n$ are linearly independent and thus the hyperplane $H = \mathrm{aff}\{y_1, \dots, y_n\}$ does not contain 0. We may therefore find an orthogonal transformation $Q \in O(n)$ such that

$$Q(H) = \{(z_1, \dots, z_n) \in \mathbf{R}^n \ : \ z_n = \sin\alpha\}$$

for some $\alpha \in (0, \pi/2)$. The points $x_i = Q(y_i)$ have the desired property. □

Lemma 8. *Consider points $x_1, \dots, x_k$ and $y_1, \dots, y_k$ in S^{n-1}. The following are equivalent.*

(1) *There is $R \in O(n)$ such that $R(x_i) = y_i$ for every $i \in \{1, \dots, k\}$.*
(2) *For every i, j in $\{1, \dots, k\}$, we have $\|x_i - x_j\| = \|y_i - y_j\|$.*

Proof. It is clear that (1) implies (2). Now assume that (2) holds. Since all vectors involved are unit, we have $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle$ for every i, j. Moreover, for every $\lambda_1, \dots, \lambda_k$ we have

$$\left\| \sum_{i=1}^k \lambda_i y_i \right\|^2 = \sum_{i=1}^k \sum_{j=1}^k \lambda_i \lambda_j \langle y_i, y_j \rangle = \sum_{i=1}^k \sum_{j=1}^k \lambda_i \lambda_j \langle x_i, x_j \rangle = \left\| \sum_{i=1}^k \lambda_i x_i \right\|^2.$$

This shows that the map $\widehat{R} : \mathrm{span}\{x_1, \dots, x_k\} \to \mathrm{span}\{y_1, \dots, y_k\}$ defined by the formula

$$\widehat{R}\left(\sum_{i=1}^k \lambda_i x_i \right) = \sum_{i=1}^k \lambda_i y_i.$$

is well-defined and isometric. Finally, we extend $\widehat{R}$ to a linear isometry $R : \mathbf{R}^n \to \mathbf{R}^n$ by choosing any isometry from $\mathrm{span}\{x_1, \dots, x_k\}^\perp$ to $\mathrm{span}\{y_1, \dots, y_k\}^\perp$. By construction, we have $R \in O(n)$ and $R(x_i) = y_i$. □

We now construct a primitive map Φ such that Φ^n is not strictly positive. When $n = 3$, an example of such a construction is depicted in Figure 1.

Consider the following $n \times n$ matrix $A = (a_{ij})$, indexed by a parameter $c \in (0, 1)$

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 1 - c^{\min(i,j)} & \text{if } i \neq j \end{cases}$$

When c approaches 1, the matrix A converges to the identity matrix. We may therefore choose a value $c \in (0, 1)$ such that the matrix A is positive definite. By Lemma 7, we may find $\alpha \in (0, \pi/2)$ and vectors $x_1, \dots, x_n$ in $E_{n,\alpha}$ such that $a_{ij} = \langle x_i, x_j \rangle$. For $i \neq j$, we have

$$\|x_i - x_j\|^2 = 2 - 2a_{ij} = 2c^{\min(i,j)}.$$

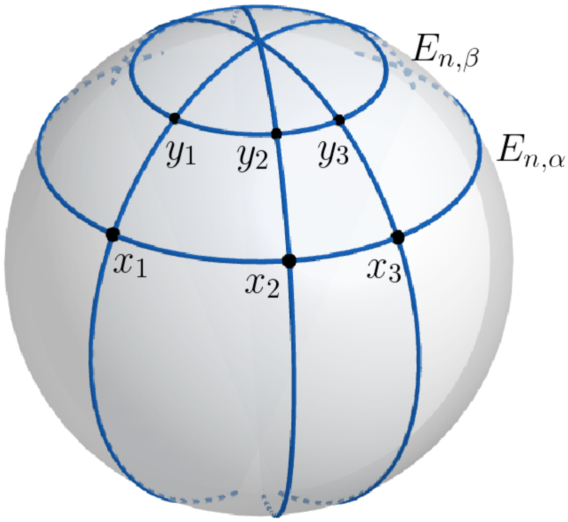

Figure 1. The affine map Φ is obtained as $R \circ \Psi$. The map Ψ is a positive affine map which preserves longitude and sends a point x_i with latitude α to a point y_i with latitude $\beta > \alpha$. The map R is a rotation chosen such that $R(y_1) = x_2$ and $R(y_2) = x_3$. It requires 4 iterations of Φ from the initial point x_1 before reaching the interior of the unit ball.

Define $\beta \in (\alpha, \pi/2)$ by the relation $\frac{\cos^2 \beta}{\cos^2 \alpha} = c$ and let Ψ be the affine map given by Lemma 6 (applied with the present values of α and β). For $1 \leqslant i \leqslant n$, set $y_i = \Psi(x_i)$. By Lemma 6 (b), we have $y_i \in E_{n,\beta}$. For $1 \leqslant i < j \leqslant n-1$, we compute using Lemma 6 (c)

$$\|y_i - y_j\|^2 = \frac{\cos^2 \beta}{\cos^2 \alpha} \|x_i - x_j\|^2 = 2c^{\min(i,j)+1} = \|x_{i+1} - x_{j+1}\|^2.$$

By Lemma 8, there exists $R \in O(n)$ such that $R(y_i) = x_{i+1}$ for $1 \leqslant i \leqslant n-1$. We define an affine bijection $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ by the formula $\Phi = R \circ \Psi$. We also set $x_{n+1} = \Phi(x_n)$, so that the relation $x_{i+1} = \Phi(x_i)$ holds for $1 \leqslant i \leqslant n$. Since $x_{n+1} = \Phi^n(x_1)$ belongs to S^{n-1}, it follows that Φ^n is not strictly positive.

Lemma 9. *The point x_0 defined as $x_0 = \Phi^{-1}(x_1)$ does not belong to B_n.*

Proof. Set $y_0 = \Psi(x_0) = R^{-1}(x_1)$. Consider the affine hyperplanes

$$V_1 = \mathrm{aff}\{y_0, \ldots, y_{n-1}\}$$
$$V_2 = \mathrm{aff}\{x_1, \ldots, x_n\}$$
$$V_3 = \mathrm{aff}\{y_1, \ldots, y_n\}$$

Since $V_2 \cap S^{n-1} = E_{n,\alpha}$ and $V_3 \cap S^{n-1} = E_{n,\beta}$ with $\alpha < \beta$, no element $S \in O(n)$ can satisfy the relation $S(V_2) = V_3$. Since $R(V_1) = V_2$, this implies that $V_1 \neq V_3$ and thus $y_0 \notin V_3$. It follows that $y_0 \in S^{n-1} \setminus E_{n,\beta}$ and therefore that $x_0 \notin B_n$ by Lemma 6 (d). □

We now show that the map Φ is primitive by proving that Φ^{n+1} is strictly positive. As in the proof of the previous section, we denote

$$A_k = C(\Phi^k) = S^{n-1} \cap \Phi^k(S^{n-1}).$$

For $1 \leqslant k \leqslant n+1$, the point x_k belongs to A_{k-1} (since $\Phi^{-(k-1)}(x_k) = x_1 \in S^{n-1}$) but not to A_k (since $\Phi^{-k}(x_k) = x_0 \notin S^{n-1}$ by Lemma 9). This shows that $A_{k-1} \neq A_k$. We have therefore a chain of strict inclusions

$$A_{n+1} \subsetneq A_n \subsetneq \cdots \subsetneq A_1 \subsetneq A_0 = S^{n-1}$$

and therefore as in the previous section (with the convention $\dim \emptyset = -1$)

$$\dim \operatorname{aff} A_{n+1} < \dim \operatorname{aff} A_n < \cdots < \dim \operatorname{aff} A_1 < \dim \operatorname{aff} A_0 = n.$$

This is only possible if $A_{n+1} = \emptyset$. It follows that Φ^{n+1} is strictly positive.

4. Maximal exponents for qubit channels

We refer to [1] for terminology from quantum information theory used in this section. Given an integer $n \geqslant 2$, let M_n be the algebra of $n \times n$ matrices with complex entries and $\mathsf{M}_n^+ \subset \mathsf{M}_n$ be the cone of positive semidefinite matrices. The maximal exponent $\gamma(\mathsf{M}_n^+)$ involves a supremum over positive maps (or, more precisely, over M_n^+-primitive maps). However in quantum information theory it is more natural to restrict the supremum to completely positive maps and to study the quantity

$$\gamma^{\mathrm{cp}}(\mathsf{M}_n^+) := \sup\{\gamma(\mathsf{M}_n^+, \Phi) \, : \, \Phi : \mathsf{M}_n \longrightarrow \mathsf{M}_n \text{ completely positive and } \mathsf{M}_n^+\text{-primitive}\}. \quad (3)$$

This quantity appears in [8, 13] in the context of the *quantum Wielandt inequality* and plays in quantum information theory the same role as the Wielandt inequality [16] plays for classical memoryless channels. By Proposition 2, since the cone M_n^+ is homogeneous (i.e., all its soles are affinely isomorphic to the set of quantum states), one may restrict the supremum in (3) to quantum channels, i.e., to maps which are completely positive and trace-preserving.

One obviously has $\gamma^{\mathrm{cp}}(\mathsf{M}_n^+) \leqslant \gamma(\mathsf{M}_n^+)$. By restricting to diagonal matrices, one has $\gamma^{\mathrm{cp}}(\mathsf{M}_n^+) \geqslant \gamma(\mathbf{R}_+^n) = (n-1)^2 + 1$. The upper bound $\gamma^{\mathrm{cp}}(\mathsf{M}_n^+) \leqslant Cn^2 \log n$ for some constant C has been proved in [8] and the improvement $\gamma^{\mathrm{cp}}(\mathsf{M}_n^+) \leqslant n^2 + Cn$ appears in the preprint [15]. To our knowledge, no upper bound on $\gamma(\mathsf{M}_n^+)$ has been proved and the only paper which considers positive but non completely-positive maps is [11].

As a byproduct of our study, we compute the exact value of the parameter in the quantum Wielandt inequality in the specific case of a qubit space ($n = 2$), both in the case of positive and completely positive maps.

Theorem 10. *We have* $\gamma(\mathsf{M}_2^+) = 4$ *and* $\gamma^{\mathrm{cp}}(\mathsf{M}_2^+) = 3$.

Proof. Since the cones M_2^+ and $\mathscr{L}_4$ are isomorphic, the fact that $\gamma(\mathsf{M}_2^+) = 4$ is an immediate consequence of Theorem 1. We now explain the inequality $\gamma^{\mathrm{cp}}(\mathsf{M}_2^+) \leqslant 3$. Let $\Phi : \mathsf{M}_2 \to \mathsf{M}_2$ be a quantum channel which is M_2^+-primitive. As in (2), let $C(\Phi)$ be the set of pure states whose image under Φ is pure. A result known as the no-pancake theorem asserts that $C(\Phi)$ cannot be a circle inside the Bloch ball (see [2, Theorem IV.9] for a precise statement), and therefore contains at most two points. Repeating the argument from Section 2 with this extra information gives the bound $\gamma(\mathsf{M}_2^+, \Phi) \leqslant 3$.

Finally, we construct a quantum channel Φ such that $\gamma(\mathsf{M}_2^+, \Phi) = 3$ by adapting the arguments from Section 3. Given α and β in $(0, \pi/2)$ such that $\alpha \neq \beta$, consider the matrices

$$A = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \sin\beta \\ \sin\alpha & 0 \end{pmatrix},$$

and the quantum channel $\Psi : \mathsf{M}_2 \to \mathsf{M}_2$ defined by $\Psi(X) = AXA^* + BXB^*$.

Define $\theta \in (0, \pi/2)$ by the relation $\tan\theta = \sqrt{\sin 2\alpha / \sin 2\beta}$ and consider the vectors ψ_+ and ψ_- in $\mathbb{C}^2$ defined as $\psi_\pm = (\cos\theta, \pm\sin\theta)$. We claim that the states ρ_+ and ρ_- defined as $\rho_\pm = |\psi_\pm\rangle\langle\psi_\pm|$ are the only states whose image under Ψ is pure. Indeed, given a unit vector $\psi \in \mathbb{C}^2$, the state

$\Psi(|\psi\rangle\langle\psi|)$ is pure if and only if the vectors $A|\psi\rangle$ and $B|\psi\rangle$ are proportional. Our claim then follows from elementary computations.

The corresponding output states are $\Psi(\rho_\pm) = |\phi_\pm\rangle\langle\phi_\pm|$, where $\phi_\pm = (\cos\delta, \pm\sin\delta)$ and $\delta \in (0,\pi/2)$ is defined by the relation $\tan\delta = \sqrt{\tan\alpha/\tan\beta}$. Since $\alpha \neq \beta$, we have $\delta \neq \pi/4$ and therefore $0 < |\langle\phi_+,\phi_-\rangle| < 1$. We now use an elementary lemma.

Lemma 11. *Let $\phi_+,\phi_-,\psi_+,\psi_-$ be unit vectors in $\mathbb{C}^2$ such that $0 < |\langle\phi_+,\phi_-\rangle| < 1$. Then there exists a unitary matrix U such that $U(\phi_+) = \psi_-$ and $U(\phi_-)$ is neither proportional to ψ_+ nor to ψ_-.*

Proof. Write $\phi_- = a\phi_+ + b\chi$ where χ is a unit vector orthogonal to ϕ_+ and a, b are complex numbers such that $|a|^2 + |b|^2 = 1$. Pick a unit vector ω orthogonal to ψ_-. Since a and b are nonzero, we may choose $\theta \in \mathbf{R}$ such that $a\psi_- + be^{i\theta}\omega$ is neither proportional to ψ_+ nor to ψ_-. The unitary matrix sending the basis (ϕ_+,χ) to the basis $(\psi_-, e^{i\theta}\omega)$ has the desired property. □

Let U be a unitary matrix given by the lemma and consider the quantum channel Φ defined by $\Phi(X) = U\Psi(X)U^*$. The only states with a pure output under Φ are ρ_+ and ρ_-. Moreover, $\Phi(\rho_+) = U|\phi_+\rangle\langle\phi_+|U^* = \rho_-$ and $\Phi(\rho_-) = U|\phi_-\rangle\langle\phi_-|U^*$ is a pure state which, by Lemma 11, is distinct from ρ_+ and ρ_-. It follows that $\Phi^2(\rho_-) = \Phi^3(\rho_+)$ is not pure. Since Φ^3 is strictly positive and Φ^2 is not, the channel Φ has a maximal index equal to 3. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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