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Asymptotic behaviour of the sectional ring

Comportement asymptotique de l'anneau canonique d'un fibré en droites

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Abstract. The theory of the Okounkov body is a usual tool for analyzing the asymptotic behaviour of the sectional ring of a line bundle over a projective manifold. In this note, combined with the algebraic reduction, we study the asymptotic behaviour of the sectional ring of a line bundle over any arbitrary compact, normal, irreducible complex space.

Résumé. La théorie du corps d'Okounkov est un outil puissant pour analyser le comportement asymptotique de l'anneau canonique d'un fibré en droites sur une variété projective. Dans cette note, combiné avec la réduction algébrique, nous étudions le comportement asymptotique de l'anneau canonique d'un fibré en droites sur tout espace complexe compact, normal et irréductible arbitraire.

Keywords. Okounkov body, canonical ring, algebraic reduction.

Mots-clés. Corps d'Okounkov, anneau canonique, réduction algébrique.

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In the general context, we consider the following question. Let X be a compact, normal, irreducible (reduced) complex space. We denote the meromorphic function field over X as $\mathscr{M}(X)$. According to [1, 20], and [21], $\mathscr{M}(X)$ is a finitely generated extension over $\mathbb{C}$. Consequently,, there exists a (reduced irreducible) projective variety, denoted by Y, such that $\mathscr{M}(X)$ is isomorphic to the rational function field of Y (referred to as a model of $\mathscr{M}(X)$). Any two models are bimeromorphic.

Now, let L be a line bundle over X (or a Cartier divisor if the space is singular). In classical terms,we define the sectional ring of L by:

$$R(X,L) := \bigoplus_{k\geq 0} H^0(X,kL)$$

We also use the notation $\mathbb{N}(L) := \{k \in \mathbb{N}, h^0(X,kL) \neq 0\}$. Throughout this note, we assume that L is $\mathbb{Q}$-effective, meaning that $\mathbb{N}(L) \neq \{0\}$. Otherwise, the Kodaira–Iitaka dimension of L is defined to be $-\infty$ (as introduced in [15]).

Let ν be a valuation of $\mathscr{M}(X)$. The theory of the Okounkov body produces a tool to study the asymptotic behaviour of the sectional ring of L via the image of the valuation. This theory was independently developed by Lazarsfeld and Mustaţă [17] and Kaveh and Khovanskii [16], offering a systematic exploration of Okounkov's construction [18], [19]. In particular, we can show that the limit

$$\lim_{k\in\mathbb{N}(L),k\to\infty}\frac{h^0(X,kL)}{k^{\kappa(L)}}$$

exists, where $\kappa(L)$ is the Kodaira–Iitaka dimension of L. By the definition of the Kodaira–Iitaka dimension of L, a priori, the limit superior

$$\limsup_{k\in\mathbb{N}(L),k\to\infty}\frac{h^0(X,kL)}{k^{\kappa(L)}}$$

exists and is strictly positive. Note that by the projection formula, the sectional ring is a bimeromorphic invariant. In other words, if $\nu:\widetilde{X}\to X$ is a modification of X, $R(X,L)\simeq R(\widetilde{X},\nu^*L)$.

In our general context, the centre of a valuation does not necessarily exist on X. It's worth mentioning that the existence of the centre in the projective setting is deduced from the valuation characterization of the properness of a scheme. Therefore, the centre exists on any model of $\mathscr{M}(X)$ since the model is projective, although it is not necessarily a bimeromorphic model for a non-projective irreducible complex space.

To study the asymptotic behaviour of the sectional ring $R(X,L)$ using the valuation approach, we opt for a model such that the sectional ring of L is isomorphic to some sectional ring of a $\mathbb{Q}-$line bundle over this model. This is achieved through the following fundamental Theorem 9 of Campana, which was communicated to the author via unpublished personal correspondence.

Recall first the following definitions due to Campana.

Definition 1 ([8, Definition 1.21]). *Let $f:X\to Y$ be a holomorphic fibration between compact manifolds (i.e. surjective with connected fibres), and S be an effective divisor on X. We define S as being partially supported on the fibres of f if $f(S)\neq Y$ and for any irreducible component T of $f(S)$ with codimension one in Y, it is the case that $f^{-1}(T)$ contains an irreducible component mapping onto T by f which is not contained in the support of S.*

We have the following basic property.

Lemma 2 ([8, Lemma 1.22]). *Let $f:X\to Y$ be a holomorphic fibration between manifolds, and S be a divisor of X that is partially supported on the fibres of f. Let L be a line bundle on Y. Then the natural injection of sheaves $L\subset f_*(f^*(L)\otimes\mathscr{O}(S))$ is an isomorphism.*

Proof. We sketch the proof for the case when L is trivial for reader's convenience. Let U be a Stein open set on Y. Since S is partially supported on the fibres of f and U is Stein, there exists an effective divisor T on U such that $\mathscr{O}(S)\subset f^*\mathscr{O}(T)$. Consequently,

$$f_*\mathscr{O}(S)\subset\mathscr{O}(T)\subset\mathscr{M}_U.$$

Any section of $f_*\mathscr{O}(S)$ on U can be regarded as the pull-back of some meromorphic function on U, which at most has poles along S. Since S is partially supported on the fibres of f, the meromorphic function must indeed be holomorphic. □

Definition 3 ([8, Definition 1.2]). *Let $f:X\to Y$ be a holomorphic fibration between (connected) compact manifolds. An irreducible divisor D on X is said to be f-exceptional if the image $f(D)$ has codimension at least 2 in Y. We say that $f:X\to Y$ is neat if there moreover exists a bimeromorphic holomorphic map $u:X\to X'$ with X' being a smooth manifold such that each $f-$exceptional divisor of X is also $u-$exceptional.*

Note that an $f-$exceptional divisor is partially supported on the fibres of f. With the help of the resolution of singularities [12, 13] and the Hironaka flattening theorem [14], we can establish the following lemma by the proof of [8, Lemma 1.3].

Lemma 4. *Let $f : X \to Y$ be a holomorphic fibration between (connected) compact manifolds. Then, there exists a base change $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ and bimeromorphic maps $u : \widetilde{X} \to X$, $v : \widetilde{Y} \to Y$ where $\widetilde{X}, \widetilde{Y}$ are smooth manifolds that result in a commuting diagram:*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{u} & X \\ \downarrow{\tilde{f}} & & \downarrow{f} \\ \widetilde{Y} & \xrightarrow{v} & Y \end{array}$$

such that each $\widetilde{f}-$exceptional divisor of $\widetilde{X}$ is also $u-$exceptional. Moreover, $\widetilde{f}$ is neat with u as a possible choice for the bimeromorphic holomorphic map.

The fundamental property of a "neat" morphism is as follows:

Lemma 5. *Assume that $f : X \to Y$ is a neat holomorphic fibration between (connected) compact manifolds. Let $u : X \to X'$ be a bimeromorphic holomorphic map with X' being a smooth manifold, and suppose that each $f-$exceptional divisor of X is also $u-$exceptional. Let E be an $f-$exceptional divisor of X (hence $u-$exceptional) and L be a line bundle over X'. Then the restriction induces an isomorphism*

$$H^0(X, u^*L) \simeq H^0(X \setminus E, u^*L).$$

In particular, multiplication with the canonical section s_E of E induces an isomorphism

$$H^0(X, u^*L) \simeq H^0(X, u^*L + E).$$

Proof. It will be enough to prove surjectivity. Since $u : X \to X'$ is bimeromorphic, there exists a closed analytic subset $S \subset X'$ with codimension at least 2 such that the restriction of u to $u^{-1}(X' \setminus S)$ is biholomorphic.

The first statement can be derived from the following diagram:

$$\begin{array}{ccccc} H^0(X', L) & \xrightarrow{u^*} & H^0(X, u^*L) & \longrightarrow & H^0(X \setminus E, u^*L) \\ \downarrow{i^*} & & & & \downarrow{j^*} \\ H^0(X' \setminus S, L) & & \xrightarrow{\simeq} & & H^0(X \setminus f^{-1}(S), u^*L) \end{array}$$

Here, $i : X' \setminus S \to X'$ and $j : X \setminus u^{-1}(S) \to X \setminus E$ represent the inclusions. Notice that j^* is injective.

Note that the morphism i^* is an isomorphism by Hartogs' theorem. For any $s \in H^0(X, u^*L+E)$, by the first statement, we have $(s/s_E)|_{X\setminus E} = s'|_{X\setminus E}$ for some $s' \in H^0(X, u^*L)$. This implies $s = s's_E$, leading to the second isomorphism. □

We also recall the definition of non-polar divisors, as defined in [11]. For more information on non polar divisors, we refer to the paper [7].

Definition 6. *An irreducible divisor on a complex manifold X is called non-polar if it is not contained in the pole of any meromorphic function on X.*

For any compact connected manifold X, we have the following algebraic reduction (cf. [22, p. 25] or [6, Lemma 1 (p. 163)]):

Definition 7. *For any compact connected manifold X, there exist morphisms represented as:*

$$\begin{array}{ccc} X' & \xrightarrow{m} & X \\ \downarrow{a} & & \\ A & & \end{array}$$

where X' is a smooth bimeromorphic model of X, m is a proper modification, A is a smooth projective variety, and a is a surjective holomorphic map with connected fibres such that $\mathcal{M}(X) \cong \mathcal{M}(A) \cong \mathcal{M}(X')$.

In this case, an irreducible divisor on X' is non-polar if and only if its image under a is A. Note that A may not be bimeromorphic to X.

Lemma 8. *Using above notations, let N be a non-polar divisor over X' and S be a divisor over X' without any common irreducible component with N. Then, for any $p \geq 0$, the multiplication with the canonical section s_{pN} of pN induces an isomorphism*

$$H^0(X', S) \simeq H^0(X', S + pN).$$

Proof. The sections of $H^0(X', S + pN)$ are the meromorphic functions $f \in \mathscr{M}(A)$ such that

$$a^* \operatorname{div}(f) + S + pN \geq 0.$$

This condition is equivalent to $a^* \operatorname{div}(f) + S \geq 0$, as N is non-polar, and S has no common irreducible component with N. Consequently, we have:

$$H^0(X', S) \simeq H^0(X', S + pN)$$

for any $p \geq 0$. □

We are now ready to prove the following theorem:

Theorem 9. *Let L be a line bundle over a compact irreducible normal complex space X. Let $k \in \mathbb{N}^*$ be such that kL is effective. There exists a smooth projective variety A (independent of k) an algebraic reduction of X such that the rational function field of A is isomorphic to $\mathscr{M}(X)$ and a $\mathbb{Q}-$effective divisor D over A such that for $m > 0$ sufficient divisible,*

$$H^0(A, mD) \cong H^0(X, mkL).$$

Such an A is unique up to bimeromorphism.

Proof. Up to a possible desingularisation of X, we can assume X to be a compact connected complex manifold. Using the notations of Definition 7, by Lemma 4, we can assume that a is neat.

We claim that there exist $\mathbb{Q}-$effective divisors (as kL is effective) such that

$$km^* L + R = a^* D + E$$

where R is an effective, a-exceptional divisor, E is a sum of non-polar divisors N and an effective divisor PSSF(a) partially supported on the fibres of a. Note that R is thus $m-$exceptional since a is neat. Thus for sufficiently divisible l,

$$H^0(X, lkl) = H^0\big(X', m^*(lkL)\big) = H^0(X', m^*(lkL) + lR)$$

by Lemma 5.

The construction of the above decomposition is as follows. Let D' be an irreducible component of $m^*(kL)$ such that $G := a(D')$ is an irreducible divisor of A. Then

$$a^* G = \sum_i k_i D_i + R$$

with R $a-$exceptional and D_i irreducible divisors such that $a(D_i) = G$. Let

$$G' = \sum_i g_i D_i$$

be the maximal effective divisor which is a linear combination of D_i such that $m^*(kL) - G'$ is effective. Note that G' is not trivial since $G' \geq D'$.

Similarly, let N be the maximal effective divisor which is a linear combination of non-polar irreducible components of $m^*(kL)$ such that $m^*(kL) - N$ is effective. Here the support of N is contained in the support of $m^*(kL)$; thus, the set of such divisors is finite. In general, the set of non-polar divisors is always finite by results of [7].

Define $m_G := \min_i \frac{g_i}{k_i}$. Then $m_G = 0$ if and only if there exists i such that $g_i = 0$. In this case, G' is partially supported on the fibres of a, and we define $R_G = 0$ in this case. If $m_G > 0$, there exists an $a-$exceptional $\mathbb{Q}-$effective divisor R_G such that $G' - m_G a^* G + R_G$ is $\mathbb{Q}-$effective and is partially supported on the fibres of a.

Consider $m^*(kL) - \sum_G m_G a^* G$ where the sum is taken over all irreducible divisors G that can be writen as $a(D')$ for some irreducible component D' of $m^*(kL)$. Define

$$D := \sum_G m_G a^* G.$$

Then $m^*(kL) - \sum_G m_G a^* G + R$, where $R := \sum_G R_G$ (which is $a-$exceptional, $\mathbb{Q}-$effective), is a sum of non-polar divisor N and a $\mathbb{Q}-$effective divisor

$$\mathrm{PSSF}(a) := m^*(kL) + R - D - N$$

partially supported on the fibres of a.

To relate the sectional ring of kL to the sectional ring of a line bundle on A, for any $p > 0$ sufficient divisible such that $p m_G \in \mathbb{Z}$ for any m_G, we consider

$$H^0(X, pkL) = H^0(X', pm^* kL + pR) = H^0\big(X', pa^* D + pN + p\mathrm{PSSF}(a)\big) = H^0(A, pD).$$

The third equality follows from Lemma 2 and 8. (In fact, it is enough to take p to be a common multiple of the set of all k_i. Note that up to $\mathbb{Q}-$linear equivalence, $\frac{1}{k}D$ is uniquely determined by L.) □

In general, we hope to use the above theorem to construct the Okounkov body over an arbitrary compact, normal, irreducible complex space. Let L be a line bundle over a compact, normal, irreducible complex space X. Let ν be a valuation of $\mathcal{M}(X)$. With the same notations as above, we hope to define the Okounkov body of (X, L) $\Delta_\nu(X, L)$ to be the Okounkov body of the algebraic reduction $(A, \frac{1}{k}D)$ which is defined in [17, Definition 4.3]. Note that the bigness condition [17, Definition 4.3] is used to show the independence of numerical equivalence of divisors which is unnecessary for the independence of linear equivalence. The difficulty is whether this definition depends on the choice of k and D and the algebraic reduction. A general construction seems to be difficult.

However, we can still study some asymptotic behaviour of the sectional ring.

For the convenience of the reader, we recall briefly the construction of the Okounkov body in the projective case following [16, Section 2.4, 3.2]. Assume that ξ is the center of ν over A. (Its existence is deduced from the properness of A.) Assume that D is a line bundle over A. For any $\sigma \in H^0(A, mD) \setminus \{0\}$, we define naturally the valuation of σ associated to ν as follows. Let e be a local trivialisation of $\mathcal{O}(D)$ near ξ. Then there exists a local holomorphic function f such that $\sigma = f \cdot e$ near ξ (over a Zariski open set). Define

$$\nu(\sigma) := \nu(f)$$

which can be easily shown to be independent of the choice of local trivialisation.

Let $\Lambda_\nu := \nu(\mathcal{M}(A)^*)$, which forms a lattice in $V_\nu := \nu(\mathcal{M}(A)^*) \otimes_{\mathbb{Z}} \mathbb{R}$. Define in V_ν, the Okounkov body $\Delta_\nu(A, D)$ associated to D as the closure (with respect to the Euclidean topology) of the set of all $\frac{1}{m}\nu(\sigma)$ for $\sigma \in H^0(A, mD) \setminus \{0\}$. It can be proven to be equal to the closure of the set of all $\nu(E)$ where E is an effective $\mathbb{Q}-$divisor $\mathbb{Q}-$linearly equivalent to D with respect to the Euclidean topology on V_ν. Here for an irreducible divisor E, we define $\nu(E)$ to be the valuation ν of any local defining function of the divisor E. We can extend by linearity to define the valuation ν of any $\mathbb{Q}-$divisor. Thus we can extend the definition of Okounkov body to the linearly equivalent class of $\mathbb{Q}-$divisors.

Example 10. Let T be a generic torus so that $\mathscr{M}(T) = \mathbb{C}$ the constant functions. Let X be the blow-up of a point in $T \times \mathbb{P}^n$. Then the composition π of the blow-up and the projection onto $\mathbb{P}^n$ gives an algebraic reduction of X. In particular, we have that

$$\mathscr{M}(X) \cong \mathscr{M}(T \times \mathbb{P}^n) \cong \mathscr{M}(\mathbb{P}^n).$$

Consider $L := \pi^*\mathscr{O}(1) \otimes \mathscr{O}(E)$ where E is the exceptional divisor of the blow-up. Using the construction from Theorem 9, we can find a $\mathbb{Q}-$divisor $\frac{1}{k}D$ which is $\mathbb{Q}-$linearly equivalent to $\mathscr{O}(1)$ for any $k > 0$ so that for any $m \geq 0$,

$$H^0(X, mL) = H^0(\mathbb{P}^n, \mathscr{O}(m)).$$

In this case, one can define the Okounkov body of (X, L) as

$$\Delta_v(X, L) := \Delta_v\big(\mathbb{P}^n, \mathscr{O}(1)\big).$$

As an application of Theorem 9, we have the following proposition.

Proposition 11. *Let L be a line bundle over a compact, normal, irreducible complex space X. Then we have that the limit*

$$\lim_{k\in\mathbb{N}(L), k\to\infty} \frac{h^0(X, kL)}{k^{\kappa(L)}}$$

exists where $\kappa(L)$ is the Kodaira–Iitaka dimension of L.

Proof. This is an application of Theorem 9 and the corresponding result in the projective case. We sketch briefly the proof of the projective case for the convenience of the reader. Here we follow the arguments in [16, Section 2.4, 3.2].

We use the same notations as in Theorem 9.

Recall that the rational rank of v is defined to be the rank of Λ_v which is the maximal size of a set of $\mathbb{Z}-$linear independent elements in Λ_v. It can be shown that the rational rank of v is less than the dimension of the algebraic reduction A which is also equal to the transcendental degree of $\mathscr{M}(X)$ over $\mathbb{C}$. Fix v a valuation with maximal rational rank. This is always possible (cf. [17, Section 5.2]).

Let μ_v be the Lebesgue measure on V_v normalised by the lattice Λ_v. By Theorem 9, there exists k_0 sufficient divisible such that $k_0 L$ is effective and there is an effective line bundle D over A such that

$$H^0(A, mD) \cong H^0(X, mk_0 L) \quad (\forall m \geq 0).$$

In particular,

$$\kappa(D) = \kappa(k_0 L) = \kappa(L).$$

By Okounkov body theory ([16, Corollary 3.11]), for any valuation v with maximal rational rank (which is called a faithful $\mathbb{Z}^{\dim_{\mathbb{C}} X}-$valued valuation for the field $\mathscr{M}(X)$ in the terminology of [16]),

$$\lim_{k\in\mathbb{N}(L), k\to\infty} \frac{h^0(A, kD)}{k^{\kappa(D)}} = \mu_v\big(\Delta_v(A, D)\big).$$

This is a consequence of the equidistribution of the sets $\frac{1}{k} v(H^0(A, kD))$ in the Okounkov body $\Delta_v(A, D)$. Thus we have

$$\lim_{kk_0\in\mathbb{N}(L), k\to\infty} \frac{h^0(X, kk_0 L)}{(kk_0)^{\kappa(L)}} = \lim_{kk_0\in\mathbb{N}(L), k\to\infty} \frac{h^0(A, kD)}{(kk_0)^{\kappa(D)}} = \mu_v\big(\Delta_v(A, D)\big) k_0^{-\kappa(L)}.$$

Since $\mathbb{N}(L)$ is a semi-group, there exists d large enough such that

$$\mathbb{N}(L) \cap [k_1 d, \infty[= k_1\mathbb{N} \cap [k_1 d, \infty[$$

for some $k_1 > 0$. Without loss of generality, we can assume that k_0 is a multiple of $k_1 d$. In particular $k_0L, k_0L + k_1L, \dots, k_0L + (k_0 - k_1)L$ are all effective and for any $k \geq 1$ large enough and any $0 \leq i < k_0/k_1$, we have inclusions

$$\mathscr{O}\big((k-1)k_0L\big) \subset \mathscr{O}(kk_0L + k_1 iL) \subset \mathscr{O}\big((k+2)k_0L\big).$$

Thus we have for $0 \leq i \leq k_0/k_1 - 1$,

$$\lim_{k\to\infty} \frac{h^0(X, kk_0L + k_1 iL)}{(kk_0 + k_1 i)^{\kappa(L)}} = \mu_\nu\big(\Delta_\nu(A, D)\big) k_0^{-\kappa(L)}.$$

This implies the conclusion

$$\lim_{k\geq d, k\to\infty} \frac{h^0(X, k_1 kL)}{(k_1 k)^{\kappa(L)}} = \lim_{k\to\infty} \frac{h^0(X, kk_0L + k_1 iL)}{(kk_0 + k_1 i)^{\kappa(L)}} = \mu_\nu\big(\Delta_\nu(A, D)\big) k_0^{-\kappa(L)}$$

since the right-hand side is independent of i. □

The existence of such a limit was previously studied in [10] for the base of a big line bundle over a projective manifold (cf. [9, Remark 15.8]).

By these methods, we can also show the differentiability of the volume function on a Moishezon manifold. To demonstrate this, we require the following observation concerning the definition of the movable intersection product in [2–4] on a compact Kähler manifold.

Remark 12. Let (Y, ω) be a compact Kähler manifold. Let $\pi : \widetilde{Y} \to Y$ be a modification. Assume that π is a composition of blow-ups of smooth centres. In particular, the cohomology classes of the irreducible components of the exceptional divisor are linearly independent. Let α_j be big classes on Y such that $\pi^*\alpha_j$ are still big classes on $\widetilde{Y}$. (For example, this holds when α_j are the first Chern classes of big line bundles over Y.) By the construction of the movable positive product over a compact Kähler manifold, we have

$$\pi_* \left\langle \pi^*\alpha_1, \dots, \pi^*\alpha_k \right\rangle = \langle \alpha_1, \dots, \alpha_k \rangle. \tag{$*$}$$

The reason is as follows. Observe that when all cohomology classes are big, in the construction of the movable intersection product described in [3], we can replace Kähler currents with logarithmic poles with positive currents that have logarithmic poles. This substitution is possible due to the continuity of the movable positive product over the big cone. Let T_j be positive currents in $\pi^*\alpha_j$ $(1 \leq j \leq k)$. Then $T_j = \pi^*\pi_* T_j$ since $\pi^*\alpha_j = \pi^*\pi_*\pi^*\alpha_j$ and the cohomology classes of the irreducible components of the exceptional divisor are linearly independent. With this, it is easy to check $(*)$.

In particular, let L be a big line bundle over a Moishezon manifold X. We can define the movable positive product of $c_1(L)$ as follows. Let $\pi : \widetilde{X} \to X$ be a modification fo X such that $\widetilde{X}$ is a projective manifold. Without loss of generality, we may assume that π is a composition of blow-ups of smooth centres.

Thus, we define for any $p > 0$,

$$\left\langle c_1(L)^p \right\rangle := \pi_* \left\langle \pi^* c_1(L)^p \right\rangle$$

in $H^{p,p}_{BC}(X, \mathbb{C})$. By the filtration property of the modification, we can easily check that the product is independent of the choice of modification using $(*)$. In other words, we have the same product for the push forward from any modification of X such that $\widetilde{X}$ is a projective manifold and that π is a composition of blow-ups of smooth centres.

Remark 13. Let L be a big line bundle over a Moishezon manifold X. Then for any $\xi \in \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, we have

$$\lim_{t\in\mathbb{Q}, t\to 0+} \frac{\mathrm{Vol}(L + t\xi) - \mathrm{Vol}(L)}{t} = \left\langle c_1(L)^{n-1} \right\rangle \cdot c_1(\xi)$$

where the movable positive product is defined as in the previous remark. The proof uses the birational invariance of the volume and reduces the case to a smooth projective bimeromorphic model. The projective case is proven in [5].

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