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Some qualitative properties of Lichnerowicz equations and Ginzburg–Landau systems on locally finite graphs

Quelques propriétés qualitatives des équations de Lichnerowicz et de Ginzburg–Landau sur les graphes localement finis

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Abstract. Let (V,E) be a locally finite weighted graph. We study some qualitative properties of positive solutions of the Lichnerowicz equation

$$v_t - \Delta v = v^{-p-2} - v^p,\ (x,t) \in V \times \mathbb{R},$$

and of (sign-changing) solutions of the Ginzburg-Landau system

$$\begin{cases} u_t - \Delta u = u - u^3 - \lambda u v^2,\ (x,t) \in V \times \mathbb{R}, \\ v_t - \Delta v = v - v^3 - \lambda v u^2,\ (x,t) \in V \times \mathbb{R}, \end{cases}$$

where $p > 0$, $\lambda > 0$ and Δ is the standard *discrete* graph Laplacian. Firstly, we prove that any positive solution v of the Lichnerowicz equation satisfies $v \geq 1$. Moreover, if we assume the boundedness of positive solution v, then it must be trivial, i.e $v \equiv 1$. We also construct a nontrivial positive solution of the Lichnerowicz equation to show that the boundedness assumption is necessary. Secondly, we obtain sharp upper bound for solutions of the Ginzburg-Landau system depending on the range of λ.

Résumé. Soit (V,E) un graphe pondéré localement fini. Nous étudions certaines propriétés qualitatives des solutions positives de l'équation de Lichnerowicz

$$v_t - \Delta v = v^{-p-2} - v^p,\ (x,t) \in V \times \mathbb{R},$$

et des solutions (avec changement de signe) du système de Ginzburg-Landau

$$\begin{cases} u_t - \Delta u = u - u^3 - \lambda u v^2,\ (x,t) \in V \times \mathbb{R}, \\ v_t - \Delta v = v - v^3 - \lambda v u^2,\ (x,t) \in V \times \mathbb{R}, \end{cases}$$

où $p > 0$, $\lambda > 0$ et Δ est le laplacien discret standard des graphes. Tout d'abord, nous prouvons que toutes les solutions positives de l'équation de Lichnerowicz satisfont à $v \geq 1$. De plus, si nous supposons qu'une solution positive v est bornée, alors elle doit être triviale, c'est-à-dire $v \equiv 1$. Nous construisons également une solution positive non bornée de l'équation de Lichnerowicz. Deuxièmement, nous obtenons une majoration précise pour les solutions du système de Ginzburg-Landau en fonction de l'étendue λ.

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Keywords. Liouville-type theorems, Lichnerowicz equations, Ginzburg–Landau system, nonexistence results, qualitative property, locally finite graphs.

Mots-clés. Théorèmes de type Liouville, équations de Lichnerowicz, système de Ginzburg–Landau, résultats de non-existence, propriété qualitative, graphes localement finis.

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1. Introduction

Throughout this paper, let (V,E) be a locally finite connected graph, where V is the set of infinite vertices and E is the set of edges. We always assume that the graph is simple, i.e., no loop and no multi-edges. Denote by

$$\mu : V \times V \longrightarrow [0,\infty)$$

the edge weight satisfying

- $\mu_{xy} = \mu_{yx}$,
- $\mu_{xy} > 0$ if and only if $(x,y) \in E$.

Given a vertex $x \in V$, we denote by $y \sim x$ if there is an edge between x and y, and in this case x and y are called neighboring vertices. The weight of a vertex $x \in V$ is defined by

$$\mu(x) = \sum_{y\sim x} \mu_{xy}.$$

Let $\ell(V) = \{u;\ u : V \to \mathbb{R}\}$. The *discrete* Laplace operator on graph $\Delta : \ell(V) \to \ell(V)$ is defined by

$$\Delta u(x) = \sum_{y\sim x} \frac{\mu_{xy}}{\mu(x)}(u(y)-u(x)) \text{ for } u \in \ell(V) \text{ and } x \in V.$$

The reader is referred to [6] for some elementary properties of weighted graphs and discrete graph Laplacian.

In this decade, elliptic and parabolic equations on weighted graphs have been extensively studied, see e.g [4–17, 21]. The optimal Liouville-type theorem for the nonnegative solutions of the elliptic inequality

$$\Delta u + u^{\sigma} \le 0 \text{ in } V$$

was established in [9]. Very recently, the result in [9] has been generalized to the system of elliptic inequalities on weighted graph

$$\begin{cases} \Delta u + v^p \le 0 \text{ in } V, \\ \Delta v + u^q \le 0 \text{ in } V, \end{cases}$$

where p and q are real numbers, see [23].

By using the calculus of variations, the existence of solutions to the Kazdan–Warner equation

$$\Delta u = c - h(x)e^u \text{ in } V$$

was studied in [7]. For elliptic equations involving the p-Laplace operator, some existence results were also obtained in [10].

In [21], the author established Kato's inequality on graph and applied it to study the boundedness of solutions of the Ginzburg–Landau equation

$$\Delta u + u(1-u^2) = 0 \text{ in } V.$$

It was proved in [21] that any bounded solution u of the Ginzburg–Landau equation satisfies $|u| \le 1$. Concerning some results on the existence, nonexistence and the blow-up property of solution for parabolic equations, we refer to [14, 15, 18, 24, 25] and references given there.

The first purpose of this paper is to study some qualitative properties of positive solutions to the parabolic Lichnerowicz equation on weighted graph

$$v_t - \Delta v = v^{-p-2} - v^p \text{ in } V \times \mathbb{R}, \tag{1}$$

where $p > 0$. The elliptic counterpart of (1) is the following

$$-\Delta u = u^{-p-2} - u^p \text{ in } V. \tag{2}$$

In [19, 22], the authors obtained some Liouville-type results for the equation (2) on the Euclidean space $\mathbb{R}^N$. Later, these results were reproved in [1] by using a simple argument. In fact, the approach in [1] is a combination of a kind of maximum principle and the Keller–Osserman theory. Moreover, some important remarks on the results in [19, 22] were also provided in [1].

Theorem A ([1]). *The following assertions hold:*

(i) *If $p > 0$ and u is a positive solution of* (2)*, then $u \ge 1$.*
(ii) *If $p > 1$, then any positive solution of* (2) *must be trivial, i.e $u = 1$.*
(iii) *If $0 < p \le 1$, the equation* (2) *has many nontrivial positive solutions.*

Very recently, the uniform lower bound of positive solution and Liouville-type theorem have been established for the Lichnerowicz equations of parabolic type on the Euclidean space $\mathbb{R}^N \times \mathbb{R}$ involving the fractional Laplacian, see [2].

In this paper, we first address similar questions for positive solutions (in the classical sense) of (1) as in [1, 2], but on the weighted graph V. In fact, we obtain the first main result as follows.

Theorem 1. *Let $p > 0$ and v is a positive solution of* (1)*. Then, the following assertions hold:*

(i) $v \ge 1$.
(ii) *If, in addition, v is bounded, then $v \equiv 1$.*

As a direct consequence of Theorem 1, when v is independent of t, we have the following.

Corollary 2. *Let $p > 0$ and u is a positive solution of* (2)*. Then, the following assertions hold:*

(i) $u \ge 1$.
(ii) *If, in addition, u is bounded, then $u \equiv 1$.*

Notice that, the boundedness assumption of solutions in Theorem 1 or in Corollary 2 is necessary. Indeed, we explicitly construct an example of an unbounded positive solution of (2) in the following example.

Example 3. Let $V = \mathbb{N} = \{0, 1, 2, \ldots\}$ and the set of edges $E = \{(n, n+1), n \in \mathbb{N}\}$, $\mu_{n,n+1} = 1$ for all $n \in \mathbb{N}$. Then, the equation (2) with $p > 0$ becomes

$$u(1) - u(0) + u^{-p-2}(0) - u^p(0) = 0$$

and

$$\frac{1}{2}(u(n+1) + u(n-1)) - u(n) + u^{-p-2}(n) - u^p(n) = 0, \; n \ge 1. \tag{3}$$

Choosing $u(0) = 2$, we have

$$u(1) = 2 + 2^p - 2^{-p-2} > 2.$$

By an induction argument and (3), we obtain an increasing sequence $(u(n))$ which is a positive unbounded solution of (2). Notice that $(u(n))$ is also a positive unbounded solution of (1) since it is independent of t.

The second purpose of this paper is to study the upper bound of solutions of the Ginzburg–Landau system of parabolic type

$$\begin{cases} u_t - \Delta u = u - u^3 - \lambda u v^2 \text{ in } V \times \mathbb{R}, \\ v_t - \Delta v = v - v^3 - \lambda v u^2 \text{ in } V \times \mathbb{R}, \end{cases} \tag{4}$$

where $\lambda > 0$. Let us recall some related results on the Ginzburg–Landau equation/system on the Euclidean setting. In [19], the author proved the boundedness of the Ginzburg–Landau equation of elliptic type

$$\Delta u + u(1-u^2) = 0 \text{ in } \mathbb{R}^N. \tag{5}$$

It was shown that any smooth solution u of (5) satisfies $|u| \le 1$. This result was then generalized for the fractional Ginzburg–Landau equation in [20], see also [2] for some important remarks. In [3], the authors have studied the Ginzburg- Landau system

$$\begin{cases} -\Delta u = u - u^3 - \lambda u v^2 \text{ in } \mathbb{R}^N, \\ -\Delta v = v - v^3 - \lambda v u^2 \text{ in } \mathbb{R}^N, \end{cases} \tag{6}$$

where $\lambda > 0$. Among other things, the following properties of solutions of (6) were proved in [3].

Theorem B ([3]). *The following statements hold:*

(i) *If* $\lambda > 0$ *and* (u, v) *is a solution of* (6), *then* $|u| \le 1$ *and* $|v| \le 1$.
(ii) *If* $\lambda \ge 1$ *and* (u, v) *is a solution of* (6), *then* $u^2 + v^2 \le 1$.
(iii) *If* $0 < \lambda < 1$ *and* (u, v) *is a solution of* (6), *then* $u^2 + v^2 \le \frac{2}{\lambda+1}$.

To the best of our knowledge, there has been no work dealing with the system (4) on graphs. Inspired by Theorem B, we prove the sharp boundedness of solutions of the system (4) on weighted graph.

The second main result in this paper reads as follows.

Theorem 4. *The following statements hold:*

(i) *If* $\lambda > 0$ *and* (u, v) *is a bounded solution of* (4), *then* $|u| \le 1$ *and* $|v| \le 1$.
(ii) *If* $\lambda \ge 1$ *and* (u, v) *is a bounded solution of* (4), *then* $u^2 + v^2 \le 1$.
(iii) *If* $0 < \lambda < 1$ *and* (u, v) *is a bounded solution of* (4), *then* $u^2 + v^2 \le \frac{2}{\lambda+1}$.

As a direct consequence of Theorem 4, we obtain similar results for the Ginzburg–Landau system of elliptic type on weighted graph

$$\begin{cases} -\Delta u = u - u^3 - \lambda u v^2 \text{ in } V, \\ -\Delta v = v - v^3 - \lambda v u^2 \text{ in } V. \end{cases} \tag{7}$$

Corollary 5. *The following statements hold:*

(i) *If* $\lambda > 0$ *and* (u, v) *is a bounded solution of* (7), *then* $|u| \le 1$ *and* $|v| \le 1$.
(ii) *If* $\lambda \ge 1$ *and* (u, v) *is a bounded solution of* (7), *then* $u^2 + v^2 \le 1$.
(iii) *If* $0 < \lambda < 1$ *and* (u, v) *is a bounded solution of* (7), *then* $u^2 + v^2 \le \frac{2}{\lambda+1}$.

For the Ginzburg–Landau system, we also give an example showing that the boundedness assumption of solutions is necessary in Theorem 4 and Corollary 5.

Example 6. Let $V = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and the set of edges $E = \{(n, n+1), n \in \mathbb{Z}\}$, $\mu_{n,n+1} = 1$ for all $n \in \mathbb{Z}$. Then, (7) becomes

$$\begin{cases} \frac{1}{2}(u(n+1) + u(n-1)) - u^3(n) - \lambda u(n) v^2(n) = 0, \\ \frac{1}{2}(v(n+1) + v(n-1)) - v^3(n) - \lambda v(n) u^2(n) = 0 \end{cases}$$

We take $u(0) = v(0) = 1$ and $u(1) = v(1) = 1$ and then

$$\begin{cases} u(n+1) = v(n+1) & = 2(1+\lambda)u(n) - u(n-1), n \geq 1 \\ u(n-1) = v(n-1) & = 2(1+\lambda)u(n) - u(n+1), n \leq 0. \end{cases}$$

By an induction argument, we also obtain increasing sequence

$$(u(n)) = (v(n)) = (\dots, -4(1+\lambda)^2 + 1, -2(1+\lambda), -1, 0, 1, 2(1+\lambda), 4(1+\lambda)^2 - 1, \dots)$$

and $(u(n)), (v(n))$ is an unbounded solution of (7) (and also of (4)).

Our idea to prove the main results is based on a kind of maximum principle, Kato's inequality on graph and especially, the Kelly–Osserman theory for parabolic inequality on graph, see Lemma 8 below.

The paper is organized as follows. In Section 2, we prove Theorem 1. The proof of Theorem 4 is given in Section 3.

2. Proof of Theorem 1

We begin this section by considering the lower bound of positive solutions of (1).

2.1. *The uniform lower bound of solutions*

In order to prove the first assertion in Theorem 1, we establish a stronger result.

Lemma 7. *Let $f : (0,\infty) \to \mathbb{R}$ be a continuous, strictly decreasing function such that there exists a positive number α satisfying $f(\alpha) = 0$. Then, any positive solution v of*

$$v_t - \Delta v \geq f(v) \text{ in } V \times \mathbb{R}, \tag{8}$$

satisfies $v \geq \alpha > 0$.

Before proving Lemma 7, we show how to use it to get Theorem 1(i). Taking $f(s) = s^{-p-2} - s^p$, $s > 0$. It is easy to verify that this function satisfies the hypotheses in Lemma 7 with $\alpha = 1$. Applying Lemma 7, we obtain that any positive solution v of (1) must be bounded from below by one, i.e $v \geq 1$.

Proof of Lemma 7. As in [2], we take

$$\phi(t) = \begin{cases} t^2 & \text{if } |t| < 1 \\ 2t - 1 & \text{if } t \geq 1 \\ -1 - 2t & \text{if } t \leq -1. \end{cases} \tag{9}$$

Then, we have $\phi \in C^1(\mathbb{R})$, $\phi \geq 0$ and

- $\phi(t) \to \infty$ as $|t| \to \infty$,
- $\sup_{\mathbb{R}} |\phi'(t)| < \infty$,
- $\phi(0) = 0$.

Let v be a positive solution of (8). Suppose that there exists $(x_0, t_0) \in V \times \mathbb{R}$ such that $v(x_0, t_0) < \alpha$. In the case where $v(x,t)$ attains its minimum $m := \inf_{V\times\mathbb{R}} v(x,t)$ at this point (x_0, t_0), the proof is easy. In fact, at this point, we have $v_t(x_0, t_0) = 0$ and $\Delta v(x_0, t_0) \geq 0$, which leads to $v_t(x_0, t_0) - \Delta v(x_0, t_0) \leq 0$. On the other hand, one has $f(v(x_0, t_0)) = f(m) > 0$ since $m < \alpha$. This is a contradiction.

Otherwise, there exists a decreasing sequence $(v(x_k, t_k))_{k\geq 0}$ such that

$$\lim_{k\to\infty} v(x_k, t_k) = m.$$

Let $0 < \epsilon_0 < \frac{f(v(x_0,t_0))}{\max_{\mathbb{R}} |\phi'(t)|}$. We construct an auxiliary function

$$w(x_0,t) = v(x_0,t) + \epsilon_0 \phi(t - t_0).$$

Since v is positive, then $w(x_0,t) \to \infty$ when $|t| \to \infty$. Consequently, there is a point $(x_0, \widetilde{t}_1) \in V \times \mathbb{R}$ such that

$$w(x_0, \widetilde{t}_1) = \min\{w(x_0,t); t \in \mathbb{R}\} \le w(x_0,t_0) = v(x_0,t_0) < \alpha.$$

By the property of local minimum, we have at $(x_0, \widetilde{t}_1)$ that $w_t = 0$ or $v_t(x_0, \widetilde{t}_1) = -\epsilon_0 \phi(\widetilde{t}_1 - t_0)$. This and (8) give

$$-\epsilon_0 \phi'(\widetilde{t}_1 - t_0) - \Delta v(x_0, \widetilde{t}_1) \ge f(v(x_0, \widetilde{t}_1)) > 0.$$

Without loss of generality, we may assume that $v(x_1, t_1) \le v(x_0, \widetilde{t}_1)$. Hence, we have the property

$$v(x_1,t_1) \le v(x_0, \widetilde{t}_1) \le v(x_0,t_0) < \alpha.$$

By repeating this argument, we obtain a decreasing sequence $(v(x_k, \widetilde{t}_k))$ such that

$$v(x_{k+1}, t_{k+1}) \le v(x_k, \widetilde{t}_{k+1}) \le v(x_k, t_k)$$

and

$$-\epsilon_k \phi'(\widetilde{t}_{k+1} - t_k) - \Delta v(x_k, \widetilde{t}_{k+1}) \ge f(v(x_k, \widetilde{t}_{k+1})) > 0, \tag{10}$$

where ϵ_k is chosen such that $0 < \epsilon_k < \frac{f(v(x_k,t_k))}{\max_{\mathbb{R}} |\phi'(t)|}$ and $\epsilon_k \to 0^+$ as $k \to \infty$.

Notice that

$$-\Delta v(x_k, \widetilde{t}_{k+1}) = \sum_{x \sim x_k} \frac{\mu_{xx_k}}{\mu_{x_k}} (v(x_k, \widetilde{t}_{k+1}) - v(x, \widetilde{t}_{k+1})) \le v(x_k, \widetilde{t}_{k+1}) - m.$$

Plugging this into (10) and letting $k \to \infty$ we obtain

$$0 + m - m \ge f(m) > f(\alpha) = 0,$$

which is impossible. The proof of Lemma 7 is complete. □

2.2. *The Liouville property*

We now prove a version of the Keller–Osserman inequality for the parabolic equation on graphs.

Lemma 8. *Given two positive constants $p > 0$ and $\beta > 0$. If v is a bounded nonnegative solution of*

$$v_t - \Delta v \le -\beta v^p \text{ in } V \times \mathbb{R}, \tag{11}$$

then $v \equiv 0$.

Remark that a version of Keller–Osserman inequality for the elliptic equation was proved in [21]. This can be seen as a consequence of Lemma 8 when v is independent of t.

Proof of Lemma 8. Suppose that there exists (x_0,t_0) such that $v(x_0,t_0) > 0$. As in the proof of Lemma 7, if v attains its maximum at this point, then $v_t(x_0,t_0) = 0$ and $\Delta v(x_0,t_0) \le 0$. This implies that

$$v_t(x_0,t_0) - \Delta v(x_0,t_0) \ge 0,$$

which contradicts (11).

Otherwise, there exists a sequence (x_n,t_n) such that $(v(x_n,t_n))$ is increasing and

$$v(x_n,t_n) \longrightarrow \sup_{V \times \mathbb{R}} v(x,t) =: M > 0.$$

Given $\varepsilon > 0$ and let ϕ be the function in the proof of Lemma 7, we construct

$$w(x_0,t) = v(x_0,t) - \varepsilon \phi(t - t_0).$$

Since $\phi(t) \to +\infty$ as $|t| \to +\infty$, there exists $\widetilde{t}_0$ such that

$$w(x_0, \widetilde{t}_0) = \max_{V \times \mathbb{R}} w(x_0,t) \ge w(x_0,t_0) = v(x_0,t_0).$$

As above, by the maximum property, $w_t(x_0, \widetilde{t}_0) = 0$ or $v_t(x_0, \widetilde{t}_0) = \varepsilon\phi'(\widetilde{t}_0 - t_0)$. Hence,

$$-\varepsilon\phi'(\widetilde{t}_0 - t_0) + \sum_{y\sim x_0} \frac{\mu_{x_0 y}}{\mu_{x_0}}(v(y, \widetilde{t}_0) - v(x_0, \widetilde{t}_0)) \geq \beta v^p(x_0, \widetilde{t}_0).$$

Without loss of generality, assume that $v(x_0, \widetilde{t}_0)) < v(x_1, t_1)$. Then, we have the property

$$v(x_0, t_0) < v(x_0, \widetilde{t}_0)) < v(x_1, t_1).$$

Repeating this argument, there exists $(x_n, \widetilde{t}_n)$ such that

$$-\varepsilon\phi'(\widetilde{t}_n - t_n) + \sum_{y\sim x_n} \frac{\mu_{x_n y}}{\mu_{x_n}}(v(y, \widetilde{t}_n) - v(x_n, \widetilde{t}_n)) \geq \beta v^p(x_n, \widetilde{t}_n) \tag{12}$$

and

$$v(x_n, t_n) < v(x_n, \widetilde{t}_n) < v(x_{n+1}, t_{n+1}).$$

The left hand side of (12) is bounded from above by

$$\varepsilon \sup_{\mathbb{R}} |\phi'| + M - v(x_n, \widetilde{t}_n)$$

and consequently

$$\varepsilon \sup_{\mathbb{R}} |\phi'| + M - v(x_n, \widetilde{t}_n) \geq \beta v^p(x_n, \widetilde{t}_n). \tag{13}$$

Letting $n \to \infty$ in (13) and choosing $0 < \varepsilon < \frac{\beta M^p}{\sup_{\mathbb{R}} |\phi'|}$, we obtain a contradiction. □

With these preparations above, we are ready to prove Theorem 1(ii). Suppose that v is a positive bounded solution of the equation (1). It results from Theorem 1(i) that $v \geq 1$. Define $w = v - 1 \geq 0$. Then, w is a nonnegative bounded solution of

$$w_t - \Delta w = (w+1)^{-p-2} - (w+1)^p \text{ in } V \times \mathbb{R}. \tag{14}$$

Put

$$M_1 = \sup_{V\times\mathbb{R}} w < \infty.$$

We now claim that

$$(s+1)^{-p-2} - (s+1)^p \leq -\frac{\gamma}{p+1} s^{p+1} \text{ for all } 0 \leq s \leq M_1, \tag{15}$$

where γ is some positive constant satisfying

$$\gamma < \frac{(p+2)(M_1+1)^{-p-3}}{M_1^p}.$$

Indeed, we put

$$g(s) = (s+1)^{-p-2} - (s+1)^p + \frac{\gamma}{p+1} s^{p+1},\ s \in [0, M_1]$$

and then

$$\begin{aligned} g'(s) &= -(p+2)(s+1)^{-p-3} - p(s+1)^{p-1} + \gamma s^p \\ &\leq -(p+2)(s+1)^{-p-3} + \gamma s^p. \end{aligned}$$

Hence, for all $0 \leq s \leq M_1$, by the choice of γ we have

$$g'(s) \leq -(p+2)(M_1+1)^{-p-3} + \gamma M_1^p < 0.$$

This implies that g is decreasing. Consequently, $g(s) \leq g(0) = 0$ for all $0 \leq s \leq M_1$, which gives (15).

From (15) and (14), we get that w is a nonnegative bounded solution of

$$w_t - \Delta w \leq -\frac{\gamma}{p+1} w^p.$$

According to Lemma 8, $w \equiv 0$ or equivalently $v \equiv 1$. The proof is complete. □

3. Proof of Theorem 4

We prove Theorem 4 in this section by using some elementary arguments and developing a kind of maximum principle.

Step 1. Proof of (i). We first show that $u \geq -1$ and $v \geq -1$. Suppose, on the contrary, that there exists $(x_0, t_0) \in V \times \mathbb{R}$ such that $v(x_0, t_0) < -1$. Let us put

$$m := \inf_{V\times\mathbb{R}} v(x, t).$$

We next split the proof into two cases.

Case 1. If $v(x, t)$ attains its minimum at this point (x_0, t_0), then we have $v_t(x_0, t_0) = 0$ and $\Delta v(x_0, t_0) \geq 0$, which leads to $v_t(x_0, t_0) - \Delta v(x_0, t_0) \leq 0$. On the other hand, one has

$$v_t(x_0, t_0) - \Delta v(x_0, t_0) = v(x_0, t_0)(1 - v^2(x_0, t_0) - \lambda u^2(x_0, t_0)) > 0$$

since $v(x_0, t_0) < -1$. This is a contradiction.

Case 2. There exists a decreasing sequence $(v(x_k, t_k))_{k\geq 0}$ such that

$$\lim_{k\to\infty} v(x_k, t_k) = m.$$

Let ϵ_0 be a small positive constant satisfying

$$0 < \epsilon_0 < \frac{v(x_0, t_0)(1 - v^2(x_0, t_0))}{\max_{\mathbb{R}} |\phi'(t)|} = \frac{|v(x_0, t_0)(1 - v^2(x_0, t_0))|}{\max_{\mathbb{R}} |\phi'(t)|}.$$

We define

$$w(x_0, t) = v(x_0, t) + \epsilon_0 \phi(t - t_0).$$

Since v is bounded, then $w(x_0, t) \to \infty$ when $|t| \to \infty$. Consequently, there is a point $(x_0, \tilde{t}_1) \in V \times \mathbb{R}$ such that

$$w(x_0, \tilde{t}_1) = \min\{w(x_0, t); t \in \mathbb{R}\} \leq w(x_0, t_0) = v(x_0, t_0) < -1.$$

By the property of local minimum, we have at $(x_0, \tilde{t}_1)$ that $w_t = 0$ or $v_t(x_0, \tilde{t}_1) = -\epsilon_0 \phi'(\tilde{t}_1 - t_0)$. This and (4) give

$$-\epsilon_0 \phi'(\tilde{t}_1 - t_0) - \Delta v(x_0, \tilde{t}_1) \geq v(x_0, \tilde{t}_1)(1 - v^2(x_0, \tilde{t}_1) - \lambda u^2(x_0, \tilde{t}_1)) > 0.$$

As above, we may assume that $v(x_1, t_1) \leq v(x_0, \tilde{t}_1)$. Hence,

$$v(x_1, t_1) \leq v(x_0, \tilde{t}_1) \leq v(x_0, t_0) < -1.$$

By repeating this argument, we obtain a sequence $(\tilde{t}_k)$ such that

$$v(x_{k+1}, t_{k+1}) \leq v(x_k, \tilde{t}_{k+1}) \leq v(x_k, t_k)$$

and

$$\begin{aligned} -\epsilon_k \phi'(\tilde{t}_{k+1} - t_k) - \Delta v(x_k, \tilde{t}_{k+1}) &\geq v(x_k, \tilde{t}_{k+1})(1 - v^2(x_k, \tilde{t}_{k+1}) - \lambda u^2(x_k, \tilde{t}_{k+1})) \\ &\geq v(x_k, \tilde{t}_{k+1})(1 - v^2(x_k, \tilde{t}_{k+1})) > 0, \end{aligned} \tag{16}$$

where ϵ_k is chosen sufficiently small such that $0 < \epsilon_k < \frac{v(x_k, t_k)(1 - v^2(x_k, t_k))}{\max_{\mathbb{R}} |\phi'(t)|}$ and $\epsilon_k \to 0^+$ as $k \to \infty$.

Notice that

$$-\Delta v(x_k, \tilde{t}_{k+1}) = \sum_{x\sim x_k} \frac{\mu_{xx_k}}{\mu(x_k)} (v(x_k, \tilde{t}_{k+1}) - v(x, \tilde{t}_{k+1})) \leq v(x_k, \tilde{t}_{k+1}) - m.$$

We plug this into (16). Since ϕ' is bounded and $\lim_{k\to\infty} v(x_k, \tilde{t}_{k+1}) = m$, we obtain by letting $k \to \infty$

$$0 + m - m \geq m(1 - m^2) > 0,$$

which is impossible. Thus $v \geq -1$.

Similarly, we also have $u \ge -1$. Note that $(-u,-v)$ is also a solution of the system (4), then we deduce that $-u \ge -1, -v \ge -1$ or equivalently $u \le 1, v \le 1$. Combining these estimates, we arrive at $|u| \le 1$ and $|v| \le 1$.

Step 2. Proof of (ii). We use again contradiction argument. Suppose, contrary to the assertion (ii), that there exists $(x_0,t_0) \in V \times \mathbb{R}$ such that $u^2(x_0,t_0)+v^2(x_0,t_0) > 1$. Let us put

$$M := \sup_{V\times\mathbb{R}} (u^2(x,t)+v^2(x,t)).$$

We again split the proof into two cases.

Case 1. If $u^2(x,t)+v^2(x,t)$ attains its maximum at this point (x_0,t_0), then we have $(u^2+v^2)_t(x_0,t_0)=0$ and $\Delta(u^2+v^2)(x_0,t_0) \le 0$, which leads to $(u^2+v^2)_t(x_0,t_0) - \Delta(u^2+v^2)(x_0,t_0) \ge 0$. On the other hand, we use [21, Lemma 2.2] to deduce that for any function $\widetilde{u} : \ell(V) \to \mathbb{R}$,

$$\Delta\widetilde{u}^2(x) = 2\widetilde{u}(x)\Delta\widetilde{u}(x) + |\nabla_{xy}\widetilde{u}|^2 \ge 2\widetilde{u}(x)\Delta\widetilde{u}(x).$$

Then, at the point (x_0,t_0) we have

$$\begin{aligned}(u^2+v^2)_t - \Delta(u^2+v^2) &\le (2u(u_t-\Delta u) + 2v(v_t - \Delta v)) \\ &= (2u(u-u^3-\lambda uv^2) + 2v(v-v^3-\lambda vu^2)) \\ &= 2(-u^4-v^4+u^2+v^2-2\lambda u^2v^2) \\ &\le 2(u^2+v^2-(u^2+v^2)^2) < 0,\end{aligned} \tag{17}$$

where we have used $\lambda \ge 1$ and $u^2+v^2 > 1$ at (x_0,t_0). Thus, we obtain a contradiction.

Case 2. There exists an increasing sequence $((u^2+v^2)(x_k,t_k))_{k\ge 0}$ such that

$$\lim_{k\to\infty}(u^2(x_k,t_k)+v^2(x_k,t_k)) = M.$$

Let ϵ_0 be a positive constant satisfying

$$0 < \epsilon_0 < \frac{|2((u^2+v^2)(x_0,t_0) - (u^2+v^2)^2(x_0,t_0))|}{\max_{\mathbb{R}}|\phi'(t)|}.$$

We put

$$w(x_0,t) = u^2(x_0,t)+v^2(x_0,t) - \epsilon_0\phi(t-t_0).$$

Since v and u are bounded, then $w(x_0,t) \to -\infty$ when $|t| \to \infty$. Consequently, there is a point $(x_0,\widetilde{t}_1) \in V \times \mathbb{R}$ such that

$$w(x_0,\widetilde{t}_1) = \max\{w(x_0,t); t \in \mathbb{R}\} \ge w(x_0,t_0) = u^2(x_0,t_0)+v^2(x_0,t_0) > 1.$$

By the property of local maximum, we have at $(x_0,\widetilde{t}_1)$ that $w_t = 0$ or $(u^2+v^2)_t(x_0,\widetilde{t}_1) = \epsilon_0\phi'(\widetilde{t}_1 - t_0)$. This and (17) give

$$\epsilon_0\phi'(\widetilde{t}_1-t_0) - \Delta(u^2+v^2)(x_0,\widetilde{t}_1) \le 2(u^2(x_0,\widetilde{t}_1)+v^2(x_0,\widetilde{t}_1) - (u^2(x_0,\widetilde{t}_1)+v^2(x_0,\widetilde{t}_1))^2).$$

As above, without loss of generality, we may assume that $(u^2+v^2)(x_1,t_1) \ge (u^2+v^2)(x_0,\widetilde{t}_1)$. Thus,

$$(u^2+v^2)(x_1,t_1) \ge (u^2+v^2)(x_0,\widetilde{t}_1) \ge (u^2+v^2)(x_0,t_0) > 1.$$

By repeating this argument, we obtain a sequence $(\widetilde{t}_k)$ such that

$$(u^2+v^2)(x_{k+1},t_{k+1}) \ge (u^2+v^2)(x_k,\widetilde{t}_{k+1}) \ge (u^2+v^2)(x_k,t_k)$$

and

$$\epsilon_k\phi'(\widetilde{t}_{k+1}-t_k) - \Delta(u^2+v^2)(x_k,\widetilde{t}_{k+1}) \le 2((u^2+v^2)(x_k,t_k) - (u^2+v^2)^2(x_k,t_k)) < 0, \tag{18}$$

where ϵ_k is chosen such that $0 < \epsilon_k < \frac{|2((u^2+v^2)(x_k,t_k)-(u^2+v^2)^2(x_k,t_k))|}{\max_{\mathbb{R}}|\phi'(t)|}$ and $\epsilon_k \to 0^+$ as $k \to \infty$.

Notice that

$$\begin{aligned}-\Delta(u^2+v^2)(x_k,\widetilde{t}_{k+1}) &= \sum_{x\sim x_k}\frac{\mu_{xx_k}}{\mu_{x_k}}((u^2+v^2)(x_k,\widetilde{t}_{k+1})-(u^2+v^2)(x,\widetilde{t}_{k+1}))\\ &\geq (u^2+v^2)(x_k,\widetilde{t}_{k+1})-M.\end{aligned}$$

Plugging this into (18) and letting $k\to\infty$ we obtain

$$0+M-M\leq 2(M-M^2)<0,$$

which is impossible.

Step 3. Proof of (iii). The proof in this case is quite similar to the Proof of (ii). So we omit the details.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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